Mahler's measure and Special Values of L-functions

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Mahler's measure

- If $P(x) = a_0x^d + \cdots + a_d = a_0 \prod_{j=1}^{d}(x - \alpha_j)$ then the *Mahler measure of P* is

$$M(P) = |a_0| \prod_{j=1}^{d} \max(|\alpha_j|, 1)),$$

  and the *logarithmic Mahler measure of P* is

$$m(P) = \log M(P) = \log |a_0| + \sum_{j=1}^{d} \log^+ |\alpha_j|.$$

- If $P \in \mathbb{Z}[x]$ then $M(P)$ is an algebraic integer. $M(P) = 1$ if and only if all the zeros of P are roots of unity (cyclotomic polynomials).

Lehmer's question

- In 1933, D.H. Lehmer was interested in the prime factors occurring in sequences of numbers of the form

$$\Delta_n(P) = a_0^n \prod_{j=1}^{d} (\alpha_j^n - 1)$$

- The growth of these numbers satisfies

$$\lim_{n\to\infty} |\Delta_n|^{1/n} = M(P).$$

- Lehmer asked whether for non-cyclotomic P, the growth rate could be smaller than $M(P) = 1.1762808\cdots = \lambda$, attained for the "Lehmer polynomial"

$$P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

Dick Lehmer 1927

$$\lambda = 1.17628081825991750\ldots$$

$$\log \lambda = 0.16235761200773813\,9\ldots$$

Mahler's measure for many variables

- Jensen's formula from Complex Analysis gives

$$m(P) = \frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{it})|dt$$

- And then, for example, if $n \to \infty$,

$$m(1+x+x^n) \to \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log|1+e^{it}+e^{iu}|dt\,du$$

- suggesting that if $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we define

$$m(P) = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n} \log|P(x_1,\ldots,x_n)|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$

$$\mathbb{T}^n = \{|x_1| = 1\} \times \cdots \times \{|x_n| = 1\}$$

Kurt Mahler

$$m(P) =$$

$$\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

What does the Mahler measure measure?

- Mahler (1962) introduced his measure as a tool in proving inequalities useful in transcendence theory.
- The main advantage of $M(P)$ over other measures of the size of P is that $M(PQ) = M(P)M(Q)$.
- A more intrinsic meaning: $P(x_1, \ldots, x_n)$ may occur as a characteristic polynomial in the description of certain discrete dynamical systems.
- In this case, $m(P)$ measures the rate of growth of configurations of a certain size as the system evolves so $m(P)$ is the *entropy* of the system, e.g. Lind, Schmidt and Ward (1990).

The set $\mathbb{L}$ of all measures

- B-Lawton (1980-1982): if $k_2 \to \infty, \ldots, k_n \to \infty$ then

$$m(P(x, x^{k_2}, \ldots, x^{k_n})) \to m(P(x_1, \ldots, x_n))$$

- so $m(P(x_1, \ldots, x_n))$ is the limit of measures of one-variable polynomials
- Conjecture (B, 1981): $\mathbb{L}$ is a closed subset of the real numbers. From this a qualitative form of "Lehmer's conjecture" would follow.
- $m(1 + x + y)$ is a limit point of $\mathbb{L}$, in fact B (1981), generalized by Condon (2012)

$$m(1 + x + x^n) = m(1 + x + y) + c(n)/n^2 + O(1/n^3),$$

where $c(n) \neq 0$ depends only on $n \ mod \ 3$

A Sign of Things to Come: Smyth's formula

- Smyth (1981), Ray (1987)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

- Notation for some basic constants

$$d_f = L'(\chi_{-f}, -1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2)$$

- e.g.

$$L(\chi_{-3}, 2) = 1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \dots$$

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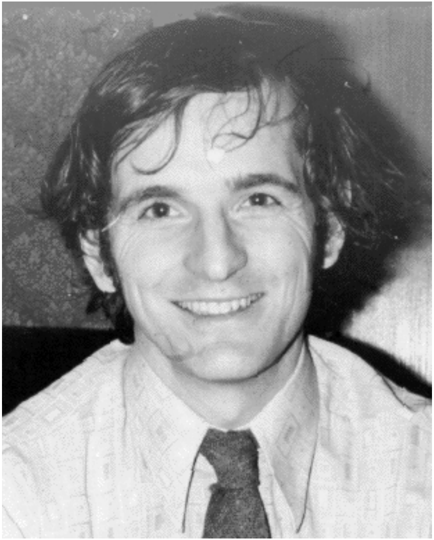
$$L(\chi_{-4}, 2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots$$

Chris Smyth

$$m(1 + x + y) = L'(\chi_{-3}, -1)$$

$$= 0.3230659472194505140\ldots$$



Aside: A possible connection with topology?

- Milnor (1982) – "Hyperbolic Geometry the first 150 Years" recalled a result of Lobachevsky:

  The number πd_3 is the volume of a hyperbolic tetrahedron T with all vertices at infinity (and thus all dihedral angles equal to $\pi/3$).

- Riley (1975) the complement of the figure-8 knot can be triangulated by 2 such equilateral tetrahedra.

- Does the appearance of d_3 in the formula for $m(1 + x + y)$ have any relationship to these facts from hyperbolic geometry?

Some small measures

$$\beta_1 = m(1+x+y) = d_3 = 0.32306594\ldots$$

$$\alpha_2 := m(x+y+1+1/x+1/y) = 0.25133043\ldots$$

$$\alpha_1 := m(xy+y+x+1+1/x+1/y+1/(xy)) = 0.22748122\ldots$$

- Notice that the polynomials in the latter two formulas are *reciprocal*, i.e. invariant under $x \to 1/x, y \to 1/y$.
- Are there formulas for α_1 and α_2 like Smyth's formula for β_1?
- Are α_1 and α_2 genuine limit points of $\mathbb{L}$?
- Are α_1 and α_2 the smallest two limit points of $\mathbb{L}$?
- Mossinghoff and B (2005) used a variety of methods to search for $m(P(x,y)) < \log(1.37) = 0.3148\ldots < \beta_1$ and found 48 of them.

Deninger's Conjecture

- Deninger (1995): Provided $P(x_1, \ldots, x_n) \neq 0$ on $\mathbb{T}^n$, $m(P)$ is related to the cohomology of the variety $\mathcal{V} = \{P(x_1, \ldots, x_n) = 0\}$.
- In particular (here $P = 0$ does intersect $\mathbb{T}^2$ but harmlessly.)

$$m(1 + x + 1/x + y + 1/y) \stackrel{?}{=} L'(E_{15}, 0)$$

E_{15} the elliptic curve of conductor $N = 15$ defined by $P = 0$.

- More notation: If E_N is an elliptic curve of conductor N, write

$$b_N = L'(E_N, 0) = \frac{N}{\pi^2} L(E_N, 2),$$

- The smallest possible conductors for elliptic curves over $\mathbb{Q}$ are 11, 14, 15, 17, 19, 20, 21 and 24 (each with 1 isogeny class).

Christopher Deninger

$$m(1 + x + 1/x + y + 1/y)$$
$$= 0.251330433713252 2\ldots$$
$$\stackrel{?}{=} L'(E_{15}, 0)$$

Elliptic curve L-functions

- $$L(E,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1}$$
  where the a_p are given by counting points on $E(\mathbb{F}_p)$

- The a_n are also the coefficients of a cusp form of weight 2 on $\Gamma_0(N)$, e.g. for $N = 15$,
  $$\sum_{n=1}^{\infty} a_n q^n = q \prod_{n=1}^{\infty} (1-q^n)(1-q^{3n})(1-q^{5n})(1-q^{15n})$$

Conjectures inspired by computation

- (B, 1996) If k is an integer then for certain specific rationals r_k,

$$F(k) := m(k + x + 1/x + y + 1/y) \stackrel{?}{=} r_k b_{N_k} \qquad (\star)$$

- (Rodriguez-Villegas, 1997) Let $q = \exp(\pi i \tau)$ be the modulus of the elliptic curve $k + x + 1/x + y + 1/y = 0$, then

$$m(k + x + 1/x + y + 1/y) = \mathrm{Re}\left(-\pi i \tau + 2 \sum_{n=1}^{\infty} \sum_{d|n} \left(\frac{-4}{d}\right) d^2 \frac{q^n}{n}\right)$$

- Hence the conjecture $(\star)$ follows (with generic rationals) from the Bloch-Beilinson conjectures.
- In fact would follow from these conjectures even for $k^2 \in \mathbb{Q}$.

Fernando Rodriguez-Villegas

$$m(k+x+1/x+y+1/y) =$$

$$\mathrm{Re}\left(-\pi i\tau + 2\sum_{n=1}^{\infty}\sum_{d|n}\left(\frac{-4}{d}\right)d^2\frac{q^n}{n}\right)$$

Conjectures become Theorems

- (Rodriguez-Villegas, 1997) $(\star)$ is true for $k^2 = 8, 18$ and 32 using some of the proven cases of Beilinson's conjecture, e.g. for CM curves
- (Lalín – Rogers, 2006) $(\star)$ is true for $k = 2$ and $k = 8$.
by establishing a number of useful functional equations for the function $F(k)$ in the LHS of $(\star)$ using calculations in $K_2(E)$
- (Rogers - Zudilin, 2012) $(\star)$ is true for $k^2 = -4, -1$ and 2,
by proving directly formulas for $L(E, 2)$ as special values of ${}_3F_2$ hypergeometric functions and then comparing directly with the corresponding formula for $F(k)$ of Rodriguez-Villegas.
- But what about the case $k = 1$?

Matilde Lalín and Mat Rogers 2006

Continuing the detour into Hyperbolic Geometry - A-polynomials

- By a result of Thurston the complement of any knot in 3-space can be triangulated by a finite collection of hyperbolic tetrahedra $T(z)$ with well-determined "shapes".
- The shape $z \in \mathbb{C}$ is equal to the cross-ratio of the sides of $T(z)$ the volume of the tetrahedron is given by $\mathcal{D}(z)$, where $\mathcal{D}$ is the
- Bloch–Wigner dilogarithm

$$\mathcal{D}(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$$

where

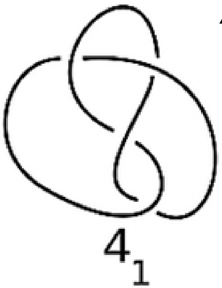
$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

The A-polynomial of a hyperbolic knot

- (Cooper, Culler, Gillet, Long and Shalen 1994) defined a new knot invariant, $A_K(x, y)$ for each knot K in 3-space
- (B and Rodriguez-Villegas, 2005) For any K, $m(A_K)$ is a finite sum of $\mathcal{D}(z)$ where the z are algebraic numbers.
- Under favourable circumstances $\pi m(A_K) = vol(\mathbb{H}^3 \backslash K)$.
- In particular this holds for the figure-8 knot where

$$A_K(x, y) = -y + x^2 - x - 1 - 1/x + 1/x^2 - 1/y$$

The Figure-8 Knot and its A-polynomial



$$A_K(x, y) = -y + x^2 - x - 1 - 1/x + 1/x^2 - 1/y$$

$$m(A_K) = \frac{1}{\pi}\, vol(\mathbb{H}^3 \backslash K) = 2d_3$$

Finally the long sought after formula for α_2!

- (Rogers - Zudilin, 2014) $(\star)$ is true for $k = 1$.
  Their method is elementary but complex. It depends on a direct and clever integration of certain modular equations of Ramanujan.

- In the midst of their calculation, they need a formula for $m(A)$ where
$$A(x, y) = -y + x^2 - x - 1 - 1/x + 1/x^2 - 1/y.$$
- $A(x, y) = 0$ defines an elliptic curve of conductor 15 but their proof requires that $m(A) = 2d_3$ not a rational multiple of b_{15}.
- However, we recognize this polynomial as exactly $A_K(x, y)$ the A-polynomial of the figure-8 knot so the result of the previous slide gives exactly what is needed!

Mat Rogers and Wadim Zudilin

More recent results about $m(k + x + y + 1/x + 1/y)$.

- Brunault (2015) uses Siegel modular units to parametrize E_N
- thus proves the conjecture $(\star)$ for $k = 3$ and $k = 12$
- with conductors $N = 21$ and $N = 48$, respectively.
- As in all of the earlier results, an individual calculation is needed for each value of k
- So we are still seeking a general method that will deal with the whole family of curves $k + x + 1/x + y + 1/y$

$F(k) = m(k + x + y + 1/x + 1/y)$ if k^2 is not rational.

- Samart (2015) has shown that if $k^2 \notin \mathbb{Q}$ so that E_k is not defined over $\mathbb{Q}$ then we can still expect formulas for $F(k)$ in certain situations.
- For example, he proves that

$$F(\sqrt{8 \pm 6\sqrt{2}}) = \frac{1}{2}(b_{64} \pm b_{32})$$

so in this case $(\star)$ does not hold – because E_k is defined over $\mathbb{Q}(\sqrt{2})$ and not over $\mathbb{Q}$

What about the limit point α_1?

- A conjecture from (B, 1996):

$$\alpha_1 = m(xy + y + x + 1 + 1/x + 1/y + 1/(xy)) \stackrel{?}{=} b_{14} \qquad (\star\star)$$

- Mellit (2012) proved this as well as 4 other of the formulas conjectured in (B, 1996) involving elliptic curves of conductor 14.
- He begins by observing that both sides of $(\star\star)$ can be expressed as linear combinations of elliptic dilogarithms.
- Then he uses a method of "parallel lines" to generate enough linear relations between values of elliptic dilogs at points of $E(\mathbb{Q})$ to deduce $(\star\star)$
- The method seems to work for $N = 20, 24$ but not for $N = 15$.

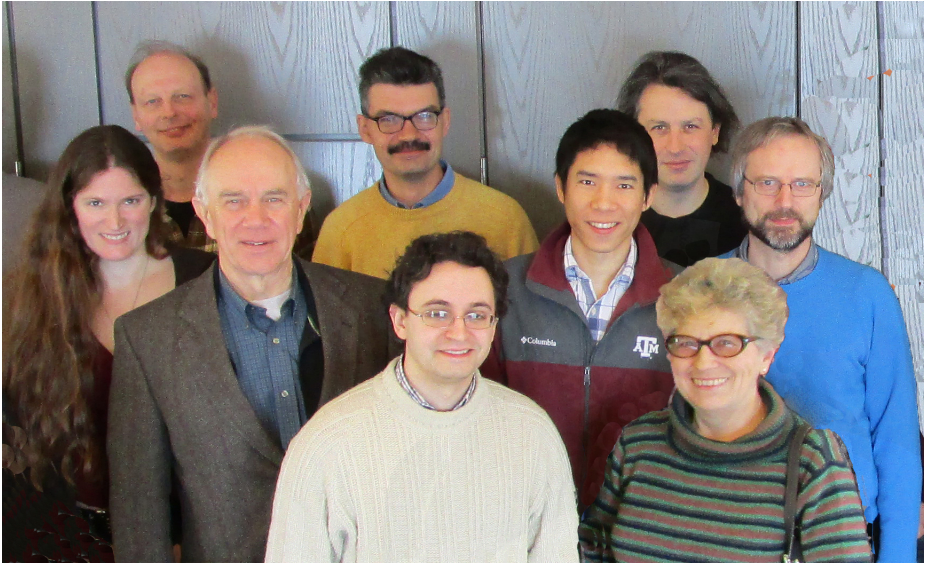
Other methods and results

- Brunault (2006) parametrizes $X_1(11)$ by modular units to prove

$$m(y^2 + (x^2 + 2x - 1)y + x^3) = 5b_{11}$$

- $X_1(11)$ has a model $y^2 + y + x^3 + x^2 = 0$
- Write $y^2 + y + x^3 + x^2 = (y - y_1(t))(y - y_2(t))$ for $x = e^{it}$, then
- $m(P) = \frac{1}{\pi}\int_0^\pi |\log|y_2(t)|| = 0.4056029\ldots$ seemingly not rb_{11}
- However $\frac{1}{\pi}\int_0^\pi \log|y_2(t)| = 0.1521471\ldots = b_{11}$ to 50 d.p.
- Fortunately (for Lehmer!) this is not a Mahler measure since $b_{11} = 0.1521471\ldots < 0.1623637\ldots = \log(\text{Lehmer's constant})$

Workshop at CRM, Montreal, February 2015



David, Matilde and Fernando - Niven Lectures 2007