Mahler's measure and L-functions of elliptic curves at $s = 3$

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Mahler's measure

- If $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the *logarithmic Mahler measure* is

$$m(P) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

$$\mathbb{T}^n = \{|x_1| = 1\} \times \cdots \times \{|x_n| = 1\}$$

- If $P(x) = a_0 \prod_{j=1}^{d}(x - \alpha_j)$, Jensen gives

$$m(P) = \log|a_0| + \sum_{j=1}^{d} \log^+ |\alpha_j|,$$

the logarithm of an algebraic integer if $P \in \mathbb{Z}[x]$.

Why the torus?

- B-Lawton (1980's)

$$m\big(P(x, x^{k_2}, \dots, x^{k_n})\big) \to m\big(P(x_1, \dots, x_n)\big),$$

if $k_2 \to \infty, \dots, k_n \to \infty$ in a suitable manner.

- so every $m(P(x_1, \dots, x_n))$ is the limit of measures of one-variable polynomials
- This would not be true if we integrated over the N-ball, for example

Examples with more variables

- Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

- Notation for some basic constants

$$d_f = L'(\chi_{-f}, -1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2), \quad z_3 = \frac{1}{\pi^2}\zeta(3)$$

Dirichlet L-functions

- $$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

  where $\chi(n)$ is a Dirichlet character

- e.g.

  $$L(\chi_{-3}, 2) = 1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \dots,$$

  the signs are given by the Legendre symbol $\left(\frac{n}{3}\right)$

Deninger's insight

- Deninger (1995): Provided $P(x_1, \ldots, x_n) \neq 0$ on $\mathbb{T}^n$, $m(P)$ is related to the cohomology of the variety $\mathcal{V} = \{P(x_1, \ldots, x_n) = 0\}$.
- In particular (here $P = 0$ does intersect $\mathbb{T}^2$ but harmlessly.)

$$m(1 + x + 1/x + y + 1/y) \stackrel{?}{=} L'(E_{15}, 0)$$

E_{15} the elliptic curve of conductor $N = 15$ defined by $P = 0$.

- More notation

$$b_N = L'(E_N, 0) = \frac{N}{\pi^2} L(E_N, 2),$$

E_N an elliptic curve of conductor N.

Elliptic curve L-functions

- $$L(E,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1}$$
  where the a_p are given by counting points on $E(\mathbb{F}_p)$

- The a_n are also the coefficients of a cusp form of weight 2 on $\Gamma_0(N)$, e.g. for $N = 15$,
  $$\sum_{n=1}^{\infty} a_n q^n = q \prod_{n=1}^{\infty} (1-q^n)(1-q^{3n})(1-q^{5n})(1-q^{15n})$$

Recipe for making conjectures

- Compute $m(P)$ for lots of $P(x, y)$
- Stir in various constants, $d_3, d_4, d_7, \ldots, b_{11}, b_{14}, \ldots$
- Apply the Lenstra–Lenstra– Lovasz algorithm (LLL)
- Publish the results

Conjectures follow from deeper conjectures

- (B, 1996)

$$m(k+x+1/x+y+1/y) \stackrel{?}{=} r_k L'(E_{N_k}, 0), \text{ for specific rationals } r_k \quad (\star)$$

- (Rodriguez-Villegas, 1997)

$$m(k+x+1/x+y+1/y) = \mathrm{Re}\left(-\pi i \tau + 2 \sum_{n=1}^{\infty} \sum_{d|n} \left(\frac{-4}{d}\right) d^2 \frac{q^n}{n}\right)$$

where $q = \exp(\pi i \tau)$ is the modulus of the curve E_{N_k}

- Hence the conjecture $(\star)$ follows (with generic rationals) from the Bloch-Beilinson conjectures.

Conjectures become Theorems

- (Rodriguez-Villegas, 1997) $(\star)$ is true for $k = 3\sqrt{2}$
- (Lalín – Rogers, 2006) $(\star)$ is true for $k = 2$ and $k = 8$
- (Brunault, 2005)

$$m(y^2 + (x^2 + 2x - 1)y + x^3) = \frac{5}{4} b_{11},$$

as conjectured in (B, 1996)

Precursors of a general idea

- (Smyth, 2002)

$$m(1 + x^{-1} + y + (1 + x + y)z) = \frac{14}{3}\frac{\zeta(3)}{\pi^2} = \frac{14}{3}z_3$$

- (Lalín, 2003)

$$m((1 + x_1)(1 + x_2)(1 + x_3) + (1 - x_2)(1 - x_3)(1 + x_4)x_5) = 93\frac{\zeta(5)}{\pi^4}$$

Maillot's insight

- (Darboux, 1875) If $P^*(\mathbf{x}) = \mathbf{x}^{\deg(P)} P(\mathbf{x}^{-1})$ is the reciprocal of P and if

$$\mathcal{V} = \{P = 0\}, \quad \text{and} \quad \mathcal{W} = \{P = 0\} \cap \{P^* = 0\},$$

  then

$$\mathcal{V} \cap \mathbb{T}^n = \mathcal{W} \cap \mathbb{T}^n$$

- (Maillot, 2003) In case $\mathcal{V}$ intersects $\mathbb{T}^n$ non-trivially, $m(P)$ is related to the cohomology of $\mathcal{W}$.

- The reason that Smyth's and Lalín's examples involve only ordinary polylogarithms can be "explained" using this observation.

Elliptic curves again

- (Rodriguez-Villegas, 2003) If $P = (1+x)(1+y)+z$, then $\mathcal{W}$ is an elliptic curve of conductor 15, so perhaps

$$m(P) \stackrel{?}{=} rL'(E_{15}, -1), \quad \text{with} \quad r \in \mathbb{Q}$$

$$L'(E_N, -1) = 2\frac{N^2}{(2\pi)^4} L(E_N, 3)$$

- (B, 2003) Yes!

$$m(P) = 2L'(E_{15}, -1), \text{to 28 decimal places}$$

- More Notation

$$L_N = L'(E_N, -1) = 2\frac{N^2}{(2\pi)^4} L(E_N, 3)$$

Recipe for making more conjectures

- Compute $m(P)$ for lots of $P(x, y, z)$ with $\mathcal{W}$ a curve of small genus
- Stir in various constants, $d_3, d_4, d_7, \ldots, z_3, L_{11}, L_{14}, \ldots$
- Apply the Lenstra–Lenstra– Lovasz algorithm (LLL)
- Show the results to Fernando, Matilde and Mat to see if they can prove them
- Publish the results

A very useful formula

- (Cassaigne–Maillot, 2000) for $a, b, c \in \mathbb{C}^*$,

$$m(a + by + cz)$$

$$= \begin{cases} \frac{1}{\pi}\left(\mathcal{D}\left(\frac{|a|}{|b|}\mathrm{e}^{\mathrm{i}\gamma}\right) + \alpha \log|a| + \beta \log|b| + \gamma \log|c|\right), & \text{if } \triangle \\ \max\{\log|a|, \log|b|, \log|c|\}, & \text{if not } \triangle \end{cases}$$

- The condition $\triangle$ means that $|a|, |b|, |c|$ form the sides of a triangle with angles α, β, γ.

- Bloch–Wigner dilogarithm

$$\mathcal{D}(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1 - x) \log|x|$$

Computing $m(P)$ for some 3-variable examples

- If $P(x,y,z) = a(x) + b(x)y + c(x)z$ then $f(t) = m(P(e^{it}, y, z))$ is given by the Cassaigne–Maillot formula
- Numerically integrate to compute

$$m(P(x,y,z)) = \frac{1}{\pi} \int_0^{\pi} f(t)\, dt$$

- On non-$\triangle$ intervals, we integrate logs and hence obtain dilogs of algebraic numbers
- On $\triangle$ intervals, we integrate dilogs and hence expect to obtain *tri-logs* perhaps *elliptic trilogs* ($\longrightarrow L(E,3)$ by Zagier's conjecture).

Computing $m(P)$ – a simple example

- $P(x, y, z) = (1 + x)(1 + y) + z$, so $a = b = 1 + x$ and $c = 1$.
- If $x = e^{it}$ then $|a| = |b| = 2\cos(\frac{t}{2})$
- If $|a| > \frac{1}{2}$, i.e. $t < t_0 = 2\cos^{-1}(\frac{1}{4}) = 2.63623\ldots$ then we are in the (isoceles) $\triangle$ case with $\gamma = 2\sin^{-1}\Big(1/(4\cos(t/2))\Big)$ and $f(t) =$ the hard part of the C-M formula.
- If $t \geq t_0$ then $f(t) = \log|c| = 0$, by the easy part of the formula.
- $m(P) = 0.4839979734786385357732733911 = 2L_{15}$ to 28 d.p.

Which polynomials?

- $P(x, y, z) = a(x) + b(x)y + c(x)z$, with a, b, c cyclotomic of degree ≤ 4
- Eliminate z from P and P^* to obtain an equation for
$$\mathcal{W} = \{Q(x, y) = 0\}$$
- $\mathcal{W}$ is the hyperelliptic curve $Y^2 = \mathrm{disc}_y\, Q$, genus g, say
- If $g \leq 2$ compute $m(P)$ and apply the recipe of the previous slide to make conjectures

Examples of genus 0

- 1. Conjectured by B (2003), proved in John Condon's thesis (2004)

$$m(x-1+(x+1)(y+z)) = \frac{28}{5} z_3$$

- 2.

$$m(x^2+1+(x^2+x+1)(y+z)) = \frac{10}{9} d_3 + \frac{35}{18} z_3$$

Matilde Lalín and Mat Rogers each have proofs of this (2006)

- 3.

$$m((x-1)^2+(x^2+1)(y+z)) = -d_3 + 2d_4$$

NB: no trilog term here – we do integrate a dilog.
The negative coefficient of d_3 is also notable (and useful).

Examples of genus 1

- 1. A mixture of a dilog and $L(E_{45}, 3)$

$$m(1 + (x^2 - x + 1)y + (x^2 + x + 1)z) \stackrel{?}{=} d_3 + \frac{1}{6} L_{45}$$

- 2. Here $\mathcal{W}$ is an elliptic curve of conductor 57, but $m(P)$ is an ordinary trilog.

$$m(x^2 + x + 1 + (x^2 - 1)(y + z)) \stackrel{?}{=} \frac{28}{5} z_3$$

- 3. Here we have both the ordinary trilog z_3 and an elliptic trilog L_{21}

$$m(x - 1 + (x^2 - 1)y + (x^2 + x + 1)z) \stackrel{?}{=} \frac{2}{3} d_3 + \frac{199}{72} z_3 + \frac{11}{24} L_{21}$$

Examples of genus 2

- Since $Q(x, y)$ is reciprocal $\mathrm{Jac}(\mathcal{W}) = E \times F$ for elliptic curves E, F (Jacobi, 1832)

- 1. Here E and F have conductors 14 and $112 = 2^4 \cdot 7$

$$m((x-1)^3 + (x+1)(y+z)) \stackrel{?}{=} 6L_{14}$$

- 2. Here E and F have conductors 108 and 36 ($E: Y^2 = X^3 + 4$)

$$m((x-1)^3 + (x+1)^3(y+z)) \stackrel{?}{=} -\frac{28}{15}z_3 + \frac{2}{15}L_{108}$$

Representing $L(E,3)$

- Combining the genus 0 example

$$m(P_1) = m(x - 1 + (x+1)(y+z)) = \frac{28}{5} z_3$$

- with the genus 2 example

$$m(P_2) = m((x-1)^3 + (x+1)^3(y+z)) \stackrel{?}{=} -\frac{28}{15} z_3 + \frac{2}{15} L_{108}$$

- we obtain

$$m(P_1 P_2^3) \stackrel{?}{=} \frac{2}{5} L_{108},$$

showing the importance of negative coefficients in these formulas

Exotic formulas

- 1. This example is of genus 1 with $E: Y^2 = X^3 + X$ of conductor 64, but we have no formula for $m(P_1)$

$$P_1 = (x-1)^2 + (x+1)^2(y+z)$$

- We also have no formula for the following genus 2 example with $\mathrm{Jac} = E \times F$ with E, F of conductors $64, 192$

$$P_2 = (x+1)^2 + (x^4+1)y + (x^2+1)(x^2+x+1)z$$

- However, the missing ingredients of the two formulas must be the same since

$$m(P_1^{12}P_2^{19}) = 12m(P_1) + 19m(P_2) \stackrel{?}{=} -19d_3 + 20d_4 + 6L_{64}$$