Chapter 1

Euclidean Spaces and Their Geometry

By Euclidean n-space, we mean the space $\mathbb{R}^n$ of all (ordered) n-tuples of real numbers. This is the domain where much, if not most, of the mathematics taught in university courses such as linear algebra, vector analysis, differential equations etc. takes place. And although the main topic of this book is algebra, the fact is that algebra and geometry can hardly be seperated: we need a strong foundation in both. The purpose of this chapter is thus to provide a succinct introduction to Euclidean space, with the emphasis on its geometry.

1.1 $\mathbb{R}^n$ and the Inner Product.

1.1.1 Vectors and n-tuples

Throughout this text, $\mathbb{R}$ will stand for the real numbers. Euclidean n-space, $\mathbb{R}^n$, is by definition the set of all (ordered) n-tuples of real numbers. An n-tuple is just a sequence consisting of n real numbers written in a column like

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}.$$

Sometimes the term sequence is replaced by the term or string or word. The entries $r_1, \ldots, r_n$ are called the *components* of the n-tuple, r_i being the ith *component*. It's important to note that the order of the components matters:

e.g.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Definition 1.1. The elements of $\mathbb{R}^n$ will be called *vectors*, and $\mathbb{R}^n$ itself will be called a *Euclidean n-space.*

Vectors will be denoted by a bold faced lower case letters $\mathbf{a}, \mathbf{b}, \mathbf{c} \dots$ and so forth. To simplify our notation a bit, we will often express a vector as a row expression by putting

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = (r_1, r_2, \dots, r_n)^T.$$

A word of explanation about the term vector is in order. In physics books and in some calculus books, a vector refers any directed line segment in $\mathbb{R}^2$ or $\mathbb{R}^3$. Of course, a vector $\mathbf{r}$ in $\mathbb{R}^n$ is a directed line segment starting at the origin $\mathbf{0} = (0, 0, \dots, 0)^T$ of $\mathbb{R}^n$. This line segment is simply the set of points of the form $t\mathbf{r} = (tr_1, tr_2, \dots tr_n)^T$, where $0 \leq t \leq 1$. More generally, the term vector may refer to the set of all directed line segments parallel to a given segment with the same length. But in linear algebra, the term vector is used to denote an element of a vector space. The vector space we are dealing with here, as will presently be explained, is $\mathbb{R}^n$, and its vectors are therefore real n-tuples.

1.1.2 Coordinates

The Euclidean spaces $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$ are especially relevant since they physically represent a line, plane and a three space respectively. It's a familiar assumption that the points on a line L can be put into a one to one correspondence with the real numbers $\mathbb{R}$. If $a \in \mathbb{R}$ (that is, if a is an element of $\mathbb{R}$), then the point on L corresponding to a has distance $|a|$ from the origin, which is defined as the point corresponding to 0. Such a one to one correspondence puts a coordinate system on L.

Next, we put a coordinate system called xy-coordinates on a plane by selecting two (usually orthogonal) lines called an x-axis and a y-axis, each having a coordinate system, and identifying a point P in the plane with the element $(p_1, p_2)^T$ of $\mathbb{R}^2$, where p_1 is the projection of P parallel to the y-axis onto the x-axis, and p_2 is the projection of P parallel to the x-axis onto the y-axis. This is diagrammed in the following figure.

FIGURE (Euclidean PLANE)

In the same manner, the points of Euclidean 3-space are parameterized by the ordered 3-tuples of real numbers, i.e. $\mathbb{R}^3$; that is, that is, every point is uniquely identified by assigning it xyz-coordinates. Thus we can also put a coordinate system on $\mathbb{R}^3$.

FIGURE (Euclidean 3-space)

But just as almost everyone eventually needs more storage space, we may also need more coordinates to store important data. For example, if we are considering a linear equation such as

$$3x + 4y + 5z + 6w = 0,$$

where the solutions are 4-tuples, we need $\mathbb{R}^4$ to store them. While extra coordinates give more degrees of freedom, our geometric intuition doesn't work very well in dimensions bigger than three. This is where the algebra comes in.

1.1.3 The Vector Space $\mathbb{R}^n$

Vector addition in $\mathbb{R}^2$ or $\mathbb{R}^3$ is probably already very familiar to you. Two vectors are added by a rule called the Parallelogram Law, which we will review below. Since n may well be bigger than 3, we define vector addition in a different, in fact much simpler, way by putting

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}. \tag{1.1}$$

Thus addition consists of adding the corresponding components of the two vectors being added.

There is a second operation called *scalar multiplication*, where a vector $\mathbf{a}$ is dilated by a real number r. This is defined (in a rather obvious way) by

$$r\mathbf{a} = r \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} ra_1 \\ ra_2 \\ \vdots \\ ra_n \end{pmatrix}. \tag{1.2}$$

These two operations satisfy the axioms which define a vector space. They will be stated explicitly in Chapter 4. Scalar multiplication has an obvious geometric interpretation. Multiplying $\mathbf{a}$ by r stretches or shrinks $\mathbf{a}$ along itself by the factor $|r|$, changing its direction if $r < 0$. The geometric interpretation of addition is the Parallelogram Law.

Parallelogram Law: *The sum* $\mathbf{a}+\mathbf{b}$ *is the vector along the diagonal of the parallelogram with vertices at* $\mathbf{0}$, $\mathbf{a}$ *and* $\mathbf{b}$.

FIGURE
(PARALLELOGRAM LAW)

Thus vector addition (1.1) agrees with the classical way of defining addition. The Parallelogram Law in $\mathbb{R}^2$ by showing that the line through $(a, b)^T$ parallel to $(c, d)^T$ meets the line through $(c, d)^T$ parallel to (a, b) at $(a+c, b+d)^T$. Note that lines in $\mathbb{R}^2$ can be written in the form $rx + sy = t$, where $r, s, t \in \mathbb{R}$, so this is an exercise in writing the equation of a line and computing where two lines meet. (See Exercise 1.29.)

Checking the Parallelogram Law in $\mathbb{R}^3$ requires that we first discuss how to represent a line in $\mathbb{R}^3$. The Parallelogram Law in $\mathbb{R}^n$, will follow in exactly the same way. We will treat this matter below.

1.1.4 The dot product

We now take up *measurements* in $\mathbb{R}^n$. The way we measure things such as length and angles is to use an important operation called either the *inner product* or the *dot product*.

Definition 1.2. The inner product of two vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)^T$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T$ in $\mathbb{R}^n$ is defined to be

$$\mathbf{a} \cdot \mathbf{b} := \sum_{i=1}^{n} a_i b_i. \tag{1.3}$$

Note that if $n = 1$, $\mathbf{a} \cdot \mathbf{b}$ is the usual product. The inner product has several important properties. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be arbitrary vectors and r any scalar (i.e., $r \in \mathbb{R}$). Then

(1) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$,

(2) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$,

(3) $(r\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (r\mathbf{b}) = r(\mathbf{a} \cdot \mathbf{b})$, and

(4) $\mathbf{a} \cdot \mathbf{a} > 0$ unless $\mathbf{a} = \mathbf{0}$, in which case $\mathbf{a} \cdot \mathbf{a} = 0$.

These properties are all easy to prove, so we will leave them as exercises.

The *length* $|\mathbf{a}|$ of $\mathbf{a} \in \mathbb{R}^n$ is defined in terms of the dot product by putting

$$\begin{aligned} |\mathbf{a}| &= \sqrt{\mathbf{a} \cdot \mathbf{a}} \\ &= (\sum_{i=1}^{n} a_i^2)^{1/2}. \end{aligned}$$

This definition generalizes the usual square root of the sum of squares definition of length for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Notice that

$$|r\mathbf{a}| = |r||\mathbf{a}| \, .$$

The *distance* between two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the length of their difference $\mathbf{a} - \mathbf{b}$. Denoting this distance by $d(\mathbf{a}, \mathbf{b})$, we see that

$$\begin{aligned} d(\mathbf{a}, \mathbf{b}) &= |\mathbf{a} - \mathbf{b}| \\ &= \big((\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})\big)^{1/2} \\ &= \big(\sum_{i=1}^{n} (a_i - b_i)^2\big)^{1/2}. \end{aligned}$$

1.1.5 Orthogonality and projections

Next we come to an important notion which involves both measurement and geometry. Two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ are said to be *orthogonal* (a fancy word for perpendicular) if $\mathbf{a} \cdot \mathbf{b} = 0$. Note that the zero vector $\mathbf{0}$ is orthogonal to every vector, and by property (4) of the dot product, $\mathbf{0}$ is the only vector orthogonal to itself. Two vectors in $\mathbb{R}^2$, say $\mathbf{a} = (a_1, a_2)^T$ and $\mathbf{b} = (b_1, b_2)^T$, are orthogonal if and only if and only if $a_1 b_1 + a_2 b_2 = 0$. Thus if $a_1, b_2 \neq 0$, then $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $a_2/a_1 = -b_1/b_2$. Thus, the slopes of orthogonal vectors in $\mathbb{R}^2$ are negative reciprocals.

For vectors in $\mathbb{R}^n$, the meaning of orthogonality follows from the following property.

Proposition 1.1. *Two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbb{R}^n$ *are orthogonal if and only if* $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$.

Let's prove this geometrically, at least for $\mathbb{R}^2$. Consider the triangle with vertices at $\mathbf{0}, \mathbf{a}, \mathbf{b}$. The hypotenuse of this triangle is a segment of length $|\mathbf{a}-\mathbf{b}|$, which follows from the Parallelogram Law. Next consider the triangle with vertices at $\mathbf{0}, \mathbf{a}, -\mathbf{b}$. The hypotenuse of this triangle is a segment of length $|\mathbf{a}+\mathbf{b}|$, which also follows from the Parallelogram Law. Now suppose $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$. Then by the side side side criterion for congruence, which says that two triangles are congruent if and only if they have corresponding sides of equal length, the two triangles are congruent. It follows that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. For the converse direction, suppose $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. Then the side angle side criterion for congruence applies, so our triangles are congruent. Thus $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$.

DIAGRAM FOR PROOF

In fact, it is much easier to use algebra (namely the dot product). The point is that $\mathbf{a}\cdot\mathbf{b} = 0$ if and only if $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$. The details are left as an exercise.

One of the most fundamental applications of the dot product is the *orthogonal decomposition* of a vector into two or more mutually orthogonal components.

Proposition 1.2. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ *be given, and suppose that* $\mathbf{b} \neq \mathbf{0}$. *Then there exists a unique scalar* r *such that* $\mathbf{a} = r\mathbf{b} + \mathbf{c}$ *where* $\mathbf{b}$ *and* $\mathbf{c}$ *are orthogonal. In fact,*

$$r = (\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}),$$

and

$$\mathbf{c} = \mathbf{a} - (\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}})\mathbf{b}.$$

Proof. We see this as follows: since we want $r\mathbf{b} = \mathbf{a} - \mathbf{c}$, where $\mathbf{c}$ has the property that $\mathbf{b}\cdot\mathbf{c} = 0$, then

$$r\mathbf{b}\cdot\mathbf{b} = (\mathbf{a}-\mathbf{c})\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b} - \mathbf{c}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b}.$$

As $\mathbf{b}\cdot\mathbf{b} \neq 0$, it follows that $r = \mathbf{a}\cdot\mathbf{b}/\mathbf{b}\cdot\mathbf{b}$. The reader should check that $\mathbf{c} = \mathbf{a} - (\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}})\mathbf{b}$ is orthogonal to $\mathbf{b}$. Thus we get the desired orthogonal decomposition

$$\mathbf{a} = (\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}})\mathbf{b} + \mathbf{c}.$$

□

FIGURE 3
ORTHOGONAL DECOMPOSITION

Definition 1.3. The vector

$$P_{\mathbf{b}}(\mathbf{a}) = (\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}})\mathbf{b}$$

will be called the *orthogonal projection* of $\mathbf{a}$ on $\mathbf{b}$.

By the previous Proposition, another way to express the orthogonal decomposition of $\mathbf{a}$ into the sum of a component parallel to $\mathbf{b}$ and a component orthogonal to $\mathbf{b}$ is

$$\mathbf{a} = P_{\mathbf{b}}(\mathbf{a}) + (\mathbf{a} - P_{\mathbf{b}}(\mathbf{a})). \tag{1.4}$$

Now suppose $\mathbf{b}$ and $\mathbf{c}$ are any two nonzero orthogonal vectors in $\mathbb{R}^2$, so that $\mathbf{b}\cdot\mathbf{c} = 0$. I claim that any vector $\mathbf{a}$ orthogonal to $\mathbf{b}$ is a multiple of $\mathbf{c}$. Reason: if $\mathbf{b} = (b_1, b_2)^T$ and $\mathbf{a} = (a_1, a_2)^T$, then $a_1b_1 + a_2b_2 = 0$. Assuming, for example, that $b_1 \neq 0$, then

$$a_1 = -\frac{b_2}{b_1}a_2 = \frac{c_1}{c_2}a_2,$$

and the claim follows.

It follows that for any $\mathbf{a} \in \mathbb{R}^2$, there are scalars r and s so that $\mathbf{a} = r\mathbf{b} + s\mathbf{c}$. We can solve for r and s by using the dot product as before. For example, $\mathbf{a}\cdot\mathbf{b} = r\mathbf{b}\cdot\mathbf{b}$. Hence we can conclude that if $\mathbf{b} \neq \mathbf{0}$, then

$$r\mathbf{b} = P_{\mathbf{b}}(\mathbf{a}),$$

and similarly, if $\mathbf{c} \neq \mathbf{0}$, then

$$s\mathbf{c} = P_{\mathbf{c}}(\mathbf{a}).$$

Therefore, we have now proved a fundamental fact which we call the **projection formula** for $\mathbb{R}^2$.

Proposition 1.3. *If* $\mathbf{b}$ *and* $\mathbf{c}$ *are two non zero mutually orthogonal vectors in* $\mathbb{R}^2$*, then any vector* $\mathbf{a}$ *in* $\mathbb{R}^2$ *can be uniquely expressed as the sum of its projections. In other words,*

$$\mathbf{a} = P_{\mathbf{b}}(\mathbf{a}) + P_{\mathbf{c}}(\mathbf{a}) = (\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}})\mathbf{b} + (\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{c}\cdot\mathbf{c}})\mathbf{c}. \tag{1.5}$$

Projections can be written much more simply if we bring in the notion of a unit vector. When $\mathbf{b} \neq \mathbf{0}$, the *unit vector along* $\mathbf{b}$ is defined to be the vector of length one given by the formula

$$\widehat{\mathbf{b}} = \frac{\mathbf{b}}{(\mathbf{b}\cdot\mathbf{b})^{1/2}} = \frac{\mathbf{b}}{|\mathbf{b}|}.$$

(Check that $\widehat{\mathbf{b}}$ is indeed of length one,) Unit vectors are also called *directions.* Keep in mind that the direction $\widehat{\mathbf{a}}$ exists only when $\mathbf{a} \neq \mathbf{0}$. It is obviously impossible to assigne a direction to the zero vector. If $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{c}}$ are unit vectors, then the projection formula (1.5) takes the simpler form

$$\mathbf{a} = (\mathbf{a}\cdot\widehat{\mathbf{b}})\widehat{\mathbf{b}} + (\mathbf{a}\cdot\widehat{\mathbf{c}})\widehat{\mathbf{c}}. \tag{1.6}$$

Example 1.1. Let $\mathbf{b} = (3,4)^T$ and $\mathbf{c} = (4,-3)^T$. Then $\widehat{\mathbf{b}} = \frac{1}{5}(3,4)^T$ and $\widehat{\mathbf{c}} = \frac{1}{5}(4,-3)^T$. Let $\mathbf{a} = (1,1)$. Thus, for example, $P_{\mathbf{b}}(\mathbf{a}) = \frac{7}{5}(3,4)^T$, and $\mathbf{a} = \frac{7}{5}(3,4)^T + \frac{1}{5}(4,-3)^T$.

1.1.6 The Cauchy-Schwartz Inequality and Cosines

If $\mathbf{a} = \mathbf{b} + \mathbf{c}$ is an orthogonal decomposition in $\mathbb{R}^n$ (which just means that $\mathbf{b}\cdot\mathbf{c} = 0$), then

$$|\mathbf{a}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2.$$

This is known as Pythagoras's Theorem (see Exercise 4).

If we apply Pythagoras' Theorem to (1.4), for example, we get

$$|\mathbf{a}|^2 = |P_{\mathbf{b}}(\mathbf{a})|^2 + |\mathbf{a} - P_{\mathbf{b}}(\mathbf{a})|^2.$$

Hence,

$$|\mathbf{a}|^2 \geq |P_{\mathbf{b}}(\mathbf{a})|^2 = (\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}})^2|\mathbf{b}|^2 = \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{b}|^2}.$$

Cross multiplying and taking square roots, we get a famous fact known as the Cauchy-Schwartz inequality.

Proposition 1.4. *For any* $\mathbf{a},\ \mathbf{b} \in \mathbb{R}^n$*, we have*

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|.$$

Moreover, if $\mathbf{b} \neq \mathbf{0}$*, then equality holds if and only if* $\mathbf{a}$ *and* $\mathbf{b}$ *are collinear.*

Note that two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *collinear* whenever one of them is a scalar multiple of the other. If either $\mathbf{a}$ and $\mathbf{b}$ is zero, then automatically they are collinear. If $\mathbf{b} \neq \mathbf{0}$ and the Cauchy-Schwartz inequality is an equality, then working backwards, one sees that $|\mathbf{a} - P_{\mathbf{b}}(\mathbf{a})|^2 = 0$, hence the validity of the second claim. You are asked to supply the complete proof in Exercise 6.

Cauchy-Schwartz says that for any two unit vectors $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{b}}$, we have the inequality

$$-1 \leq \widehat{\mathbf{a}} \cdot \widehat{\mathbf{b}} \leq 1.$$

We can therefore define the angle θ between any two non zero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ by putting

$$\theta := \cos^{-1}(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}).$$

Note that we don't try to define the angle when either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$. (Recall that if $-1 \leq x \leq 1$, then $\cos^{-1} x$ is the unique angle θ such that $0 \leq \theta \leq \pi$ with $\cos\theta = x$.) With this definition, we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \tag{1.7}$$

provided $\mathbf{a}$ and $\mathbf{b}$ are any two non-zero vectors in $\mathbb{R}^n$. Hence if $|\mathbf{a}| = |\mathbf{b}| = 1$, then the projection of $\mathbf{a}$ on $\mathbf{b}$ is

$$P_{\mathbf{b}}(\mathbf{a}) = (\cos\theta)\mathbf{b}.$$

Thus another way of expressing the projection formula is

$$\widehat{\mathbf{a}} = (\cos\beta)\widehat{\mathbf{b}} + (\cos\gamma)\widehat{\mathbf{c}}.$$

Here β and γ are the angles between $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{c}$ respectively, and $\cos\beta$ and $\cos\gamma$ are the corresponding direction cosines.

In the case of $\mathbb{R}^2$, there is already a notion of the angle between two vectors, defined in terms of arclength on a unit circle. Hence the expression $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ is often (especially in physics) taken as definition for the dot product, rather than as definition of angle, as we did here. However, defining $\mathbf{a} \cdot \mathbf{b}$ in this way has the disadvantage that it is not at all obvious that elementary properties such as the identity $(\mathbf{a}+\mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$ hold. Moreover, using this as a definition in $\mathbb{R}^n$ has the problem that the angle between two vectors must also be defined. The way to solve this is to use arclength, but this requires bringing in an unnecessary amount of machinery. On the other hand, the algebraic definition is easy to state and remember, and it works for any dimension. The Cauchy-Schwartz inequality, which is valid in $\mathbb{R}^n$, tells us that it possible two define the angle θ between $\mathbf{a}$ and $\mathbf{b}$ via (1.7) to be $\theta := \cos^{-1}(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})$.

1.1.7 Examples

Let us now consider a couple of typical applications of the ideas we just discussed.

Example 1.2. A film crew wants to shoot a car moving along a straight road with constant speed x km/hr. The camera will be moving along a straight track at y km/hr. The desired effect is that the car should appear to have exactly half the speed of the camera. At what angle to the road should the track be built?

Solution: Let θ be the angle between the road and the track. We need to find θ so that the projection of the velocity vector $\mathbf{v}_R$ of the car on the track is exactly half of the velocity vector $\mathbf{v}_T$ of the camera. Thus

$$(\frac{\mathbf{v}_R \cdot \mathbf{v}_T}{\mathbf{v}_T \cdot \mathbf{v}_T})\mathbf{v}_T = \frac{1}{2}\mathbf{v}_T$$

and $\mathbf{v}_R \cdot \mathbf{v}_T = |\mathbf{v}_R||\mathbf{v}_T|\cos\theta$. Now $|\mathbf{v}_R| = x$ and $|\mathbf{v}_T| = y$ since speed is by definition the magnitude of velocity. Thus

$$\frac{xy}{y^2}\cos\theta = \frac{1}{2}$$

Consequently, $\cos\theta = y/2x$. In particular the camera's speed cannot exceed twice the car's speed.

Example 1.3. What we have seen so far can be applied to finding a formula for the distance from a point $\mathbf{v} = (v_1, v_2)^T$ in $\mathbb{R}^2$ to a line $ax + by = c$. Of course this problem can be solved algebraically by considering the line through $(v_1, v_2)^T$ orthogonal to our line. A more illuminating way to proceed, however, is to use projections since they will give a method which can be used in any $\mathbb{R}^n$, whereas it isn't immediately clear how to extend the first method. The way to proceed, then, is to begin by converting the line into a more convenient form. The way we will do this is to choose two points $(x_0, y_0)^T = \mathbf{a}$ and $(x_1, y_1)^T = \mathbf{b}$ on the line. Then the line can also be represented as the set of all points of the form $\mathbf{a}+t\mathbf{c}$, where $\mathbf{c} = \mathbf{b}-\mathbf{a}$. Since distance is invariant under translation, we can replace our original problem with the problem of finding the distance d from $\mathbf{w} = \mathbf{v} - \mathbf{a}$ to the line $t\mathbf{c}$. Since this distance is the length of the component of $\mathbf{w}$ orthogonal to $\mathbf{c}$, we get the formula

$$\begin{aligned} d &= |\mathbf{w} - P_{\mathbf{c}}(\mathbf{w})| \\ &= \left|\mathbf{w} - \left(\frac{\mathbf{w}\cdot\mathbf{c}}{\mathbf{c}\cdot\mathbf{c}}\right)\mathbf{c}\right| \end{aligned}$$

We will give another way to express this distance below.

Example 1.4. Suppose ℓ is the line through $(1,2)^T$ and $(4,-1)^T$. Let us find the distance d from $(0,6)^T$ to ℓ. Since $(4,-1)^T - (1,2)^T = (3,-3)^T$, we may as well take

$$\mathbf{c} = 1/\sqrt{2}(1,-1)^T.$$

We can also take $\mathbf{w} = (0,6)^T - (1,2)^T$, although we could also use $(0,6)^T - (4,-1)^T$. The formula then gives

$$\begin{aligned} d &= \left|(-1,4)^T - \left(\frac{(-1,4)^T\cdot(1,-1)^T}{\sqrt{2}}\right)\frac{(1,-1)^T}{\sqrt{2}}\right| \\ &= \left|(-1,4)^T - \left(\frac{-5}{2}\right)(1,-1)^T\right| \\ &= \left|\left(\frac{3}{2},\frac{3}{2}\right)^T\right| \\ &= \frac{3\sqrt{2}}{2}. \end{aligned}$$

Exercises

Exercise 1.1. Verify the four properties of the dot product on $\mathbb{R}^n$.

Exercise 1.2. Verify the assertion that $\mathbf{b}\cdot\mathbf{c} = 0$ in the proof of Theorem 1.2.

Exercise 1.3. Prove the second statement in the Cauchy-Schwartz inequality that $\mathbf{a}$ and $\mathbf{b}$ are collinear if and only if $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}|$.

Exercise 1.4. A nice application of Cauchy-Schwartz is that if $\mathbf{a}$ and $\mathbf{b}$ are unit vectors in $\mathbb{R}^n$ such that $\mathbf{a}\cdot\mathbf{b} = 1$, then $\mathbf{a} = \mathbf{b}$. Prove this.

Exercise 1.5. Show that $P_{\mathbf{b}}(r\mathbf{x} + s\mathbf{y}) = rP_{\mathbf{b}}(\mathbf{x}) + sP_{\mathbf{b}}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $r, s \in \mathbb{R}$. Also show that $P_{\mathbf{b}}(\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot P_{\mathbf{b}}(\mathbf{y})$.

Exercise 1.6. Prove the vector version of Pythagoras's Theorem. If $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{a}\cdot\mathbf{b} = 0$, then $|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

Exercise 1.7. Show that for any $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$,

$$|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 = 4\mathbf{a}\cdot\mathbf{b}.$$

Exercise 1.8. Use the formula of the previous problem to prove Proposition 2.1, that is to show that $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$ if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.

Exercise 1.9. Prove the law of cosines: If a triangle has sides with lengths a, b, c and θ is the angle between the sides of lengths a and b, then $c^2 = a^2 + b^2 - 2ab\cos\theta$. (Hint: Consider $\mathbf{c} = \mathbf{b} - \mathbf{a}$.)

Exercise 1.10. Another way to motivate the definition of the projection $P_{\mathbf{b}}(\mathbf{a})$ is to find the minimum of $|\mathbf{a} - t\mathbf{b}|^2$. Find the minimum using calculus and interpret the result.

Exercise 1.11. Orthogonally decompose the vector $(1, 2, 2)$ in $\mathbb{R}^3$ as $\mathbf{p} + \mathbf{q}$ where $\mathbf{p}$ is required to be a multiple of $(3, 1, 2)$.

Exercise 1.12. Use orthogonal projection to find the vector on the line $3x + y = 0$ which is nearest to $(1, 2)$. Also, find the nearest point.

Exercise 1.13. How can you modify the method of orthogonal projection to find the vector on the line $3x + y = 2$ which is nearest to $(1, -2)$?

1.2 Lines and planes.

1.2.1 Lines in $\mathbb{R}^n$

Let's consider the question of representing a line in $\mathbb{R}^n$. First of all, a line in $\mathbb{R}^2$ is cut out by a singlle linear equation $ax + by = c$. But a single equation $ax + by + cz = d$ cuts out a plane in $\mathbb{R}^3$, so a line in $\mathbb{R}^3$ requires at least two equations, since it is the intersection of two planes. For the general case, $\mathbb{R}^n$, we need a better approach. The point is that every line is determined two points. Suppose we want to express the line through $\mathbf{a}$ and $\mathbf{b}$ $\mathbb{R}^n$. Notice that the space curve

$$\mathbf{x}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = (1 - t)\mathbf{a} + t\mathbf{b}, \tag{1.8}$$

where t varies through $\mathbb{R}$, has the property that $\mathbf{x}(0) = \mathbf{a}$, and $\mathbf{x}(1) = \mathbf{b}$. As you can see from the Parallelogram Law, this curve traces out the line through $\mathbf{a}$ parallel to $\mathbf{b} - \mathbf{a}$ as in the diagram below.

Equation (1.8) lets us define a line in any dimension. Hence suppose $\mathbf{a}$ and $\mathbf{c}$ are any two vectors in $\mathbb{R}^n$ such that $\mathbf{c} \neq \mathbf{0}$.

Definition 1.4. The *line through* $\mathbf{a}$ *parallel to* $\mathbf{c}$ is defined to be the path traced out by the curve $\mathbf{x}(t) = \mathbf{a} + t\mathbf{c}$ as t takes on all real values. We will refer to $\mathbf{x}(t) = \mathbf{a} + t\mathbf{c}$ as the *vector form* of the line.

In this form, we are defining $\mathbf{x}(t)$ as a vector-valued function of t. The vector form $\mathbf{x}(t) = \mathbf{a} + t\mathbf{c}$ leads directly to *parametric form* of the line. In the parametric form, the components $x_1, \dots, x_n$ of $\mathbf{x}$ are expressed as linear functions of t as follows:

$$x_1 = a_1 + tc_1,\ x_2 = a_2 + tc_2, \dots, x_n = a_n + tc_n. \tag{1.9}$$

Letting $\mathbf{a}$ vary while $\mathbf{b}$ is kept fixed, we get the family of all lines of the form $\mathbf{x} = \mathbf{a} + t\mathbf{c}$. Every point of $\mathbb{R}^n$ is on one of these lines, and two lines either coincide or don't meet at all. (The proof of this is an exercise.) We will say that two lines $\mathbf{a} + t\mathbf{c}$ and $\mathbf{a}' + t\mathbf{c}'$ are *parallel* if $\mathbf{c}$ and $\mathbf{c}'$ are collinear. We will also say that the line $\mathbf{a} + t\mathbf{c}$ is parallel to $\mathbf{c}$.

Example 1.5. Let's find an expression for the line in $\mathbb{R}^4$ passing through $(3, 4, -1, 0)$ and $(1, 0, 9, 2)$. We apply the trick in (1.8). Consider

$$\mathbf{x} = (1 - t)(3, 4, -1, 0) + t(1, 0, 9, 2).$$

Clearly, when $t = 0$, $\mathbf{x} = (3, 4, -1, 0)$, and when $t = 1$, then $\mathbf{x} = (1, 0, 9, 2)$. We can also express $\mathbf{x}$ in the vector form $\mathbf{x} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ where $\mathbf{a} = (3, 4, -1, 0)$ and $\mathbf{b} = (1, 0, 9, 2)$. The parametric form is

$$x_1 = -2t + 1, \quad x_2 = -4t + 4, \quad x_3 = 10t + 1, \quad x_4 = 2t.$$

Example 1.6. The intersection of two planes in $\mathbb{R}^3$ is a line. Let's show this in a specific example, say the planes are $x+y+z = 1$ and $2x-y+2z = 2$. By inspection, $(1, 0, 0)$ and $(0, 0, 1)$ lie on both planes, hence on the intersection. The line through these two points is $(1-t)(1, 0, 0) + t(0, 0, 1) = (1-t, 0, t)$. Setting $x = 1 - t$, $y = 0$ and $z = t$ and substituting this into both plane equations, we see that this line does indeed lie on both planes, hence is in the intersection. But by staring at the equations of the planes (actually by subtracting twice the first equation from the second), we see that every point (x, y, z) on the intersection has $y = 0$. Thus all points on the intersection satisfy $y = 0$ and $x + z = 1$. But any point of this form is on our line, so we have shown that the intersection of the two planes is the line.

Before passing to planes, let us make a remark about the Parallelogram Law for $\mathbb{R}^n$, namely that $\mathbf{a} + \mathbf{b}$ is the vector along the diagonal of the parallelogram with vertices at $\mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$. This is valid in any $\mathbb{R}^n$, and can be seen by observing (just as we noted for $n = 2$) that the line through $\mathbf{a}$ parallel to $\mathbf{b}$ meets the line through $\mathbf{b}$ parallel to $\mathbf{a}$ at $\mathbf{a} + \mathbf{b}$. We leave this as an exercise.

1.2.2 Planes in $\mathbb{R}^3$

The solution set of a linear equation

$$ax + by + cz = d \tag{1.10}$$

in three variables x, y and z is called a *plane in* $\mathbb{R}^3$. The linear equation (1.10) expresses that the dot product of the vector $\mathbf{a} = (a, b, c)^T$ and the variable vector $\mathbf{x} = (x, y, z)^T$ is the constant d:

$$\mathbf{a} \cdot \mathbf{x} = d.$$

If $d = 0$, the plane passes through the origin, and its equation is said to be *homogeneous*. In this case it is easy to see how to interpret the plane equation. The plane $ax + by + cz = 0$ consists of all $(r, s, t)^T$ orthogonal to $\mathbf{a} = (a, b, c)^T$. For this reason, we call $(a, b, c)^T$ a *normal* to the plane. (On a good day, we are normal to the plane of the floor.)

Example 1.7. Find the plane through $(1,2,3)^T$ with nomal $(2,3,5)^T$. Now $\mathbf{a} = (2,3,5)^T$, so in the equation (1.10) we have $d = (2,3,5)^T \cdot (1,2,3)^T = 23$. Hence the plane is $2x + 3y + 5z = 23$.

Holding $\mathbf{a} \neq \mathbf{0}$ constant and varying d gives a family of planes filling up $\mathbb{R}^3$ such that no two distinct planes have any points in common. Hence the family of planes $ax+by+cz = d$ (a, b, c fixed and d arbitrary) are all *parallel.* By drawing a picture, one can see from the Parallelogram Law that every vector $(r, s, t)^T$ on $ax + by + cz = d$ is the sum of a fixed vector $(x_0, y_0, z_0)^T$ on $ax + by + cz = d$ and an arbitrary vector $(x, y, z)^T$ on the parallel plane $ax + by + cz = 0$ through the origin.

FIGURE

1.2.3 The distance from a point to a plane

A nice application of our projection techniques is to be able to write down a simple formula for the distance from a point to a plane P in $\mathbb{R}^3$. The problem becomes quite simple if we break it up into two cases. First, consider the case of a plane P through the origin, say with equation $ax + by + cz = 0$. Suppose $\mathbf{v}$ is an arbitrary vector in $\mathbb{R}^3$ whose distance to P is what we seek. Now we can decompose $\mathbf{v}$ into orthogonal components where one of the components is along the normal $\mathbf{n} = (a, b, c)^T$, say

$$\mathbf{v} = P_{\mathbf{n}}(\mathbf{v}) + (\mathbf{v} - P_{\mathbf{n}}(\mathbf{v})), \tag{1.11}$$

where

$$P_{\mathbf{n}}(\mathbf{v}) = (\frac{\mathbf{v} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}})\mathbf{n}.$$

It's intuitively clear that the distance we're looking for is

$$d = |P_{\mathbf{n}}(\mathbf{v})| = |\mathbf{v} \cdot \mathbf{n}|/\sqrt{\mathbf{n} \cdot \mathbf{n}},$$

but we need to check this carefully. First of all, we need to say that the distance from $\mathbf{v}$ to P means the minimum value of $|\mathbf{v} - \mathbf{r}|$, where $\mathbf{r}$ is on P. To simplify notation, put $\mathbf{p} = P_{\mathbf{n}}(\mathbf{v})$ and $\mathbf{q} = \mathbf{v} - \mathbf{p}$. Since $\mathbf{v} = \mathbf{p} + \mathbf{q}$,

$$\mathbf{v} - \mathbf{r} = \mathbf{p} + \mathbf{q} - \mathbf{r}.$$

Since P contains the origin, $\mathbf{q} - \mathbf{r}$ lies on P since both $\mathbf{q}$ and $\mathbf{r}$ do, so by Pythagoras,

$$|\mathbf{v} - \mathbf{r}|^2 = |\mathbf{p}|^2 + |\mathbf{q} - \mathbf{r}|^2.$$

But $\mathbf{p}$ is fixed, so $|\mathbf{v}-\mathbf{r}|^2$ is minimized by taking $|\mathbf{q}-\mathbf{r}|^2 = 0$. Thus $|\mathbf{v}-\mathbf{r}|^2 = |\mathbf{p}|^2$, and the distance $D(\mathbf{v}, P)$ from $\mathbf{v}$ to P is indeed

$$D(\mathbf{v}, P) = |\mathbf{p}| = \frac{|\mathbf{v} \cdot \mathbf{n}|}{(\mathbf{n} \cdot \mathbf{n})^{\frac{1}{2}}} = |\mathbf{v} \cdot \widehat{\mathbf{n}}|.$$

Also, the point on P nearest $\mathbf{v}$ is $\mathbf{q}$. If $\mathbf{v} = (r, s, t)^T$, the distance is

$$D(\mathbf{v}, P) = \frac{|ar + bs + ct|}{\sqrt{a^2 + b^2 + c^2}}.$$

We now attack the general problem by reducing it to the first case. We want to find the distance $D(\mathbf{v}, Q)$ from $\mathbf{v}$ to an arbitrary plane Q in $\mathbb{R}^3$. Suppose the equation of Q is $ax + by + cz = d$, and let $\mathbf{c}$ be a vector on Q. I claim that the distance from $\mathbf{v}$ to Q is the same as the distance from $\mathbf{v} - \mathbf{c}$ to the plane P parallel to Q through the origin, i.e. the plane $ax + by + cz = 0$. Indeed, we already showed that every vector on Q has the form $\mathbf{w} + \mathbf{c}$ where $\mathbf{w}$ is on P. Thus let $\mathbf{r}$ be the vector on Q nearest $\mathbf{v}$. Since $d(\mathbf{v}, \mathbf{r}) = |\mathbf{v} - \mathbf{r}|$, it follows easily from $\mathbf{r} = \mathbf{w} + \mathbf{c}$ that $d(\mathbf{v}, \mathbf{r}) = d(\mathbf{v} - \mathbf{c}, \mathbf{w})$. Hence the problem amounts to minimizing $d(\mathbf{v} - \mathbf{c}, \mathbf{w})$ for $\mathbf{w} \in P$, which we already solved. Thus

$$D(\mathbf{v}, Q) = |(\mathbf{v} - \mathbf{c}) \cdot \widehat{\mathbf{n}}|,$$

which reduces to the formula

$$D(\mathbf{v}, Q) = \frac{|ar + bs + ct - d|}{\sqrt{a^2 + b^2 + c^2}},$$

since

$$\mathbf{c} \cdot \widehat{\mathbf{n}} = \frac{\mathbf{c} \cdot \mathbf{n}}{(\mathbf{n} \cdot \mathbf{n})^{\frac{1}{2}}} = \frac{d}{\sqrt{a^2 + b^2 + c^2}}.$$

In summary, we have

Proposition 1.5. *Let Q be the plane in $\mathbb{R}^3$ defined by $ax + by + cz = d$, and let $\mathbf{v}$ be any vector in $\mathbb{R}^3$, possibly lying on Q. Let $D(\mathbf{v}, Q)$ be the distance from $\mathbf{v}$ to Q. Then*

$$D(\mathbf{v}, Q) = \frac{|ar + bs + ct - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

In fact, the problem we just solved has a far more general version known as the least squares problem. We will come back to this topic in a later chapter.

Ii is as an exercises to find a formula for the distance from a point to a line. A more challenging exercise is to find the distance between two lines. If one of the lines is parallel to **a** and the other is parallel to **b**, then it turns out that what is needed is a vector orthogonal to both **a** and **b**. This is the same problem encountered if one wants to find the plane through three non collinear points. What is needed is a vector orthogonal to $\mathbf{q} - \mathbf{p}$ and $\mathbf{r} - \mathbf{p}$. Both of these problems are solved by using the cross product, which we take up in the next section.

Exercises

Exercise 1.14. Express the line $ax + by = c$ in $\mathbb{R}^2$ in parametric form.

Exercise 1.15. Express the line with vector form $(x, y)^T = (1, -1)^T + t(2, 3)^T$ in the form $ax + by = c$.

Exercise 1.16. Find the line through the points $\mathbf{a}$ and $\mathbf{b}$ in the following cases:

(i) $\mathbf{a} = (1, 1, -3)^T$ and $\mathbf{b} = (6, 0, 2)^T$, and

(ii) $\mathbf{a} = (1, 1, -3, 4)^T$ and $\mathbf{b} = (6, 0, 2, -3)^T$.

Exercise 1.17. Prove the Parallelogram Law in $\mathbb{R}^n$ for any n.

Exercise 1.18. Find the line of intersection of the planes $3x - y + z = 0$ and $x - y - z = 1$ in parametric form.

Exercise 1.19. Do the following:

(a) Find the equation in vector form of the line through $(1, -2, 0)^T$ parallel to $(3, 1, 9)^T$.

(b) Find the plane perpendicular to the line of part (a) passing through $(0, 0, 0)^T$.

(c) At what point does the line of part (a) meet the plane of part (b)?

Exercise 1.20. Determine whether or not the lines $(x, y, z)^T = (1, 2, 1)^T + t(1, 0, 2)^T$ and $(x, y, z)^T = (2, 2, -1)^T + t(1, 1, 0)^T$ intersect.

Exercise 1.21. Consider any two lines in $\mathbb{R}^3$. Suppose I offer to bet you they don't intersect. Do you take the bet or refuse it? What would you do if you knew the lines were in a plane?

Exercise 1.22. Use the method of § 1.2.2 to find an equation for the plane in $\mathbb{R}^3$ through the points $(6, 1, 0)^T$, $(1, 0, 1)^T$ and $(3, 1, 1)^T$

Exercise 1.23. Compute the intersection of the line through $(3, -1, 1)^T$ and $(1, 0, 2)^T$ with the plane $ax + by + cz = d$ when

(i) $a = b = c = 1$, $d = 2$,

(ii) $a = b = c = 1$ and $d = 3$.

Exercise 1.24. Find the distance from the point $(1, 1, 1)^T$ to

(i) the plane $x + y + z = 1$, and

(ii) the plane $x - 2y + z = 0$.

Exercise 1.25. Find the orthogonal decomposition $(1, 1, 1)^T = \mathbf{a}+\mathbf{b}$, where $\mathbf{a}$ lies on the plane P with equation $2x + y + 2z = 0$ and $\mathbf{a} \cdot \mathbf{b} = 0$. What is the orthogonal projection of $(1, 1, 1)^T$ on P?

Exercise 1.26. Here's another bet. Suppose you have two planes in $\mathbb{R}^3$ and I have one. Furthermore, your planes meet in a line. I'll bet that all three of our planes meet. Do you take this bet or refuse it. How would you bet if the planes were all in $\mathbb{R}^4$ instead of $\mathbb{R}^3$?

Exercise 1.27. Show that two lines in $\mathbb{R}^n$ (any n) which meet in two points coincide.

Exercise 1.28. Verify that the union of the lines $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{b}$ is fixed but $\mathbf{a}$ is arbitrary is $\mathbb{R}^n$. Also show that two of these lines are the same or have no points in common.

Exercise 1.29. Verify the Parallelogram Law (in $\mathbb{R}^n$) by computing where the line through $\mathbf{a}$ parallel to $\mathbf{b}$ meets the line through $\mathbf{b}$ parallel to $\mathbf{a}$.

1.3 The Cross Product

1.3.1 The Basic Definition

The cross product of two non parallel vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$ orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ defined geometrically as follows. Let P denote the unique plane through the origin containing both $\mathbf{a}$ and $\mathbf{b}$, and let $\mathbf{n}$ be the choice of unit vector normal to P so that the thumb, index finger and middle finger of your right hand can be lined up with the three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{n}$ without breaking any bones. In this case we call $(\mathbf{a}, \mathbf{b}, \mathbf{n})$ a *right handed triple*. (Otherwise, it's a left handed triple.) Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$, so $0 < \theta < \pi$. Then we put

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \mathbf{n}. \tag{1.12}$$

If $\mathbf{a}$ and $\mathbf{b}$ are collinear, then we set $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. While this definition is very pretty, and is useful because it reveals the geometric properties of the cross product, the problem is that, as presented, it isn't computable unless $\mathbf{a} \cdot \mathbf{b} = 0$ (since $\sin\theta = 0$). For example, one sees immediately that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

To see a couple of examples, note that $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $(\mathbf{i}, -\mathbf{j}, -\mathbf{k})$ both are right handed triples, but $(\mathbf{i}, -\mathbf{j}, \mathbf{k})$ and $(\mathbf{j}, \mathbf{i}, \mathbf{k})$ are left handed. Thus $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, while $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$. Similarly, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$. In fact, these examples point out two of the general properties of the cross product:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a},$$

and

$$(-\mathbf{a}) \times \mathbf{b} = -(\mathbf{a} \times \mathbf{b}).$$

The question is whether or not the cross product is computable. In fact, the answer to this is yes. First, let us make a temporary definition. If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, put

$$\mathbf{a} \wedge \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

We call $\mathbf{a} \wedge \mathbf{b}$ the *wedge product* of $\mathbf{a}$ and $\mathbf{b}$. Notice that $\mathbf{a} \wedge \mathbf{b}$ is defined without any restrictions on $\mathbf{a}$ and $\mathbf{b}$. It is not hard to verify by direct computation that $\mathbf{a} \wedge \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, so $\mathbf{a} \wedge \mathbf{b} = r(\mathbf{a} \times \mathbf{b})$ for some $r \in \mathbb{R}$.

The key fact is the following

Proposition 1.6. *For all* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbb{R}^3$,

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \wedge \mathbf{b}.$$

This takes care of the computability problem since $\mathbf{a} \wedge \mathbf{b}$ is easily computed. An outline the proof goes as follows. The wedge product and the dot product are related by the following identity:

$$|\mathbf{a} \wedge \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = (|\mathbf{a}||\mathbf{b}|)^2. \tag{1.13}$$

The proof is just a calculation, and we will omit it. Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, and since $\sin\theta \geq 0$, we deduce that

$$|\mathbf{a} \wedge \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta. \tag{1.14}$$

It follows that $\mathbf{a} \wedge \mathbf{b} = \pm|\mathbf{a}||\mathbf{b}| \sin\theta\mathbf{n}$. The fact that the sign is + proven by showing that

$$(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{n} > 0.$$

The proof of this step is a little tedious so we will omit it.

1.3.2 Some further properties

Before giving applications, we let us give some of the algebraic properties of the cross product.

Proposition 1.7. *Suppose* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. *Then :*

(i) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$,

(ii) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$, *and*

(iii) *for any* $r \in \mathbb{R}$,
$$(r\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (r\mathbf{b}) = r(\mathbf{a} \times \mathbf{b}).$$

Proof. The first and third identities are obvious from the original definition. The second identity, which says that the cross product is distributive, is not at all obvious from the definition. On the other hand, it is easy to check directly that

$$(\mathbf{a} + \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c},$$

so *(ii)* has to hold also since $\otimes = \wedge$. □

Recalling that $\mathbb{R}^2$ can be viewed as the complex numbers, it follows that vectors in $\mathbb{R}^1 = \mathbb{R}$ and $\mathbb{R}^2$ can be multiplied, where the multiplication is both associative and commutative. Proposition 1.7 says that the cross product gives a multiplication on $\mathbb{R}^3$ which is distributive, but not commutative. It is in fact anti-commutative. Also, the cross product isn't associative: $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ are not in general equal. Instead of the usual associative law for multiplication, the cross product satisfies a famous identity known as the Jacobi identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}.$$

The Jacobi Identity and the anti-commutativity $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ are the basic axioms for what is called a Lie algebra, which is an important structure in abstract algebra with many applications in mathematics and physics. The next step in this progression of algebras (that is, a product on $\mathbb{R}^4$ is given by the the quaternions, which are fundamental, but won't be considered here.

1.3.3 Examples and applications

The first application is to use the cross product to find a normal $\mathbf{n}$ to the plane P through $\mathbf{p}, \mathbf{q}, \mathbf{r}$, assuming they don't all lie on a line. Once we have $\mathbf{n}$, it is easy to find the equation of P. We begin by considering the plane Q through the origin parallel to P. First put $\mathbf{a} = \mathbf{q} - \mathbf{p}$ and $\mathbf{b} = \mathbf{r} - \mathbf{p}$. Then $\mathbf{a}, \mathbf{b} \in Q$, so we can put $\mathbf{n} = \mathbf{a} \times \mathbf{b}$. Suppose $\mathbf{n} = (a, b, c)^T$ and $\mathbf{p} = (p_1, p_2, p_3)^T$. Then the equation of Q is $ax + by + cz = 0$, and the equation of P is obtained by noting that

$$\mathbf{n} \cdot ((x, y, z)^T - (p_1, p_2, p_3)^T) = 0,$$

or, equivalently,

$$\mathbf{n} \cdot (x, y, z)^T = \mathbf{n} \cdot (p_1, p_2, p_3)^T.$$

Thus the equation of P is

$$ax + by + cz = ap_1 + bp_2 + cp_3.$$

Example 1.8. Let's find an equation for the plane in $\mathbb{R}^3$ through $(1, 2, 1)^T$, $(0, 3, -1)^T$ and $(2, 0, 0)^T$. Using the cross product, we find that a normal is $(-1, 2, 1)^T \times (-2, 3, -1)^T = (-5, -3, 1)^T$. Thus the plane has equation $-5x - 3y + z = (-5, -3, 1)^T \cdot (1, 2, 1)^T = -10$. One could also have used $(0, 3, -1)^T$ or $(2, 0, 0)^T$ on the right hand side with the same result, of course.

The next application is the area formula for a parallelogram.

Proposition 1.8. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two noncollinear vectors in* $\mathbb{R}^3$*. Then the area of the parallelogram spanned by* $\mathbf{a}$ *and* $\mathbf{b}$ *is* $|\mathbf{a} \times \mathbf{b}|$.

We can extend the area formula to 3-dimensional (i.e. solid) parallelograms. Any three noncoplanar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ determine a solid parallelogram called a parallelepiped. This parallelepiped $\mathcal{P}$ can be explicitly defined as

$$\mathcal{P} = \{r\mathbf{a} + s\mathbf{b} + t\mathbf{c} \mid 0 \le r, s, t \le 1\}.$$

For example, the parallelepiped spanned by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ is the unit cube in $\mathbb{R}^3$ with vertices at $\mathbf{0}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{i}+\mathbf{j}$, $\mathbf{i}+\mathbf{k}$, $\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+\mathbf{j}+\mathbf{k}$. A parallelepiped has 8 vertices and 6 sides which are pairwise parallel.

To get the volume formula, we introduce the *triple product* $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Proposition 1.9. *Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three noncoplanar vectors in $\mathbb{R}^3$. Then the volume of the parallelepiped they span is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.*

Proof. We leave this as a worthwhile exercise. $\square$

By the definition of the triple product,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) - a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

The right hand side of this equation is a 3×3 determinant which is written

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

We'll see somewhat later how the volume of n-dimensional parallelepiped is expressed as the absolute value of a certain $n \times n$ determinant, which is a natural generalization of the triple product.

Example 1.9. We next find the formula for the distance between two lines. Consider two lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$ parameterized as $\mathbf{a}_1 + t\mathbf{b}_1$ and $\mathbf{a}_2 + t\mathbf{b}_2$ respectively. We want to show that the distance between ℓ_1 and ℓ_2 is

$$d = |(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|/|\mathbf{b}_1 \times \mathbf{b}_2|.$$

This formula is somewhat surprising because it says that one can choose any two initial points $\mathbf{a}_1$ and $\mathbf{a}_2$ to compute d. First, let's see why $\mathbf{b}_1 \times \mathbf{b}_2$ is involved. This is in fact intuitively clear, since $\mathbf{b}_1 \times \mathbf{b}_2$ is orthogonal to the directions of both lines. But one way to see this concretely is to take a tube of radius r centred along ℓ_1 and expand r until the tube touches ℓ_2. The point $\mathbf{v}_2$ of tangency on ℓ_2 and the center $\mathbf{v}_1$ on ℓ_1 of the disc (orthogonal to ℓ_1) touching ℓ_2 give the two points so that $d = d(\mathbf{v}_1, \mathbf{v}_2)$, and, by construction, $\mathbf{v}_1 - \mathbf{v}_2$ is parallel to $\mathbf{b}_1 \times \mathbf{b}_2$. Now let $\mathbf{v}_i = \mathbf{a}_i + t_i\mathbf{b}_i$

for $i = 1, 2$, and denote the unit vector in the direction of $\mathbf{b}_1 \times \mathbf{b}_2$ by $\widehat{\mathbf{u}}$. Then

$$\begin{aligned} d &= |\mathbf{v}_1 - \mathbf{v}_2| \\ &= (\mathbf{v}_1 - \mathbf{v}_2) \cdot \frac{(\mathbf{v}_1 - \mathbf{v}_2)}{|\mathbf{v}_1 - \mathbf{v}_2|} \\ &= |(\mathbf{v}_1 - \mathbf{v}_2) \cdot \widehat{\mathbf{u}}| \\ &= |(\mathbf{a}_1 - \mathbf{a}_2 + t_1\mathbf{b}_1 - t_2\mathbf{b}_2) \cdot \widehat{\mathbf{u}}| \\ &= |(\mathbf{a}_1 - \mathbf{a}_2) \cdot \widehat{\mathbf{u}}|. \end{aligned}$$

The last equality is due to the fact that $\mathbf{b}_1 \times \mathbf{b}_2$ is orthogonal to $t_1\mathbf{b}_1 - t_2\mathbf{b}_2$ plus the fact that the dot product is distributive. This is the formula we sought.

For other applications of the cross product, consult *Vector Calculus* by Marsden and Tromba.

Exercises

Exercise 1.30. Using the cross product, find the plane through the origin that contains the line through $(1,-2,0)^T$ parallel to $(3,1,9)^T$.

Exercise 1.31. Using the cross product, find

(a) the line of intersection of the planes $3x+2y-z=0$ and $4x+5y+z=0$, and

(b) the line of intersection of the planes $3x+2y-z=2$ and $4x+5y+z=1$.

Exercise 1.32. Is $\mathbf{x}\times\mathbf{y}$ orthogonal to $2\mathbf{x}-3\mathbf{y}$? Generalize this property.

Exercise 1.33. Find the distance from $(1,2,1)^T$ to the plane containing $1,3,4)^T$, $(2,-2,-2)^T$, and $(7,0,1)^T$. Be sure to use the cross product.

Exercise 1.34. Formulate a definition for the angle between two planes in $\mathbb{R}^3$. (Suggestion: consider their normals.)

Exercise 1.35. Find the distance from the line $\mathbf{x}=(1,2,3)^T+t(2,3,-1)^T$ to the origin in two ways:

(i) using projections, and

(ii) using calculus, by setting up a minimization problem.

Exercise 1.36. Find the distance from the point $(1,1,1)^T$ to the line $x=2+t, y=1-t, z=3+2t$,

Exercise 1.37. Show that in $\mathbb{R}^3$, the distance from a point $\mathbf{p}$ to a line $\mathbf{x}=\mathbf{a}+t\mathbf{b}$ can be expressed in the form

$$d=\frac{|(\mathbf{p}-\mathbf{a})\times\mathbf{b}|}{|\mathbf{b}|}.$$

Exercise 1.38. Prove the identity

$$|\mathbf{a}\times\mathbf{b}|^2+(\mathbf{a}\cdot\mathbf{b})^2=(|\mathbf{a}||\mathbf{b}|)^2.$$

Deduce that if $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then

$$|\mathbf{a}\times\mathbf{b}|^2+(\mathbf{a}\cdot\mathbf{b})^2=1.$$

Exercise 1.39. Show that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

Deduce from this $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is not necessarily equal to $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. In fact, can you say when they are equal?