Lecture 6

PROPERTIES OF THE TRUNCATION OPERATOR

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The most important property of Λ^T is that it converts smooth slowly increasing functions into rapidly decreasing functions but we begin by studying its formal properties.

Recall that Λ^T is defined for T suitably regular in $\mathfrak{a}_0^+$ and that it is defined first of all for continuous or, better, bounded measurable φ by

$$\Lambda^T\varphi(g) = \sum_P (-1)^{a_P} \sum_{\delta \in P\backslash G} \int_{N\backslash \mathbb{N}} \varphi(n\delta g)\hat{\tau}_P(H(\delta g) - T) \quad ,$$

where

$$a_P = \dim \mathfrak{a}_P/\mathfrak{a}_G \quad .$$

By Lemma 2.1 the sums appearing on the right are finite.

PROPOSITION 6.1. <u>The operator</u> Λ^T <u>is an idempotent, so that</u>

$$\Lambda^T(\Lambda^T\varphi) = \Lambda^T\varphi \quad .$$

This proposition is of course an immediate consequence of the following lemma.

LEMMA 6.2. <u>If</u> φ <u>is bounded measurable then</u>

$$\int_{N_1\backslash \mathbb{N}_1} \Lambda^T\varphi(n_1 g)dn_1 = 0$$

<u>unless</u> $\varpi(H(g) - T) \leq 0$ <u>for every</u> $\varpi \in \hat{\Delta}_1$.

We first consider

$$(1) \qquad \int_{N_1 \backslash \mathbf{N}_1} \sum_{\delta \in P \backslash G} \int_{N \backslash \mathbf{N}} \varphi(n\delta n_1 g)\hat{\tau}_P(H(\delta n_1 g) - T)\,dn\,dn_1 \quad .$$

Let $\Omega(\mathfrak{a}_0, P)$ be the set of s in $\Omega(\mathfrak{a}_0, \mathfrak{a}_0)$ such that $s^{-1}\alpha > 0$ for all $\alpha \in \Delta_0^P$. The Bruhat decomposition assures us that $P \backslash G$ is a disjoint union

$$\bigcup_{s \in \Omega(\mathfrak{a}_0, P)} P w_s N_0 \quad ,$$

w_s being a representative of s.

Thus the expression (1) is equal to the sum over $\Omega(\mathfrak{a}_0, P)$ of

$$(2) \quad \int_{N_1 \backslash \mathbf{N}_1} \sum_{\nu \in w_s^{-1} N_0 w_s \cap N_0 \backslash N_0} \int_{N \backslash \mathbf{N}} \varphi(n w_s \nu n_1 g)\hat{\tau}_P(H(w_s \nu n_1 g) - T)\,dn\,dn_1 \quad .$$

The outer integral and the sum can be fused to obtain an integral over

$$w_s^{-1} N_0 w_s \cap N_0 \backslash N_0 \mathbf{N}_1 \quad ,$$

which we then decompose as an iterated integral, so that (2) becomes a triple integral

$$\int_{w_s^{-1}\mathbf{N}_0 w_s \cap N_0 \mathbf{N}_1 \backslash N_0 \mathbf{N}_1} \int_{w_s^{-1} N_0 w_s \cap N_0 \backslash w_s^{-1} \mathbf{N}_0 w_s \cap N_0 \mathbf{N}_1} \int_{N \backslash \mathbf{N}}$$

The domain of integration in the outer integral depends on the choice of N_1 and on s but not on P. Since it is the alternation over P that will force the vanishing we ignore the final integration and concentrate on the inner double integral. A little reflection convinces one that

$$w_s^{-1}N_0w_s \cap N_0 \backslash w_s^{-1}\mathbf{N}_0w_s \cap N_0\mathbf{N}_1 = w_s^{-1}N_0w_s \cap N_1 \backslash w_s^{-1}\mathbf{N}_0w_s \cap \mathbf{N}_1 \quad .$$

Since $s \in \Omega(\mathfrak{a}_0, P)$ the intersection $w_sN_0w_s^{-1} \cap M$ is $N_0 \cap M$. Thus $w_sP_1w_s^{-1} \cap M$ is a parabolic subgroup of M with unipotent radical $w_sN_1w_s^{-1} \cap M$. If we pass the variable in $w_s^{-1}\mathbf{N}_0w_s \cap \mathbf{N}_1$ through w_s we obtain a variable in $\mathbf{N}_0 \cap w_s\mathbf{N}_1w_s^{-1} = (\mathbf{N} \cap w_s\mathbf{N}_1w_s^{-1})(\mathbf{M} \cap w_s\mathbf{N}_1w_s^{-1})$. Thus the second integration in the double integral can be taken over the product

$$(N \cap w_sN_1w_s^{-1}\backslash \mathbf{N} \cap w_s\mathbf{N}_1w_s^{-1}) \times (M \cap w_sN_1w_s^{-1}\backslash \mathbf{M} \cap w_s\mathbf{N}_1w_s^{-1} \quad .$$

The volume of the first factor is 1 and since the first integration is taken over $N\backslash\mathbf{N}$ the integral does not depend on the first variable in the product.

Thus the double integral becomes finally

$$\int_{M \cap w_sN_1w_s^{-1}\backslash \mathbf{M} \cap w_s\mathbf{N}_1w_s^{-1}} \int_{N\backslash\mathbf{N}} \quad .$$

However

$$(M \cap w_sN_1w_s^{-1}).N$$

is the unipotent radical of a parabolic subgroup P_s of G. So the double integral becomes a single integral over $N_s \backslash \mathbf{N}_s$, which we now write out explicitly.

(3) $$\int_{N_s \backslash \mathbf{N}_s} \varphi(n w_s n_1 g)\hat{\tau}_P(H(w_s n_1 g) - T)\, dn \quad ,$$

the n_1 being the variable for the outer integration, which does not concern us at the moment.

The group P_s is contained in P. The group N_1 is fixed but s varies over $\Omega(\mathfrak{a}_0, P)$ and we are to sum over P and $\Omega(\mathfrak{a}_0, P)$. What we do is fix s and a $P^o \supseteq P_0$ and sum over all P with $s \in \Omega(\mathfrak{a}_0, P)$ and $P_s = P^o$.

The set $\{\alpha \in \Delta_0 | s^{-1}\alpha > 0\}$ is the disjoint union of two subsets, the first S^1 consisting of those α in it for which $s^{-1}\alpha$ is orthogonal to $\mathfrak{a}_1$ and the second S_1 of those for which it is not. It is clear that $\Delta_0^{P_s} \subset \Delta_0^P$ and that

$$\Delta_0^{P_s} = \Delta_0^P \cap S^1 \quad ,$$

for $\alpha \in S_1$ if and only if $s^{-1}\alpha$ is a root in N_1. Thus the freedom of P is that the intersection of Δ_0^P with S_1 can be chosen at will.

The dependence of (3) on P is through the function $\hat{\tau}_P(H(w_s n_1 g) - T)$. The sum

$$\sum (-1)^{a_P} \hat{\tau}_P(H(w_s n_1 g) - T)$$

There is considerable overlap with Clozel's 1st lecture

over the allowed P is therefore 0 unless

$$\varpi_\alpha(H(w_s n_1 g) - T) > 0$$

for $\alpha \notin \Delta_0^{P^o} \cup S_1$ and

$$\varpi_\alpha(H(w_s n_1 g) - T) \le 0$$

for $\alpha \in S_1$.

To complete the proof of the lemma we have to show that these inequalities imply that

$$\varpi(H(g) - T) \le 0$$

for $\varpi \in \hat{\Delta}_1$. We have

$$s^{-1}(H(w_s n_1 g) - T) = H(g) - T + s^{-1}H(w_s v) + T - s^{-1}T$$

with $v \in N_0(\mathbf{A})$.

We write, identifying $\mathfrak{a}_0$ and its dual,

$$H(w_s n_1 g) - T = \sum_{\alpha \in \Delta_0} t_\alpha \alpha$$

with $t_\alpha > 0$ for $\alpha \notin \Delta_0^{P^o} \cup S_1$ and $t_\alpha \le 0$ for $\alpha \in S_1$. Then

$$\begin{aligned}\varpi(s^{-1}(H(w_s n_1 g) - T)) &= \sum t_\alpha \varpi(s^{-1}\alpha) \\ &= \sum_{\alpha \notin S^1} t_\alpha \varpi(s^{-1}\alpha)\end{aligned}$$

for $s^{-1}\alpha$ is orthogonal to $\mathfrak{a}_1$ if $\alpha \in S^1$. If $\alpha \notin S^1 \cup S_1$ then $t_\alpha > 0$ and $\varpi(s^{-1}\alpha) \leq 0$ and if $\alpha \in S_1$ then $t_\alpha \leq 0$ and $\varpi(s^{-1}\alpha) \geq 0$. Thus this expression is less than or equal to zero.

To complete the proof of the lemma we need only show that for sufficiently regular T

$$\varpi(s^{-1}H(w_s v)) + \varpi(T - s^{-1}T) \geq 0 \quad .$$

There is certainly no harm in replacing G by a Levi factor of the smallest standard parabolic containing s, which to simplify the notation we suppose is G itself. Then given any constant C we can take T sufficiently regular and suppose that

$$\varpi(T-s^{-1}T) \geq C \quad .$$

It therefore remains to show that there exists a constant C such that

$$\varpi(s^{-1}H(w_s v)) \geq - C \tag{4}$$

for all $v \in N_0(\mathbb{A})$. This is a statement which is easily seen to be independent of the choice of K. Indeed it is enough to prove it over a field which splits G. So we can suppose G is split and semi-simple.

Then one has the usual optimal choice of K and for this one proves by induction on the length of s the following lemma.

LEMMA 6.3. _If_ v _lies in_ $\mathbb{N}_0$ _then_

$$s^{-1}H(w_s v) = \sum_{\substack{\alpha>0 \\ s\alpha<0}} c_\alpha \alpha$$

with $c_\alpha \geq 0$.

This gives the relation (4) with $C = 0$. To prove the lemma one begins with $SL(2)$, taking $w_s \in K$. So, for the non-trivial s,

$$w_s v w_s^{-1} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} .$$

Moreover $H(w_s v w_s^{-1})$ is the sum of its local contributions and these are

(i) v real

$$-\tfrac{1}{2}\ln(1+|x_v|^2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(ii) v complex

$$-\ln(1+|x_v|^2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(iii) v non-archimedean

$$-\ln \max\{|1|, |x_v|\}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Thus for $SL(2)$ and hence in general the lemma is proved for an s of length one.

For a Chevalley group and an optimal choice of K we may take $w_s \in K$. If $s = s_1 s_2$ with s_1 a reflection associated to the root β

and $1 + \text{length } s_2 = \text{length } s$ then

$$s^{-1}H(w_s v) = s_2^{-1}s_1^{-1}H(w_{s_1}w_{s_2}v) = s_2^{-1}s_1^{-1}H(w_{s_1}v') + s_2^{-1}s_1^{-1}(s_1H(w_{s_2}v)) \quad .$$

The induction assumption allows us to write this as

$$s_2^{-1}d_\beta\beta + \sum_{\substack{\alpha>0 \\ s_2\alpha<0}} c_\alpha\alpha$$

with $d_\beta \geq 0$, $c_\alpha \geq 0$. Since

$$\{\alpha > 0 | s\alpha < 0\} = \{\alpha > 0 | s_2\alpha < 0\} \cup \{s_2^{-1}\beta\}$$

the lemma follows.

PROPOSITION 6.4. <u>Suppose that</u> φ_1 <u>and</u> φ_2 <u>are continuous functions on</u> $G\backslash\mathbf{G}$ <u>and that on</u>

$$|\varphi_1(g)| \leq c|g|^N$$

<u>for some</u> N <u>and that on any Siegel domain in</u> $\mathbf{G}^1$ <u>we have an inequality</u>

$$|\varphi_2(g)| \leq c_N|g|^{-N}$$

<u>for all</u> N. <u>Then</u>

$$\int_{G\backslash\mathbf{G}^1} \Lambda^T\varphi_1(g)\varphi_2(g)dg = \int_{G\backslash\mathbf{G}^1} \varphi_1(g)\Lambda^T\varphi_2(g)dg \quad .$$

This clearly reduces to showing that

$$\int_{G\backslash \mathbf{G}^1} \left\{ \sum_{\delta \in P\backslash G} \int_{N\backslash \mathbf{N}} \varphi_1(n\delta g)\,dn\,\hat{\tau}_P(H(\delta g) - T) \right\} \varphi_2(g)\,dg$$

is equal to

$$\int_{G\backslash G^1} \varphi_1(g) \left\{ \sum_{\delta \in P\backslash G} \int_{N\backslash \mathbf{N}} \varphi_2(n\delta g)\,dn\,\hat{\tau}_P(H(\delta g) - T) \right\} dg \quad .$$

It follows readily from Lemma 7.8 of the next lecture that the second integral is absolutely convergent when φ_1 and φ_2 are replaced by their absolute values. Thus a formal proof of the equality assures us of both the equality and the convergence of the first integral.

The formal proof is of course easy, the second expression reducing to

$$\int_{P\backslash G} \varphi_1(g)\hat{\tau}_P(H(g) - T) \left\{ \int_{N\backslash \mathbf{N}} \varphi_2(ng)\,dn \right\} dg$$

which equals

$$\int_{\mathbf{N}P\backslash \mathbf{G}} \hat{\tau}_P(H(g) - T) \left\{ \int_{N\backslash \mathbf{N}} \varphi_1(ng)\,dn \right\} \left\{ \int_{N\backslash \mathbf{N}} \varphi_2(ng)\,dn \right\} dg \quad ,$$

an expression symmetric in φ_1 and φ_2.

COROLLARY 6.5. Λ^T <u>extends to an orthogonal projection on the Hilbert space</u> L.

We will not need any of these assertions in the next two lectures. What we will need is the fact that Λ^T transforms smooth slowly increasing functions into rapidly decreasing functions. For now we content ourselves with a relatively simple statement.

To any element Y of the universal enveloping algebra of the Lie algebra of G we can associate a left-invariant differential operator $R(Y)$ on $\mathbf{G}$.

LEMMA 6.6. Suppose T is sufficiently regular. Let $\mathfrak{S}$ be a Siegel domain on $\mathbf{G}^1$. For any pair of positive numbers N and N' and any open compact subgroup K_0 of $G(\mathbf{A}^f)$ we can find a finite subset $\{Y_1, \ldots, Y_r\}$ in the universal enveloping algebra such that

$$|\Lambda^T \varphi(g)|\,|g|^{N'} \leq \sum_i \sup_{h \in \mathbf{G}^1 g} |R(Y_i)\varphi(h)|\,|h|^{-N}$$

for $g \in \mathfrak{S}$ provided φ is invariant on the right under K_0 and sufficiently smooth that all the operators $R(Y_i)$ can be applied to it.

This is proved by an argument similar to that used for the proof of the σ-expansion. Its structure is more transparent, many of the incidental difficulties met with the σ-expansion no longer arising. However the alternating sum is used in a slightly different way and it is best to dispose of the necessary technical lemma immediately.

For this purpose we fix $P_1 \subset P_2$ and consider a continuous function ψ on $N_1 \backslash \mathbf{N}_1$. If $P_1 \subset P \subset P_2$ then

$$\Pi_P \psi : n_1 \longrightarrow \int_{N \backslash \mathbf{N}} \psi(n n_1)\, dn_1$$

is also a function on $N_1 \backslash \mathbf{N}_1$ because N is a normal subgroup of N_1. We want to consider

$$\prod \psi = \sum_P (-1)^{a_P} \prod_P \psi \quad .$$

Let $\Delta_0^2 - \Delta_0^1 = \{\alpha_1, \ldots, \alpha_s\}$ and let $\sum_i$ be the set of positive roots α of the form

$$(5) \qquad \alpha = \sum_{\beta \in \Delta_0} b_\beta \beta$$

with $b_\beta \neq 0$ for $\beta = \alpha_i$ or $\beta \in \Delta_0 - \Delta_0^2$. There is a parabolic P^i between P_1 and P_2 such that the Lie algebra of N^i is spanned by the root vectors attached to the roots α in $\sum_i$. For any P between P_1 and P_2 there is a unique subset $\sum_P$ of $\{\alpha_1, \ldots, \alpha_r\}$ such that Δ_0^2 is the disjoint union of $\sum_P$ and Δ_0^P. Moreover

$$N = \prod_{i \in \sum_P} N^i \quad .$$

It follows easily, all the groups N being normal in N_1, that

$$(6) \qquad \prod = \prod_{i=1}^r \left(\prod_{P_2} - \prod_i \right) \quad ,$$

where for simplicity of notation we have set $\prod_{P^i} = \prod_i$.

Let $\sum_i^o$ be the set of positive roots which when written as in (5) have $b_{\alpha_i} \neq 0$. Let an integer $r \geq 0$ be given. For the purposes of the next lemma we define a left-invariant differential operator of type r to be a product

$$\prod_{i=1}^r \prod_{j=1}^r X_{ij} \quad ,$$

the order being immaterial and X_{ij} being a root vector of type α with $\alpha \in \sum_i^o$.

LEMMA 6.7. For any integer $r \geq 0$ and any open compact subgroup U of $\mathbb{N}_1^f = N_1(\mathbb{A}^f)$ there is a constant $c = c(r, U)$ and a finite collection $Y_1, \ldots, Y_m$ of differential operators of type r, the collection depending on r alone and not on U, such that

$$\|\prod \psi\|_\infty \leq c \sum_i \|R(Y_i)\psi\|_\infty$$

for any function ψ on $N_1 \backslash \mathbb{N}_1 / U$ which has continuous derivations up to order rs.

The norms in the inequality are of course L_∞-norms. A little reflection shows that we can make a number of simplifications. First of all replacing ψ by $\prod_{P_2} \psi$ we can work in the group M_2 rather than in G. In other words we may suppose that $G = P_2$. Then the formula (6) reduces to the case that P_1 is a maximal proper parabolic of G over $\mathbb{Q}$.

We choose a composition series of groups over $\mathbb{Q}$

$$N_1 = V_\ell \supseteq V_{\ell-1} \supseteq \cdots \supseteq V_0 = \{1\}$$

with V_{i+1}/V_i isomorphic to the additive group. Since

$$(1 - \prod_{P_1})\psi(n_1) = \sum_{i=0}^{\ell-1} \int_{V_i \backslash \mathbb{V}_i} \psi(vn_1)\,dv - \int_{V_{i+1} \backslash \mathbb{V}_{i+1}} \psi(vn_1)\,dv$$

it is enough to prove the following lemma.

LEMMA 6.8. *Let* $r \geq 0$ *be an integer and let* U *be an open subgroup of* $\mathbb{A}$. *There is a constant* $c = c(r, U)$ *such that for any function* ψ *on* $Q\backslash\mathbb{A}/U$ *which is continuously differentiable of order* r

$$\sup_{K} |\psi(x) - \int_{Q\backslash\mathbb{A}} \psi(Y)\,dy| \leq c\|\frac{\partial^r \psi}{\partial x^r}\|_\infty .$$

To be a function on $Q\backslash\mathbb{A}/U$ is to be a function on a quotient $L\backslash\mathbb{R}$ where $L = L(U)$ is a lattice in $\mathbb{R}$. The inequality thus follows readily from

$$\sum_{n\neq 0} |a_n| \leq \left(\sum_{n\neq 0} |n^r a_n|^2\right)^{1/2}\left(\sum_{n\neq 0} \frac{1}{n^{2r}}\right)^{1/2} ,$$

at least for $r > 0$, but the case $r = 0$ is quite trivial.

We shall apply Lemma 6.7 to a function

$$n \longrightarrow \psi(na)$$

where ψ is a function on $G\backslash\mathbb{G}$ and $a \in A_0(\mathbb{A})$. If we want to regard the Y_i as left-invariant differential operators on $\mathbb{G}$ we must write the inequality of Lemma 6.7 as

$$(7) \qquad \sup_{n_1} |\prod \psi(n_1 a)| \leq c \sum_i \sup_i |R(ada^{-1}(Y_i))\psi(n_1 a)| .$$

This will be to our advantage.

We now take up the proof of Lemma 6.8. The first step is to replace

$$\hat{\tau}_P(H(x) - T)\int_{N\backslash \mathbf{N}} \varphi(ng)\,dn$$

by

$$\sum_{P_1\subset P\subset P_2} \sum_{P_1\backslash P} F^1_P(x, T)\sigma^2_1(H(x) - T)\int_{N\backslash\mathbf{N}} \varphi(ng)\,dn \quad ,$$

the sum being over the pairs P_1, P_2. There is then a sum over $P\backslash G$ and an alternating sum over P. The final result is a sum over pairs $P_1\subset P_2$ of

$$\sum_{P_1\backslash G} \sum_{\{P\backslash P_1\subset P\subset P_2\}} (-1)^{a_P} F^1_P(\delta g, T)\sigma^2_1(H(\delta g) - T)\int_{N\backslash\mathbf{N}} \psi(n\delta g)\,dn \quad .$$

However Corollary 4.1.2 allows us to replace F^1_P by F^1_2. The upshot is that we are forced to estimate

$$\sum_{P_1\backslash G} F^1_2(\delta g, T)\sigma^2_1(H(\delta g) - T)\left|\sum_P (-1)^{a_P} \int_{N\backslash\mathbf{N}} \varphi(n\delta g)\right| \quad .$$

Lemma 2.1 shows that

$$\sum_{P_1\backslash G} F^1_2(\delta g, T)\sigma^2_1(H(\delta g) - T) \le c|g|^M$$

for some M, T being held constant. Thus the problem is to estimate

$$\left|\sum_P (-1)^{a_P} \int_{N\backslash\mathbf{N}} \varphi(n\delta g)\right| \quad .$$

It is now best to be more precise about Siegel domains. In contrast to the previous definition the elements g of $\mathfrak{S}_P(T_0)$ will now be required to have all of the following properties:

(i) If $g = pk$ and $a = a(g)$ is the projection of p on A_0 then $g \in a\Omega$ where Ω is a fixed compact set. ⟵ OK for G but not for arbitrary P

(ii) $\alpha(H(g) - T_0) > 0$ for all $\alpha \in \Delta_0^P$.

(iii) There are constants c_1 and c_2 so that   Consider [illegible] N_P

$$|\ln|a|| \leq c_1(1+\|H(g)\|) \leq c_2(1+|\ln|a||) \quad .$$

The final condition is easily seen to force the component of a in $A_0(\mathbb{A}^f)$ to lie in a compact set. This modification entails a modification in $\mathfrak{S}_P^1(T_0, T)$ but the set

$$P_1\mathfrak{S}_P^1(T_0, T)$$

and thus the function F_P^1 is not changed; provided of course c_1, c_2 and Ω, which affect the size of $\mathfrak{S}_P(T_0)$, are all chosen large enough.

This definition has the advantage that for a given $\mathfrak{S}_G(T_0)$, for example that of Lemma 6.6, there are positive constants c_1 and ε such that

$$|\delta g|^{-1} \leq c|g|^{-\varepsilon}$$

for all $\delta \in G$ and all $g \in \mathfrak{S}_G(T_0)$. (I know no reference for this fact. It can be deduced from Prop. II.1.5 of A. Borel, Ensembles fondamentaux

<u>pour les groupes arithmétiques et formes automorphes</u>, Cours à l'IHP (1964).)

Thus all we need do is show that if $g \in \mathfrak{G}_2^1(T_0, T)$ and $\sigma_1^2(H(g) - T) \neq 0$ then, for a suitable choice of g modulo P_1,

$$(8) \quad |g|^{M+N'} |\sum_P (-1)^{a_P} \int_{N \backslash \mathbf{N}} \varphi(ng)| \leq \sum_i \sup_{h \in \mathbf{G}^1 g} |R(Y_i)\varphi(h)| |h|^{-N} \quad .$$

The suitable choice of g will be an element in $\mathfrak{G}^1(T_0, T)$. Then conditions (i) and (iii) yield

$$|g|^{M+N'} \leq ce^{M'\|H\|} \quad ,$$

with $H = H(g) = H(a)$. Thus denoting the right side of (8) by A we need only show that

$$(9) \qquad ce^{M'\|H\|} |\sum_P (-1)^{a_P} \int_{N \backslash \mathbf{N}} \varphi(ng)| \leq A \quad .$$

Since we can readily deal with right translations by elements from a compact set in $\mathbf{G}^1$ we may suppose that $g = a(g) = a$. As in Lemma 4.2 we may write $H = H_1 + H_2$ with $H_1 \in \mathfrak{a}_1^2$ and $H_2 \in \mathfrak{a}_2$ and deduce from the fact that $\sigma_1^2(H-T) \neq 0$ that

$$\|H_2\| \leq c(1+\|H_1\|) \quad ,$$

the constant c depending of course on T, but that is of no consequence. Thus it will be enough to prove (9) with H replaced by H_1 but with a larger

M'. This is an easy consequence of (7), for the inequality (ii) applied with P_2 replacing P assures us that the coefficients of $\mathrm{ad}\,a^{-1}(Y)$ with respect to a fixed basis of the universal enveloping algebra are bounded by $ce^{-M''\|H_1\|}$, where $M'' \longrightarrow \infty$ with r.

Appendix

Truncation has been seen to have two essential properties. It is an idempotent and it converts slowly increasing smooth functions to rapidly decreasing functions. It may be worthwhile to see how this comes to pass in a simple case.

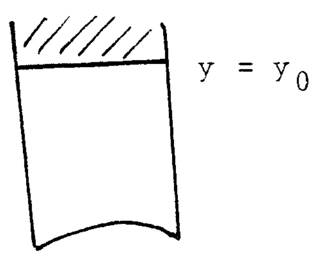
A function f on the upper half-plane which is invariant under $SL(2, \mathbf{Z})$ may also be considered as a function on $SL(2, \mathbf{Z})\backslash SL(2, \mathbf{R})$ if we set

$$\phi(g) = f\left(\frac{ai+b}{ci+d}\right), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

In particular

$$\phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = f(a^2 i + x) .$$

The function f is determined by its values on a a fundamental domain



Truncation is achieved by leaving f untouched below a certain line $y = y_0$ in the fundamental domain and by removing the constant term of its Fourier expansion above the line. So it is clearly idempotent.

The inequality

$$\sum_{n\neq 0} |a_n| \leq \sqrt{\sum_{n\neq 0} n^{2r} |a_n|^{2r}} \sqrt{\sum_{n\neq 0} \frac{1}{n^{2r}}}$$

shows that for $r \geq 1$ and $y > y_0$

$$(1) \qquad |\Lambda f(u+iy)| \leq c \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{d^r}{dx^r} f(x+iy)\right|^2 dx \quad .$$

However if X is the element

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in the Lie algebra then

$$\frac{d^r}{dx^r} f(x+iy) = \frac{d^r}{dx^r} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right)$$

with $y = a^2$ and right side of this equality is

$$a^{-2r} R(X)^r \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) \quad .$$

Thus bounds on $R(X)^r\phi$ of the form

$$\left|R(X)^r \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right)\right| \leq c(r) a^{2s} \quad ,$$

where s is a constant independent of r - and this is the kind of bound that will be available to us - yield

$$\left|\frac{d^r}{dx^r} f(x+iy)\right| \le c(r)y^{s-r} \quad .$$

The inequality (1) then implies that

$$|\Lambda f(u+iy)| \le cc(r)y^{s-r}$$

for any r.