MORNING SEMINAR ON TRACE FORMULA

References

I. Truncation and basic identity:

1. J. Arthur, A trace formula for reductive groups I: terms associated to classes in $G(\mathbb{Q})$, Duke Math. J. 45 (1978).

2. ———, A trace formula for reductive groups II: applications of a truncation operator, Comp. Math. 40 (1980).

II. Formalities of non-invariance:

3. ———, The trace formula in invariant form, Ann. of Math. 114 (1981).

III. The $\mathfrak{o}$-expansion: (1) and

4. ———, in preparation.

IV. The χ-expansion: (2) and

5. ———, A Paley-Wiener theorem for real reductive groups, to appear in Acta Math.

6. ———, On the inner product of truncated Eisenstein series, Duke Math. J. 49 (1982).

7. ———, On a family of distributions obtained from Eisenstein series I: Applications of the Paley-Wiener theorem, Amer. Jour. of Math. 104 (1982).

8. ———, On a family of distributions obtained from Eisenstein series II: Explicit formulas, Amer. Jour. of Math. 104 (1982).

V. Expository

9. ———, Eisenstein series and the trace formula, Proc. Symp. in Pure Math. 33.

10. ———, The trace formula for reductive groups, Journées automorphes, Publ. Math. de l'Univ. Paris VII (15).

This seminar has two purposes: To expound the work of Arthur systematically and with any luck to extend it to the twisted case.

Lecture 1

INTRODUCTION

R. Langlands

Basic notation

G: reductive group over $\mathbf{Q}$.

P: parabolic subgroup of G with unipotent radical N and Levi factor M.

P_0: fixed minimal over $\mathbf{Q}$ parabolic subgroup of G. If $P \supseteq P_0$ then P is called standard.

ε: automorphism of G of finite order ℓ which fixes P_0.

E: group of order ℓ generated by ε.

G': $G \rtimes E$.

For simplicity I will also usually denote $G(\mathbf{Q})$, $P(\mathbf{Q})$ and so on by G, P, ... while designating $G(\mathbf{A})$, $P(\mathbf{A})$, ... by $\mathbf{G}$, $\mathbf{P}$, ...

Z: connected center of G.

$\mathbf{Z}_0$: closed ε-invariant subgroup of $\mathbf{Z}$ with $\mathbf{Z}_0 G$ closed. We fix once and for all a unitary character ξ of $\mathbf{Z}_0$ trivial on $Z_0 = \mathbf{Z}_0 \cap G$. It will always be there but will ultimately disappear from the notation. The group $\mathbf{Z}_0$ is in many applications $\{1\}$ and we urge the reader to fix his attention on this case.

L: space of measurable functions ψ on $G \backslash \mathbf{G}$ satisfying the following two conditions

(i) $$\psi(zg) = \xi(z)\psi(g) \qquad \forall z \in \mathbf{Z}_0$$

(ii) $$\int_{\mathbf{Z}_0 G \backslash \mathbf{G}} |\psi(g)|^2 dg < \infty .$$

We define a unitary representation R of G on L by

$$R(g)\psi : h \longrightarrow \psi(hg) \quad .$$

Let ω be a unitary character of $\mathbf{G}$ trivial on G and satisfying

$$\omega(\varepsilon^{-1}(z))\xi(\varepsilon^{-1}(z)) = \xi(z) \quad .$$

Then the operator $R(\varepsilon)$ defined by

$$R(\varepsilon)\psi : h \longrightarrow \omega(\varepsilon^{-1}(h))\psi(\varepsilon^{-1}(h))$$

acts on L. This definition does not necessarily yield a representation of the group $\mathbf{G}'$. The relations satisfied are:

$$R(\varepsilon)R(g) = \omega^{-1}(g)R(\varepsilon(g))R(\varepsilon)$$

and

$$R(\varepsilon^{\ell})\psi(h) = \omega(\varepsilon^{-1}(h)\varepsilon^{-2}(h) \ \ldots \ \varepsilon^{-\ell}(h))\psi(h) \quad .$$

If ϕ is a continuous compactly supported function on $\mathbf{G}$ then $R(\phi)$ is the operator defined by

$$R(\phi)\psi(h) = \int_{\mathbf{G}} \phi(g)\psi(hg)dg \quad .$$

The operator of interest is however $R(\phi)R(\varepsilon)$ and this is given by

$$\begin{aligned} R(\phi)R(\varepsilon)\psi(h) &= \int_{\mathbf{G}} \omega(\varepsilon^{-1}(hg))\phi(g)\psi(\varepsilon^{-1}(hg))dg \\ &= \int_{\mathbf{Z}_0 G\backslash\mathbf{G}} \left\{ \sum_{\gamma\in Z_0\backslash G_{\phi}} \int_{\mathbf{Z}_0} \omega(g)\xi(z)\phi(h^{-1}\gamma z\varepsilon(g))dz \right\} \psi(g)dg \ . \end{aligned}$$

Thus it is an integral operator with kernel

$$K(h, g) = \sum_{\gamma \in Z_0 \backslash G} \int_{\mathbf{Z}_0} \omega(g)\xi(z)\phi(h^{-1}\gamma z\varepsilon(g))dz \quad .$$

In order to simplify the formulas it is convenient to denote the function

$$g \longrightarrow \int_{\mathbf{Z}_0} \xi(z)\phi(zg)dz$$

by ϕ, the original function playing no further role. Then the kernel may be written

$$K(h, g) = \omega(g) \sum_{Z_0 \backslash G} \phi(h^{-1}\gamma\varepsilon(g)) \quad .$$

Recall: If the quotient $Z_0 G \backslash \mathbf{G}$ is not compact then $R(\phi)R(\varepsilon)$ is usually not of trace class even for smooth ϕ.

Truncation

This is a process for transforming sufficiently smooth slowly increasing functions on $Z_0 G \backslash \mathbf{G}$ into rapidly decreasing functions. Composition of the truncation operator with K yields an operator of trace class. For now I content myself with a formal description of the operator, postponing proofs and a precise description of its properties until later.

If $P \supseteq P_0$ let $\mathfrak{a}$ be $X_*(P) \otimes \mathbf{R}$, $X_*(P)$ being the lattice dual to the lattice $X^*(P)$ of rational characters of P.
We have $P_0 \longrightarrow P$, thus $X^*(P) \longrightarrow X^*(P_0)$, $X_*(P_0) \longrightarrow X_*(P)$,

and so $\mathfrak{a}_0 \longrightarrow \mathfrak{a}$. On the other hand if A_0 is a maximal split torus over $\mathbf{Q}$ in P_0 we can identify $\mathfrak{a}_0$ with $X_*(A_0) \otimes \mathbf{R}$ or with the Lie algebra of $A_0(\mathbf{R})$ and we can choose A in A_0 to be a maximal split torus in P. This yields $\mathfrak{a} \longrightarrow \mathfrak{a}_0$. Thus we have a natural decomposition (the center of a Levi factor M of P)

$$\mathfrak{a}_0 = \mathfrak{a}_P \oplus \mathfrak{a}_0^P \quad ,$$

where to emphasize its dependence on P we have written $\mathfrak{a} = \mathfrak{a}_P$. It is convenient to fix A_0 once and for all.

Let Δ_0 be the set of simple roots of $\mathfrak{a}_0$ and let Δ_0^P be the simple roots in M. Thus $\Delta_0^P \subseteq \Delta_0$. On $\mathfrak{a}_0^G$ we introduce an inner product compatible with the root system Δ_0 and let $\hat{\Delta}_0$ be the dual basis. Thus

$$\langle \varpi_\alpha, \beta \rangle = \delta_{\alpha\beta} \qquad \alpha, \beta \in \Delta_0, \ \varpi_\alpha \in \hat{\Delta}_0 \quad .$$

We let $\hat{\tau}_P$ be the characteristic function of

$$^+\mathfrak{a}_P = \{H \in \mathfrak{a}_P \mid \varpi_\alpha(H) > 0, \ \alpha \in \Delta_0 - \Delta_0^P\} \quad .$$

Observe that if $H = H_P + H^P$, $H_P \in \mathfrak{a}_P$, $H^P \in \mathfrak{a}_0^P$ then

$$\hat{\tau}_P(H) = \hat{\tau}_P(H_P) \quad .$$

We choose <u>once and for all</u> a maximal compact subgroup K of $\mathbf{G}$ such that $\mathbf{G} = \mathbf{P}_0 K = \mathbf{N}_0 \mathbf{M}_0 K$. It is important to observe that many operations and many formulas, including the trace formula itself, contain K implicitly. We define $H(g)$, $g \in \mathbf{G}$ by $g = pk$ and

$$|\chi(p)| = e^{\langle H(g), \chi \rangle}, \quad \chi \in X^*(P_0) \quad . \qquad H(g) \in \mathfrak{a}_0$$

In order to define the truncation operator we have to choose $T \in \mathfrak{a}_0$. This done we define $\Lambda^T \varphi$, φ a continuous function on $G \backslash \mathbf{G}$, by

$$\Lambda^T \varphi(g) = \sum_P (-1)^{\dim \mathfrak{a}_P/\mathfrak{a}_G} \sum_{\delta \in P \backslash G} \int_{N \backslash \mathbf{N}} \varphi(n\delta g) dn \hat{\tau}_P(H(\delta g) - T) \quad .$$

Facts (to be provided later)

(a) Each of the inner sums is finite.

(b) $\Lambda^T(\Lambda^T \varphi) = \Lambda^T \varphi$.

(c) Λ^T transforms sufficiently smooth slowly increasing functions into rapidly decreasing functions.

(d) Λ^T extends to an orthogonal projection on L.

If φ is a continuous function on $P_1 \backslash \mathbf{G}$ we can more generally introduce a truncated function Λ^{T,P_1} given by

$$\Lambda^{T,P_1} \varphi(g) = \sum_{P_0 \subset R \subset P_1} (-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1} \sum_{\delta \in R \backslash P_1} \int_{N_R \backslash \mathbf{N}_R} \phi(n\delta g) \hat{\tau}_R^{P_1}(H(\delta g) - T) dn .$$

Here $\hat{\tau}_R^{P_1}$ is a function on $\mathfrak{a}_0 = \mathfrak{a}_{P_1} \oplus \mathfrak{a}_0^{P_1}$ and $\hat{\tau}_R^{P_1}(H) = \hat{\tau}_R^{P_1}(H^{P_1})$. On $\mathfrak{a}_{R}^{P_1}$ the function is defined by a dual basis $\hat{\Delta}_{R}^{P_1}$ to $\Delta_{0,R}^{P_1}$. It is the characteristic function of

$$^{+}\mathfrak{a}_R^{P_1} = \{H \in \mathfrak{a}_R^{P_1} \mid \varpi_\alpha(H) > 0, \varpi_\alpha \in \hat{\Delta}_R^{P_1}\} .$$

Some modification of the exposition is called for.

It is often convenient to define $\Delta_R^{P_1}$, $\hat{\Delta}_R^{P_1}$, and $\mathfrak{a}_R^{P_1}$. Here $\mathfrak{a}_R^{P_1}$ is the orthogonal complement of $\mathfrak{a}_0^R$ in $\mathfrak{a}_0^{P_1}$ and $\Delta_R^{P_1}$ is the collection of restrictions of $\alpha \in \Delta_0^{P_1} - \Delta_0^R$ to $\mathfrak{a}_R^{P_1}$. The dual basis to $\Delta_R^{P_1}$ is $\hat{\Delta}_R^{P_1}$ which may be identified with $\{\varpi_\alpha \mid \alpha \in \Delta_0^{P_1} - \Delta_0^R\} \subseteq \hat{\Delta}_0^{P_1}$ for $\varpi_\alpha|\mathfrak{a}_0^R = 0$ if $\varpi_\alpha \in \hat{\Delta}_0^{P_1}$, $\alpha \notin \Delta_0^R$.

The operator Λ^{T,P_1} has properties similar to those of Λ^T.

The basic identity

If P is an ε-invariant standard parabolic subgroup we define a kernel K_P by

$$K_P(h, g) = \sum_{\gamma \in Z_0 \backslash M} \int_{\mathbf{N}} \phi(h^{-1}\gamma n \varepsilon(g))\,dn \quad .$$

Thus K_P is a function on $\mathbf{N}P\backslash\mathbf{G} \times \mathbf{N}P\backslash\mathbf{G}$ and $K_G = K$.

If $P_1 \subseteq P_2$ are two standard parabolic subgroups we let σ_1^2 be the characteristic function of the set of H in $\mathfrak{a}_1$ $(= \mathfrak{a}_{P_1})$ or $\mathfrak{a}_0$ (depending on one's point of view) for which

(i) $\varpi_\alpha(H) > 0 \;\forall \varpi_\alpha \in \hat{\Delta}_1 (= \hat{\Delta}_{P_1} = \hat{\Delta}_{P_1}^G)$

(ii) $\alpha(H) > 0 \;\forall \alpha \in \Delta_{P_1}^{P_2}$ $(= \Delta_1^2)$

(iii) $\alpha(H) \leq 0 \;\forall \alpha \in \Delta_{P_1}$ $(= \Delta_{P_1}^G) - \Delta_{P_1}^{P_2}$.

Modify: $\varepsilon^{\varpi}_\alpha(H) > 0$ $\forall$ ϖ_α, $\alpha \in \Delta_0^G - \Delta_0'$?

The basic identity is the equality

$$\sum_{P_0 \subset P} (-1)^{\dim \mathfrak{a}_P^{\varepsilon}/\mathfrak{a}_G^{\varepsilon}} \sum_{\delta \in P\backslash G} K_P(\delta g, \delta g)\hat{\tau}_P(H(\delta g) - T)$$

$$= \sum_{P_0 \subset P_1 \subset P_2} \sum_{\delta \in P_1\backslash G} \sigma_1^2(H(\delta g) - T)\left(\sum_{P_1 \subset P \subset P_2} (-1)^{\dim \mathfrak{a}_P^{\varepsilon}/\mathfrak{a}_G^{\varepsilon}} \Lambda^{T,P_1} K_P(\delta g, \delta g)\right) .$$

The sum over P_1 and P_2 is over all standard parabolic subgroups, but the sum over P is on both sides the sum over all ε-invariant standard parabolics. The symbol $\mathfrak{a}_P^{\varepsilon}$ denotes the space of ε-invariants in $\mathfrak{a}_P$ and the truncation Λ^{T,P_1} is carried out on the first variable in $K_P(h, g)$ before substitution of $(\delta g, \delta g)$ for (h, g). The symbol $\hat{\tau}_P$ is an abbreviation for $\hat{\tau}_P^G$.

Let

$$\mathbf{G}^1 = \{g \in \mathbf{G} \,|\, |\chi(g)| = 1 \;\forall \chi \in X^*(G)\} \quad .$$

We shall expand the integral of the left side over $G\backslash \mathbf{G}^1$ as a sum over conjugacy classes $\mathfrak{o}$, obtaining finally the fine $\mathfrak{o}$-expansion and the integral of the right as a sum over automorphic representations obtaining ultimately the fine χ-expansion. The resulting equality is the (twisted) trace formula. Observe that all integrals that arise will be shown to be convergent.

Since the twisted case remains to be worked out as we go along I will confine myself on the whole in the remainder of this introduction to the ordinary trace formula. Even here it remains uncertain that the formal statements have the form given until various papers in preparation are written.

The $\mathfrak{o}$-expansions

The fine $\mathfrak{o}$-expansion that we ultimately obtain will be formed by sums over conjugacy classes, but the first step is to obtain a coarse $\mathfrak{o}$-expansion and this runs over semi-simple conjugacy classes. If $\gamma \in G' = G \rtimes E$ then γ may be written as $\gamma = \gamma_s \gamma_u$ with γ_s semi-simple and γ_u unipotent. The two elements γ and γ_s have the same projection on E and we are interested only in those γ which project on ε. Two such elements γ and γ' are in the same conjugacy class if $\gamma' = \delta^{-1}\gamma\delta$, $\delta \in G$. They are the same semi-simple classes if $\gamma'_s = \delta^{-1}\gamma_s\delta$, $\delta \in G$.

The fine $\mathfrak{o}$-expansion will have the form

$$\sum_P \sum_{\mathfrak{o} \in \mathcal{O}(M)} c_{\mathfrak{o}} (T_M^G \Phi_{\mathfrak{o}})(\phi) \quad .$$

The sum is over (ε-invariant) standard parabolics, $\mathcal{O}(M)$ is the set of conjugacy classes in M, and $c_{\mathfrak{o}}$ is a constant. It is 0 if $\mathfrak{o}$ is not elliptic, the class of γ in M being elliptic if the maximal split component of the center of the centralizer of γ_s is contained in $\mathfrak{a}_M = \mathfrak{a}_P$. ~~split~~

$\Phi_{\mathfrak{o}}$ is the (twisted) orbital integral over the adelic orbit of γ and $T_M^G \Phi_{\mathfrak{o}}$ is a distribution associated to $\Phi_{\mathfrak{o}}$ and is a weighted (twisted) orbital integral over the class in G induced from $\mathfrak{o}$ in the sense of Lusztig-Spaltenstein. For $M = G$ we have $T_M^G \Phi_{\mathfrak{o}} = \Phi_{\mathfrak{o}}$. Thus the distribution

$$\sum_{\mathfrak{o} \in \mathcal{O}(G)} c_{\mathfrak{o}} T_G^G \Phi_{\mathfrak{o}}$$

is (twisted-) invariant.

The χ-expansion

The χ-expansion is also obtained in two stages. The coarse expansion is derived first. It is a sum over cuspidal pairs. A cuspidal pair consists of a standard parabolic subgroup P and a cuspidal representation ρ of $\mathbf{M}^1$. Two pairs (ρ, P) and (ρ', P') are said to be equivalent if there is an $s \in \Omega(\mathfrak{a}, \mathfrak{a}')$ with representative w_s such that the representations ρ' and

$$m' \longrightarrow \rho(w_s^{-1} m' w_s)$$

are equivalent. $\Omega(\mathfrak{a}, \mathfrak{a}')$ is the set of linear transformations from $\mathfrak{a}$ to $\mathfrak{a}'$ obtained by restriction of some element $w_s \in G(\mathbf{Q})$.

The fine χ-expansion has the form

$$\sum_P \int_{\prod(\mathbf{M})} d(\pi)(T_M^G \sigma_\pi)(\phi) d\pi \quad .$$

The sum is over all (ε-invariant?) standard parabolics and the integral is over all (ε-invariant?) unitary automorphic representations of $\mathbf{M}$, or at least a part of them which will be described later together with the measure $d\pi$. In the integrand appear a function $d(\pi)$ and $T_M^G \sigma_\pi$. Here σ_π is the (twisted) trace of σ_π but $T_M^G \sigma_\pi$ is a distribution associated to σ_π by means of derivatives of intertwining operators on $\mathrm{Ind}_P^G \pi$. Like $T_M^G \Phi_{\mathfrak{o}}$ the distribution $T_M^G \sigma_\pi$ will in general not be (twisted) invariant for $M \neq G$. But if $M = G$ then $T_G^G \sigma_\pi = \sigma_\pi$ and

$$\int_{\Pi(\mathbf{G})} d(\pi) T_G^G \sigma_\pi d\pi$$

is a (twisted) invariant distribution.

Thus apart from the explicit determination of the functions $c(\boldsymbol{\sigma})$ and $d(\pi)$, problems which have not yet been solved completely, the final form of the trace formula, from an analytic point of view and before stabilization, is

$$\sum_P \sum_{\boldsymbol{\sigma} \in \boldsymbol{\mathcal{O}}(M)} c_{\boldsymbol{\sigma}}(T_M^G \Phi_{\boldsymbol{\sigma}})(\phi) = \sum_P \int_{\Pi(\mathbf{M})} d(\pi)(T_M^G \sigma_\pi)(\phi) d\pi \quad .$$

Any further modification, especially any transfer of terms from one side to the other to obtain an identity between invariant distributions, will probably be determined by the problem to be solved.

<u>A final remark</u>. Let L^d be the direct sum of all irreducible invariant subspaces of L. In the course of deriving the fine χ-expansion one has to show that the restriction of $R(\phi)$ to L^d is of trace class for sufficiently smooth ϕ. As I indicated the proof of this has not yet been completely worked out. So this result and its consequences remain for the moment uncertain.

Make some remarks about the trace formula over arbitrary number fields (of finite degree over $\mathbb{Q}$)