Lecture 13

SOME FORMAL PROPERTIES OF THE TERMS IN THE TRACE FORMULA

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13.1. Some combinatorics.

Given two parabolic subgroups P and Q such that $P \subset Q$ we have defined τ_P^Q (resp. $\hat{\tau}_P^Q$) to be the characteristic function of the set of $H \in \mathfrak{a}_P^Q$ such that $\alpha(H) > 0$ for all $\alpha \in \Delta_P^Q$ (resp. $\varpi(H) > 0$ for all $\varpi \in \hat{\Delta}_P^Q$). By abuse of notation we also consider them as functions on $\mathfrak{a}_0$ depending only on the projection on $\mathfrak{a}_P^Q$.

When P and Q are ε-invariant we define ${}_\varepsilon\tau_P^Q$ (resp. ${}_\varepsilon\hat{\tau}_P^Q$) to be the restriction to $(\mathfrak{a}_P^Q)^\varepsilon$ the subset of ε-invariant vectors. They will also be considered as functions on $\mathfrak{a}_0^\varepsilon$ and even on $\mathfrak{a}_0$. We introduce a new functions on $(\mathfrak{a}_P^Q)^\varepsilon \times (\mathfrak{a}_P^Q)^\varepsilon$:

$$ {}_\varepsilon\Gamma_P^Q(H, X) = \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^\varepsilon - a_Q^\varepsilon} \, {}_\varepsilon\tau_P^R(H) \, {}_\varepsilon\hat{\tau}_R^Q(H-X) \quad . $$

The key observation for all that follows is the

LEMMA 13.1.1.

(i) <u>Assume that</u> X <u>remains in a compact subset</u> ω <u>then</u>

$$ H \longrightarrow {}_\varepsilon\Gamma_P^Q(H, X) $$

<u>is supported in a compact subset of</u> $(\mathfrak{a}_P^Q)^\varepsilon$ <u>independent of</u> $X \in \omega$.

(ii) <u>If</u> X <u>is regular then</u>

$$H \longrightarrow {}_{\varepsilon}\Gamma_P^Q(H, X)$$

<u>is the characteristic function of the set of</u> $H \in (\mathfrak{a}_P^Q)^{\varepsilon}$ <u>such that</u>

$$\alpha(H) > 0 \quad \underline{\text{for all}} \quad \alpha \in \Delta_P^Q$$
$$\varpi(H) \leq \omega(X) \quad \underline{\text{for all}} \quad \varpi \in \hat{\Delta}_P^Q \quad .$$

(iii) ${}_{\varepsilon}\Gamma_P^Q(H, 0) = \delta_P^Q$ (<u>the Kronecker symbol</u>).

Given H we define $S = S_H$ to be the ε-invariant parabolic subgroup S between P and Q such that

$$\Delta_P^S = \{\alpha \in \Delta_P^Q | \alpha(H) > 0\} \quad .$$

We have

$$ {}_{\varepsilon}\Gamma_P^Q(H, X) = \sum_{\substack{P \subset R \subset S \\ \varepsilon(R)=R}} (-1)^{a_R^{\varepsilon} - a_Q^{\varepsilon}} \, {}_{\varepsilon}\hat{\tau}_R^Q(H-X) \quad .$$

This is non zero only if $\varpi(H-X) > 0$ for all $\varpi \in \hat{\Delta}_S^Q$ and $\varpi(H-X) \leq 0$ for all $\varpi \in \hat{\Delta}_P^Q - \hat{\Delta}_S^Q$. Choose $X_1 \in (\mathfrak{a}_P^Q)^{\varepsilon}$ such that

$$\alpha(X_1) \leq \inf_{X \in \omega \cup \{0\}} \alpha(X)$$

for all $\alpha \in \Delta_P^Q$. Since $\alpha(H) > 0 \geq \alpha(X_1)$ for $\alpha \in \Delta_P^S$ and $\varpi(H) > \varpi(X) \geq \varpi(X_1)$ for $\varpi \in \hat{\Delta}_S^Q$ we have $\varpi(H) > \varpi(X_1)$ for all $\varpi \in \hat{\Delta}_P^Q$. In the same way, replacing Inf by max and changing the sense of inequalities we define X_2; then for all $\varpi \in \hat{\Delta}_P^Q$ we have

$$\varpi(X_1) < \varpi(H) \leq \varpi(X_2)$$

whenever ${}_\varepsilon\Gamma_P^Q(H, X) \neq 0$ and $X \in \omega$. Assertion (i) follows.

Consider now a fixed X such that $\alpha(X) \geq 0$ for all $\alpha \in \Delta_P^Q$, then we may take $X_1 = 0$ and $X_2 = X$; this implies $S_H = Q$ if ${}_\varepsilon\Gamma_P^Q(H, X) \neq 0$ and assertion (ii) follows.

If $X = 0$ we may take $X_1 = X_2 = 0$ and this implies $S_H = Q$ and $S_H = P$ if ${}_\varepsilon\Gamma_P^Q(H, 0) \neq 0$. This yields assertion (iii). □

Remark: Assertion (iii) above has already been proved, with other notations, in Lecture 2; see 13.1.2. below.

We now introduce matrices of function on $\mathfrak{a}_0^\varepsilon$ whose entries are indexed by pairs of ε-invariant parabolic subgroups: let ${}_\varepsilon\tau = ({}_\varepsilon\tau_{P,Q})$ be such that

$$
\begin{aligned}
{}_\varepsilon\tau_{P,Q} &= 0 \quad \text{if} \quad P \not\subset Q \\
{}_\varepsilon\tau_{P,Q} &= (-1)^{a_P^\varepsilon}\, {}_\varepsilon\tau_P^Q \quad \text{if} \quad P \subset Q
\end{aligned}
$$

considered as functions on $\mathfrak{a}_0^\varepsilon$. In the same way we define ${}_\varepsilon\hat{\tau}$. Assertion (iii) in the above lemma yields the

COROLLARY 13.1.2. ${}_\varepsilon\tau\,{}_\varepsilon\hat{\tau} = 1$. □

We introduce a matrix ${}_\varepsilon\Gamma = ({}_\varepsilon\Gamma_{P,Q})$ whose entries are such that

$$ {}_\varepsilon\Gamma_{P,Q} = 0 \quad \text{if} \quad P \not\subset Q $$
$$ {}_\varepsilon\Gamma_{P,Q} = (-1)^{a_P^\varepsilon - a_Q^\varepsilon}\, {}_\varepsilon\Gamma_P^Q \quad \text{if} \quad P \subset Q \quad . $$

Using the definition of ${}_\varepsilon\Gamma_P^Q$ we see that

$$ {}_\varepsilon\Gamma(H,\ X) = {}_\varepsilon\tau(H)\, {}_\varepsilon\hat{\tau}(H-X) \quad . $$

LEMMA 13.1.3.

$$ {}_\varepsilon\tau_P^Q(H-X) = \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^\varepsilon - a_Q^\varepsilon}\, {}_\varepsilon\tau_P^R(H)\, {}_\varepsilon\Gamma_R^Q(H,\ X) \quad . $$

Using Corollary 13.1.2 we see that

$$ {}_\varepsilon\hat{\tau}(H-X) = {}_\varepsilon\tau(H)^{-1}\, {}_\varepsilon\Gamma(H,\ X) = {}_\varepsilon\hat{\tau}(H)\, {}_\varepsilon\Gamma(H,\ X) \quad . \ \square $$

Since $H \longrightarrow {}_\varepsilon\Gamma_P^Q(H,\ X)$ is compactly supported on $(\mathfrak{a}_P^Q)^\varepsilon$ the integral

$$ {}_\varepsilon\gamma_P^Q(\lambda,\ X) = \int_{(\mathfrak{a}_P^Q)^\varepsilon} {}_\varepsilon\Gamma_P^Q(H,\ X) e^{\lambda(H)} dH $$

is convergent for all $\lambda \in \mathfrak{a}_0^* \otimes \mathbf{C}$ and defines an analytic function. We want to compute ${}_\varepsilon\gamma_P^Q$. We define ${}_\varepsilon\Delta_P^Q$ to be the set of restrictions to

$(\mathfrak{a}_P^Q)^\varepsilon$ of ε-orbits of elements in Δ_P^Q. Given $\alpha \in {}_\varepsilon\Delta_P^Q$ the coroot $\check{\alpha}$ lies in $(\mathfrak{a}_P^Q)^\varepsilon$. We define

$$ {}_\varepsilon c_P^Q = |\det(\check{\alpha}, \check{\beta})|^{\frac{1}{2}} \qquad \alpha, \beta \in {}_\varepsilon\Delta_P^Q $$

and

$$ {}_\varepsilon\theta_P^Q(\lambda) = ({}_\varepsilon c_P^Q)^{-1} \prod_{\alpha \in {}_\varepsilon\Delta_P^Q} \lambda(\check{\alpha}) \quad . $$

Now assume that $\mathrm{Re}(\lambda(\check{\alpha})) < 0$ for all $\alpha \in {}_\varepsilon\Delta_P^Q$, then

$$ \int_{(\mathfrak{a}_P^Q)^\varepsilon} {}_\varepsilon\hat{\tau}_P^Q(H) e^{\lambda(H)} dH = \theta_P^Q(\lambda)^{-1} \quad . $$

Replacing roots by weights we define ${}_\varepsilon\hat{\Delta}_P^Q$, ${}_\varepsilon\hat{c}_P^Q$, and ${}_\varepsilon\hat{\theta}_P^Q$ is the Laplace transform of ${}_\varepsilon\tau_P^Q$. This yields the following expression for ${}_\varepsilon\gamma_P^Q$:

LEMMA 13.1.4.

$$ {}_\varepsilon\gamma_P^Q(\lambda, X) = \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^\varepsilon - a_Q^\varepsilon} e^{\lambda({}_\varepsilon X_R^Q)} \hat{\theta}_P^R(\lambda)^{-1} \theta_R^Q(\lambda)^{-1} $$

where ${}_\varepsilon X_R^Q$ *is the projection of* X on $(\mathfrak{a}_R^Q)^\varepsilon$.

The left-hand side is analytic, the right-hand side is meromorphic and hence they are equal everywhere and the singularities of the right-hand side cancel. □

Letting ${}_{\varepsilon}\gamma_P^Q(X) = {}_{\varepsilon}\gamma_P^Q(0, X)$ we have the

LEMMA 13.1.5. The function

$$X \longrightarrow {}_{\varepsilon}\gamma_P^Q(X)$$

is a homogeneous polynomial of degree $k = a_P^{\varepsilon} - a_Q^{\varepsilon}$ given by

$$\frac{1}{k!} \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^{\varepsilon} - a_Q^{\varepsilon}} \lambda({}_{\varepsilon}X_R^Q)^k \; {}_{\varepsilon}\hat{\theta}_P^R(\lambda)^{-1} \; {}_{\varepsilon}\theta_R^Q(\lambda)^{-1}$$

well defined if λ is not a singular value of $\hat{\theta}(\lambda)^{-1}$ or $\theta(\lambda)^{-1}$, and independent of λ.

It is clear that $X \longrightarrow {}_{\varepsilon}\gamma_P^Q(X)$ is analytic and homogeneous of degree $k = a_P^{\varepsilon} - a_Q^{\varepsilon}$ and it is easy to compute the limit

$${}_{\varepsilon}\gamma_P^Q(0, X) = \lim_{\substack{t \to 0 \\ t \in \mathbf{R}}} {}_{\varepsilon}\gamma_P^Q(t\lambda, X)$$

when λ is not a singular value for ${}_{\varepsilon}\theta(\lambda)^{-1}$ or ${}_{\varepsilon}\hat{\theta}(\lambda)^{-1}$. $\square$

13.2. The trace formula as a polynomial.

The left-hand side of the trace formula for the group G and the function ϕ is a sum over $\mathfrak{o} \in \mathcal{O}$ of terms ${}_{\varepsilon}J_{\mathfrak{o}}^{G,T}(\phi)$ which are the integral over $G \backslash \mathbf{G}_{\varepsilon}^{1}$ of ${}_{\varepsilon}J_{\mathfrak{o}}^{G,T}(\phi, x)$ which in turn are the sums over ε-invariant parabolic subgroups $P \subset G$ (standard) of

$$(-1)^{a_P^{\varepsilon} - a_Q^{\varepsilon}} \sum_{\delta \in P \backslash G} {}_{\varepsilon}\hat{\tau}_P(H(\delta x) - T) K_{P,\mathfrak{o}}^{\varepsilon,\phi}(\delta x, \delta x)$$

where

$$K_{P,\sigma}^{\varepsilon,\phi}(x, y) = \sum_{\gamma \in M_P \cap \sigma} \int_{\mathbb{N}_P} \phi(x^{-1}\gamma n \varepsilon(y))\, dn \quad .$$

It was proved in Lecture 4 that the integral over $G\backslash \mathbf{G}'_\varepsilon$ is convergent provided T is suitably regular uniformly if ϕ varies in some compact set of functions.

We want to compute $J^{G,T+X}$ in terms of $J^{Q,T}$ where Q runs over ε-invariant parabolic subgroups. Using 13.1.3 we see that

$${}_\varepsilon J_\sigma^{T+X}(\phi, x) = \sum_{\substack{P \subset Q \\ \varepsilon(P)=P \\ \varepsilon(Q)=Q}} (-1)^{a_P^\varepsilon - a_Q^\varepsilon}$$

$$\sum_{\xi \in Q\backslash G} \sum_{\delta \in P\backslash Q} {}_\varepsilon\Gamma_Q^G(H(\xi x), X)\hat{\tau}_P^Q(H(\delta\xi x)-T)$$

$$K_{P,\sigma}^{\varepsilon,\phi}(\delta\xi x, \delta\xi x) \quad .$$

But if $x = nmk$ with $n \in \mathbb{N}_Q$, $m \in \mathbb{M}_Q$ and $k \in K$ we have (if $P \subset Q$)

$$K_{P,\sigma}^{\varepsilon,\phi}(x, x) = K_{P,\sigma\cap Q}^{\varepsilon,\phi_Q^k}(m, m)$$

where

$$\phi_Q^k(m) = \delta_Q(m)^{\frac{1}{2}} \int_{\mathbb{N}_Q} \phi(k^{-1}mn\varepsilon(k))\, dn \quad .$$

Using the fact that the left-hand side of the trace formula is convergent for (Q, ϕ_Q^k) uniformly for $k \in K$ provided T is suitably regular we get when T and X are suitably regular

$$ {}_\varepsilon J_{\mathfrak{o}}^{G,T+X}(\phi) = \sum_{\varepsilon(Q)=Q} {}_\varepsilon\gamma_Q^G(X)\, {}_\varepsilon J_{\mathfrak{o} \cap Q}^{Q,T}(\phi_Q) $$

where

$$ \phi_Q = \int_K \phi_Q^k dk \quad . $$

The right-hand side is a polynomial in X and this allows one to define ${}_\varepsilon J_{\mathfrak{o}}^{G,T}(\phi)$ for all T as a polynomial in T of degree $a_R^\varepsilon - a_G^\varepsilon$ where R is any ε-invariant parabolic subgroup whose rank is minimal for the property $K_{R,\mathfrak{o}}^{\varepsilon,\phi} \neq 0$.

A cuspidal datum χ is a conjugacy class of pairs (π, M_P) where π is a cuspidal automorphic representation for M_P the Levi subgroup of a standard parabolic subgroup. If one considers the partial spectral decomposition indexed by cuspidal data one is led to introduce partial kernels $K_{P,\chi}(x, y)$ and one can show, using a refinement of the results in Lectures 7 and 8, that provided T is sufficiently regular

$$ {}_\varepsilon J_\chi^{G,T}(\phi, x) = \sum_{\varepsilon(P)=P} (-1)^{a_P^\varepsilon - a_G^\varepsilon} \sum_{\delta \in P\backslash G} {}_\varepsilon\hat{\tau}_P(H(\delta x)-T) K_{P,\chi}^\varepsilon(\delta x, \delta x) $$

is integrable over $G\backslash \mathbf{G}_\varepsilon^1$; we shall denote by ${}_\varepsilon J_\chi^{G,T}$ its integral. As above we get

$$ {}_{\varepsilon}J_{\chi}^{G,T+X}(\phi) = \sum_{\varepsilon(Q)=Q} {}_{\varepsilon}\gamma_Q^G(x)\, {}_{\varepsilon}J_{\chi}^{Q,T}(\phi_Q) $$

provided T and X are suitably regular. The right-hand side is a polynomial in X of degree $a_R^{\varepsilon} - a_G^{\varepsilon}$ where R is any ε-invariant parabolic subgroup whose rank is minimal for the property $K_{R,\chi} \neq 0$.

13.3. <u>Changing the minimal parabolic</u>.

Let $\Omega^{G,\varepsilon}$ be the subgroup of ε-invariant elements in the Weyl group; let $w \in G$ be an element which represents $s \in \Omega^{G,\varepsilon}$. Simple changes of variable yield

$$ {}_{\varepsilon}J^{T}(\phi) = \int_{G\backslash \mathbf{G}'_{\varepsilon}} \sum_{\varepsilon(P)=P} (-1)^{a_P^{\varepsilon}} \sum_{\delta \in w^{-1}(P)\backslash G} $$

$$ {}_{\varepsilon}\hat{\tau}_P(H(w\delta x)-T) K^{\varepsilon}_{w^{-1}(P)}(\delta x, \delta x) $$

where $w^{-1}(P) = w^{-1}Pw$ and where $K_{w^{-1}(P)}$ is defined in an obvious way. It is natural to define ${}_{\varepsilon}\hat{\tau}_{w^{-1}(P)}$ such that

$$ {}_{\varepsilon}\hat{\tau}_P(H) = {}_{\varepsilon}\hat{\tau}_{w^{-1}(P)}(w^{-1}(H)) \quad . $$

If $y = nmak$ is a Langlands-Iwasawa decomposition corresponding to $Q = w^{-1}(P_0)$ we define H_Q such that $H_Q(y) = H(a)$ and hence

$$ w^{-1}H(wy) = H_Q(y) + w^{-1}H(w) $$

and

$$ {}_{\varepsilon}\hat{\tau}_P(H(wy)-T) = {}_{\varepsilon}\hat{\tau}_{w^{-1}(P)}(H_Q(y)-T_Q) $$

where $T_Q = w^{-1}(T-H(w))$. With these notations we get

$$ {}_\varepsilon J^T(\phi) = \int_{G\backslash \mathbf{G}'_\varepsilon} \sum_{\substack{\varepsilon(R)=R \\ R\supset Q}} \sum_{\delta\in R\backslash G} {}_\varepsilon\hat{\tau}_R(H_Q(\delta x)-T_Q)K_R^\varepsilon(\delta x,\ \delta x) $$

which can be written

$$ {}_\varepsilon J^T(\phi) = {}_\varepsilon J_Q^{T_Q}(\phi) $$

where ${}_\varepsilon J_Q$ is the trace formula computed using the minimal ε-invariant parabolic subgroup Q in place of P_0.

13.4. Action of conjugacy.

We now want to compare $J^T(\phi)$ with $J^T(\phi^y)$ where

$$ \phi^y(x) = \phi(yx\varepsilon(y)^{-1}) \quad . $$

We have

$$ J^T(\phi^y) = \int_{G\backslash \mathbf{G}'_\varepsilon} \sum_{\substack{\varepsilon(P)=P \\ P\supset P_0}} \sum_{\delta\in P\backslash G} {}_\varepsilon\hat{\tau}_P(H(\delta xy)-T)K_P^\varepsilon(\delta x,\ \delta x) $$

but

$$ H(\delta xy) = H(\delta x) + H(k(\delta x)y) $$

where $k(\delta x)$ is the K-component of an Iwasawa decomposition of (δx). Using 13.1.3 we are led to introduce

$$ {}_{\varepsilon}u_P^Q(x, y) = \int_{{}^{\varepsilon}_{Q} \setminus \mathfrak{a}^{\varepsilon}_{P}} {}_{\varepsilon}\Gamma_P^Q(H, -H(k(x)y))\, dH $$

and

$$ \phi_{Q,y}(m) = \delta_Q(m)^{\frac{1}{2}} \int_K \int_{\mathbf{N}_Q} \phi(k^{-1} m n \varepsilon(k))\, {}_{\varepsilon}u_Q^G(k, y)\, dk\, dn $$

with these notations we obtain as in 13.2

$$ {}_{\varepsilon}J^{G,T}(\phi^y) = \sum_{\varepsilon(Q)=Q} {}_{\varepsilon}J^{Q,T}(\phi_{Q,y}) \quad . $$

13.5. <u>On some regularity property</u>.

In 13.1 we introduced

$$ {}_{\varepsilon}\gamma_P^Q(\lambda, X) = \int_{(\mathfrak{a}_P^Q)^{\varepsilon}} {}_{\varepsilon}\Gamma_P^Q(H, X) e^{\lambda(H)}\, dH \quad . $$

We shall now study this function when λ is imaginary. Consider D a differential operator with constant coefficients on $i(\mathfrak{a}_P^Q)^{\varepsilon *}$ then if $\lambda \in i(\mathfrak{a}_P^Q)^{\varepsilon *}$ we have

$$ |D\, {}_{\varepsilon}\gamma_P^Q(\lambda, X)| \le \int_{(\mathfrak{a}_P^Q)^{\varepsilon}} |P_D(H)\, {}_{\varepsilon}\Gamma_P^Q(H, X)|\, dH $$

where P_D is the polynomial associated to D. Using that

$$ \Gamma(tH, tX) = \Gamma(H, X) $$

for $t \in \mathbf{R}_+^\times$ and Lemma 13.1.1(i) it is not difficult to see that

LEMMA 13.5.1.

$$|D_\varepsilon \gamma_P^Q(\lambda, X)| < c(1 + \|X\|)^N$$

for some N independent of λ when λ is imaginary. □

In other words, $X \longrightarrow \gamma(\lambda, X)$ is a "slowly increasing" function.

Now consider φ a Schwartz-Bruhat function on $i(\mathfrak{a}_P^Q)^{*\varepsilon}$, let $\hat{\varphi}$ be its Fourier transform so that

$$\varphi(\lambda) = \int_{(\mathfrak{a}_P^Q)^\varepsilon} \hat{\varphi}(H) e^{\lambda(H)} dH \quad .$$

We define

$$_\varepsilon\gamma_P^Q(\lambda, \varphi) = \int_{(\mathfrak{a}_P^Q)^\varepsilon} \hat{\varphi}(X)\, {}_\varepsilon\gamma_P^Q(\lambda, X) dX \quad .$$

This makes sense also when $\hat{\varphi}$ is a "rapidly decreasing" distribution. Lemma 13.5.1 above shows that on $i(\mathfrak{a}_P^Q)^{\varepsilon *}$ the function

$$\lambda \longrightarrow {}_\varepsilon\gamma_P^Q(\lambda, \varphi)$$

is smooth and by 13.1.4 we obtain the following expression

$$_\varepsilon\gamma_P^Q(\lambda, \varphi) = \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^\varepsilon - a_Q^\varepsilon} \varphi({}_\varepsilon\lambda_R^Q)$$

$$_\varepsilon\hat{\theta}_P^R(\lambda)^{-1}\, {}_\varepsilon\theta_R^Q(\lambda)^{-1}$$

which is valid at least when λ is imaginary and not a singular value of ${}_{\varepsilon}\hat{\theta}(\lambda)^{-1}$ or ${}_{\varepsilon}\theta(\lambda)^{-1}$ and where ${}_{\varepsilon}\lambda_R^Q$ is the projection of λ on $(\mathfrak{a}_R^Q)^{\varepsilon *} \otimes \mathbf{C}$.

The left-hand side is smooth and hence the singularities of the right-hand side cancel when φ is any Schwartz-Bruhat function. This implies that more generally we have the

LEMMA 13.5.2. <u>Given any smooth function</u> φ

$$\sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^\varepsilon - a_Q^\varepsilon} \varphi({}_{\varepsilon}\lambda_R^Q)\, {}_{\varepsilon}\hat{\theta}_P^R(\lambda)^{-1}\, {}_{\varepsilon}\theta_R^Q(\lambda)^{-1}$$

<u>extends to a smooth function of</u> $\lambda \in i(\mathfrak{a}_P^Q)^{\varepsilon *}$. □