Lecture 2

THE BASIC IDENTITY PROVED

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Recall that the identity asserts the equality of

$$\sum_{P_0 \subset P} (1)^{\dim \mathfrak{a}_P^\varepsilon / \mathfrak{a}_G^\varepsilon} \sum_{\delta \in P \backslash G} K_P(\delta g, \delta g) \hat{\tau}_P(H(\delta g) - T)$$

and

$$\sum_{P_0 \subset P_1 \subset P_2} \sum_{P_1 \backslash G} \sigma_1^2(H(\delta g)-T) \left(\sum_{P_1 \subset P \subset P_2} (-1)^{\dim \mathfrak{a}_P^\varepsilon / \mathfrak{a}_G^\varepsilon} \Lambda^{T,P_1} K_P(\delta g, \delta g) \right) .$$

Once it is shown that the sums occurring on both sides are finite, the proof will be a purely combinatorial matter.

Recall that we can define a height function on $\mathbf{A}^n$ by setting

$$\|x\|_v = \begin{cases} \sqrt{\sum_i |x_i|_v^2} & v \text{ archimedean} \\ \max |x_i| & v \text{ non-archimedean} \end{cases}$$

$$\|x\| = \prod_v \|x\|_v \quad .$$

Then we can choose a height function on $V(\mathbf{A})$ for any vector space over $\mathbf{Q}$ simply by choosing a basis for V over $\mathbf{Q}$ and then identifying $V(\mathbf{A})$ with $\mathbf{A}^n$. It will be useful to recall briefly the properties of these height functions and other functions derived from them.

(a) If $\|\cdot\|_1$ and $\|\cdot\|_2$ are the height functions associated to different bases of $V(\mathbf{Q})$ then there is a positive constant c such that

$$\frac{1}{c}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1$$

for all x.

(b) If a is an idèle then $\|ax\| = |a|\,\|x\|$.

(c) $\|\cdot\|$ is bounded on compact sets.

(d) There is a constant c such that for all $x_1, \ldots, x_n$

$$\|x_1 \otimes \ldots \otimes x_n\| \leq c\|x_1\| \ldots \|x_n\| \quad .$$

(e) If $\varphi : V \longrightarrow W$ is linear over $\mathbf{Q}$ then there is a positive constant c such that

$$\|\varphi(x)\| \leq c\|x\| \quad .$$

(f) If $v \in V(\mathbf{Q})$ then $\|v\| \geq 1$, provided $v \neq 0$.

A basis of V defines in a natural way a basis of the space $M(V)$ of linear transformations of V which we use to introduce a height function on $M(V, \mathbf{A})$ and on $GL(V, \mathbf{A})$. We have

(g) $$\|xy\| \leq \|x\|\|y\| \qquad x, y \in M(V, \mathbf{A})$$

(h) $$\|xv\| \leq \|x\|, \|v\| \qquad x \in M(V, \mathbf{A}),\ v \in V(\mathbf{A}) \quad .$$

We introduce a set $\rho_1, \ldots, \rho_r$ of rational representations of G over $\mathbf{Q}$ with the following two properties:

(i) Every representation of G over $\mathbf{Q}$ can be obtained from $\rho_1, \ldots, \rho_r$ by the formation of tensor products and direct summands.

(ii) For each $\alpha \in \Delta_0$ there is an $i = i(\alpha)$, a vector v_α in V_i, the space of ρ_i, and a positive integer d_α such that

$$\rho(p)v_\alpha = \xi_{d_\alpha \varpi_\alpha}(p)v_\alpha$$

for all $p \in P$.

We let ρ be the direct sum of the ρ_i. It acts on $V = \oplus V_i$. We set

$$|g| = |g|_\rho = \|\rho(g)\| \quad .$$

This height function on $G(\mathbf{A})$ has several obvious properties.

(i) $$|hg| \leq |h||g|$$

(j) If $\varphi : G \longrightarrow H$ then

$$|\varphi(g)| \leq c|g|^N$$

where c and N depend upon φ alone.

(k) If $G = GL(n)$ then

$$\|g\| \leq c|g|^N$$

where c and N are independent of g.

(ℓ) There are constants c and N such that the number of elements in

$$\{g \mid |g| \leq M\}$$

is at most cM^N.

It is enough to prove this for $GL(n) \subseteq M(n)$, the space of $n \times n$ matrices. We have a morphism from $GL(n)$ to $\mathbf{P}^1 \times \mathbf{P}M(n)$ given by

$$g \longrightarrow (\deg g, \det^{-1} g) \times g$$

and the assertion is a consequence of standard properties of heights in projective spaces, the inverse image of a point in $\mathbf{P}^1 \times \mathbf{P}M(n)$ consisting of at most two in $GL(n)$.

(m) If A is a split torus we have a homomorphism $H : A \longrightarrow X_*(A) \otimes \mathbf{R}$ such that

$$e^{\lambda(H(a))} = |\xi_\lambda(a)|$$

for all $\lambda \in X^*(A)$. There are constants c and N such that

$$\{a \mid \|H(a)\| \leq M\} \subseteq A(\mathbf{Q})\{a \mid |a| \leq cM^N\} \quad .$$

It is enough to prove this for $GL(1)$ where it is clear.

To prove that the sum occurring on the left of the basic identity if finite we need only establish the following lemma.

LEMMA 2.1. <u>There are constants</u> c <u>and</u> N <u>such that the number of</u> δ <u>in</u> $P\backslash G$ <u>for which</u>

$$\varpi_\alpha(H(\delta g) - T) > 0$$

<u>for all</u> $\alpha \in \hat{\Delta}_P$ <u>is at most</u> $c(|g|e^{\|T\|})^N$.

To prove the lemma we need only show that we can find a set of representatives for these δ each of which satisfies $|\delta| \le c'(|g|e^{\|T\|})^{N'}$. According to reduction theory we can find a compact set C and a $T_0 \in \mathfrak{a}_0$ and representatives δ for which $\delta g = am$, $a \in A_0(\mathbf{A})$,

Same mistake as on p. 6.15

$$\alpha(H(a)) - \alpha(T_0) > 0 \quad , \tag{1}$$

$\alpha \in \Delta_0^P$ and $m \in C$. Since

$$|\delta| \le |\delta g|\,|g|^{-1} = |am|\,|g|^{-1} \le c\,|a|\,|g|^{-1} \quad ,$$

all we need show, according to (m), is that

$$\|H(a)\| \le c(1 + \ln|g| + \|T\|) \quad . \tag{2}$$

Observe that

$$1 \le \|\rho(\delta^{-1})v_\alpha\| = \|\rho(g)\rho(\delta g)^{-1}v_\alpha\| \le |g|\,\|\rho(\delta g)^{-1}v_\alpha\| \le c\,|g|\,e^{-d_\alpha\varpi_\alpha(H(a))} \quad .$$

Consequently

$$\varpi_\alpha(H(a)) \le c(1 + |g|) \quad , \tag{3}$$

the constant c varying from time to time.

Moreover by assumption

$$(4) \qquad \varpi_\alpha(H(a)) > \varpi_\alpha(T), \qquad \alpha \in \hat{\Delta}_P .$$

Also

$$(5) \qquad |\lambda(a)| = |\lambda(g)|, \quad \lambda \in X_G^*$$

for $\alpha \in \hat{\Delta}_P$. The inequality (2) is an immediate consequence of (1), (3), (4), and (5) and standard geometric properties of root systems which we now recall. They are actually valid for each of the systems $\Delta_{P_1}^{P_2}$.

The first is

$$(a) \qquad (\alpha, \beta) \leq 0 \quad \text{if} \quad \alpha \neq \beta, \quad \text{and} \quad \alpha, \beta \in \Delta_{P_1}^{P_2} .$$

It implies

(b) $(\varpi_\alpha, \varpi_\beta) \geq 0 \ \forall \alpha, \beta,$ where $(\varpi_\alpha, \beta) = \delta_{\alpha\beta}$. We denote the set $\{\varpi_\alpha\}$ by $\hat{\Delta}_{P_1}^{P_2}$.

As a consequence

$$(c) \quad \mathfrak{a}_1^{2+} = \{H \in \mathfrak{a}_1^2 \mid \alpha(H) > 0 \ \forall \alpha \in \Delta_{P_1}^{P_2}\} \subseteq {}^+\mathfrak{a}_1^2 = \{H \in \mathfrak{a}_1^2 \mid \varpi_\alpha(H) > 0 \ \forall \varpi_\alpha \in \hat{\Delta}_{P_1}^{P_2}\} .$$

Moreover if $P_1 \subset P \subset P_2$ and $\alpha \in \Delta_{P_1}^P$ then

$$\varpi_\alpha = \varpi'_\alpha + \sum_{\Delta_P^2} c_\beta \omega_\beta \qquad (\Delta_P^2 = \Delta_P^{P_2})$$

with $\varpi'_\alpha \in \hat{\Delta}_{P_1}^P$, $\varpi_\alpha \in \hat{\Delta}_{P_1}^{P_2}$. Applying (c) to $\hat{\Delta}_{P_1}^P$ we see that

$$\varpi'_\alpha = \sum_{\gamma \in \Delta^P_{P_1}} d_\gamma \gamma$$

with $d_\gamma \geq 0$. Thus

$$(\varpi'_\alpha, \beta) \leq 0$$

for $\beta \in \Delta_{P_1} - \Delta^P_{P_1}$ and

$$0 = (\varpi'_\alpha, \beta) + c_\beta \quad ,$$

so that $c_\beta \geq 0$.

We conclude

(d) $\{H \,|\, \alpha(H) > 0,\ \alpha \in \Delta^P_{P_1},\ \varpi_\alpha(H) > 0,\ \alpha \in \hat{\Delta}^{P_2}_P\} \subseteq \{H \,|\, \omega_\alpha(H) > 0 \,\forall \alpha \in \hat{\Delta}^{P_2}_{P_1}\}$.

This is all that is necessary to complete the proof of the lemma.

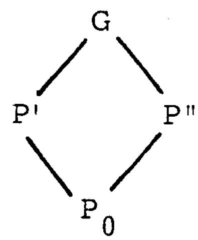
Returning to the basic identity we show that the sums on the right are finite. This will be an immediate consequence of the following lemma.

LEMMA 2.2. <u>For a given</u> P_1,

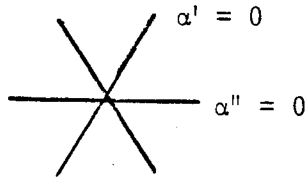
$$\sum_{P_1 \subset P_2} \sigma^2_1 = \hat{\tau}_1 \qquad\qquad (\hat{\tau}_1 = \hat{\tau}_{P_1}) \quad .$$

This lemma is clear because if $\varpi_\alpha(H) > 0 \,\forall \alpha \in \Delta_1$ then there is a unique P_2 such that $\alpha(H) > 0 \,\forall \alpha \in \Delta^{P_2}_{P_1}$ and $\alpha(H) \leq 0 \,\forall \alpha \in \Delta_{P_1} - \Delta^{P_2}_{P_1}$.

For the group $SL(3)$ we can easily describe these sets geometrically. The lattice of standard parabolic subgroups is



and the diagram of chambers is



$\{\alpha'\} = \Delta'_0 \quad \{\alpha''\} = \Delta''_0$

Then

σ_0 :

σ'_0 :

σ''_0 :

$\hat{\tau}_0$:

Moreover

$\hat{\tau}_{P'} = \sigma^G_{P'}$: 

$\hat{\tau}_{P''} = \sigma^G_{P''}$: 

We return to the identity and on the right side consider a fixed pair $P_1 \subset P$ and sum over all $P_2 \supset P$. This means we have to consider

$$\sum_{P_2 \supset P} \sigma_1^2(H) \quad .$$

LEMMA 2.3. <u>For fixed</u> $P_1 \subset P$,

$$\sum_{P_2 \supset P} \sigma_1^2 = \tau_{P_1}^P \hat{\tau}_P \quad .$$

The sum on the left is a characteristic function, namely of

$$\{H \,|\, \varpi_\alpha(H) > 0,\ \alpha \in \Delta_{P_1},\ \alpha(H) > 0,\ \alpha \in \Delta_{P_1}^P\} \ ,$$

Modify $\varepsilon\varpi_\alpha(H) > 0$ $\alpha \in \Delta_0^{G} - \Delta_0^{P_1}$

while the function on the right is the characteristic function of

$$\{H \,|\, \varpi_\alpha(H) > 0,\ \alpha \in \Delta_P,\ \alpha(H) > 0,\ \alpha \in \Delta_{P_1}^P\} \quad .$$

Modify $\varepsilon\varpi_\alpha(H) > 0$ $\alpha \in \Delta_P$

The first conditions clearly imply the second. So we need only show that the second imply the first. This is however (d).

Second ⇒ (taking $H \in \mathfrak{a}_1$) $\alpha(H) \geq 0$ $\forall \alpha \in \Delta_0^{P_1}$. Now project on $\mathfrak{a}_0^\varepsilon$ and apply (d) for ε-invariant roots. $\underset{\varepsilon}{\sum} {}^{\varepsilon}H = H_0$. Obtain $\varpi_\alpha(H_0) \geq 0$ $\forall \alpha \in \Delta_0$ ⇒ $\varepsilon\varpi_\alpha(H) \geq 0$ $\forall \alpha$. ⇒ (by openness) $\varepsilon\varpi_\alpha(H) > 0$

This leaves us with a sum over P on the left and a sum over P_1 and P on the right. All we need do is show that for a fixed P the contribution from P on the left is equal to the sum over P_1 with the same fixed P on the right. Dropping factors which are obviously equal we see that we are reduced to showing that

$$\sum_{\delta \in P\backslash G} K_P(\delta g, \delta g)\hat{\tau}_P(H(\delta g) - T)$$

$$= \sum_{P_1} \sum_{P_1\backslash G} \tau_{P_1}^{P}(H(\delta g) - T)\hat{\tau}_P(H(\delta g) - T)\Lambda^{T,P_1} K_P(\delta g, \delta g) \quad .$$

The inner sum on the right may be written as a double sum, first over $P_1\backslash P$ and then over $P\backslash G$. Since

$$\hat{\tau}_P(H(\delta_1 \delta g) - T) = \hat{\tau}_P(H(\delta g) - T), \quad \delta_1 \in P$$

and

$$\Lambda^{T,P_1} K(\delta_1 \delta g, \delta_1 \delta g) = \Lambda^{T,P_1} K(\delta_1 \delta g, \delta g)$$

we need only show that

$$(1) \quad \sum_{P_1} \sum_{P_1\backslash P} \tau_{P_1}^{P}(H(\delta_1 g) - T)\,\Lambda^{T,P_1} K_P(\delta_1 g, h) = K_P(g, h) \quad .$$

LEMMA 2.4. Suppose P is a standard parabolic subgroup and ϕ a continuous function on $P\backslash \mathbf{G}$. Then

$$\sum_{P_1 \subset P} \sum_{P_1\backslash P} \Lambda^{T,P_1} \phi(\delta, g)\tau_1^{P}(H(\delta, g) - T) = \int_{N\backslash \mathbb{N}} \phi(ng)dn \quad .$$

Recalling the definition of Λ^{T,P_1} we see that the left side is

$$\sum_{R\subset P_1\subset P}\sum_{\delta\in P_1\backslash P}\sum_{\gamma\in R\backslash P_1}(-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1}\int_{N_R\backslash \mathbf{N}_R}\phi(n\gamma,\delta,g)\,dn\quad \hat{\tau}_R^1(H(\gamma\delta,g)-T)\ \tau_1^P(H(\delta,g)-T)\ .$$

In the double sum over R and P_1 we fix R and sum over P_1. Thus we have

$$\sum_R\sum_{R\backslash P}\left\{\sum_{R\subset P_1\subset P}(-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1}\hat{\tau}_R^1(H)\tau_1^P(H)\right\}\int_{N_R\backslash \mathbf{N}_R}\phi(n\gamma g)\,dn$$

with

$$H = H(\gamma g) - T\ .$$

Observe that if $\gamma\in P_1$ then

$$\tau_1^P(H(\gamma\delta_1 g) - T) = \tau_1^P(H(\delta_1 g) - T)\ .$$

If $R = P$ the sum over P_1 in the parentheses is clearly 1. We need to show that it is 0 otherwise. Once this is done the left side of (1) will have been shown to equal

$$\int_{N\backslash \mathbf{N}} K_p(ng,\ h)\ .$$

Since $K_p(ng,\ h) = K_p(g,\ h)$ and

$$\int_{N\backslash \mathbf{N}} dn = 1\ ,$$

the basic identity is proved.

We prove now a more general combinatorial statement, of which the desired identity is a special case. We fix R and P, $R \subset P$, and a Λ in $\mathfrak{a}_R^P$. Let $\varepsilon_R^{P_1}(\Lambda)$ be $+1$ or -1 according as the number of roots $\alpha \in \Delta_R^{P_1}$ such that $(\alpha, \Delta) \leq 0$ is even or odd. Let $\phi_R^{P_1}(\Lambda, H)$ be the characteristic function of those H in $\mathfrak{a}_R^P$ such that $\varpi_\alpha(H) > 0$ if $(\alpha, \Lambda) \leq 0$ and $\varpi_\alpha(H) \leq 0$ if $(\alpha, \Lambda) > 0$.

LEMMA 2.5.

$$\sum_{R \subset P_1 \subset P} \varepsilon_R^{P_1}(\Lambda)\phi_R^{P_1}(\Lambda, H)\tau_{P_1}^P(H)$$

is 0 if $(\Lambda, \alpha) \leq 0$ for some $\alpha \in \Delta_R^P$ and is 1 otherwise.

The identity we need is the special case that $(\Lambda, \alpha) \leq 0$ for all $\alpha \in \Delta_R^P$. We observe first of all that an identity very similar to the one we need is easy to prove, namely that if $R \neq P$ then

$$(1) \qquad \sum_{R \subset P_1 \subset P} (-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1} \tau_R^1(H)\hat{\tau}_1^P(H) = 0 \quad .$$

For a given H all terms are 0 unless $\varpi_\alpha(H) > 0$ for all $\alpha \in \Delta_R^P$. If $\varpi_\alpha(H) > 0$ for all $\alpha \in \Delta_R^P$ then

$$\Delta = \{\alpha \in \Delta_R^P \mid \alpha(H) > 0\}$$

is not empty. For this H the sum on the left is

$$\sum_{\Delta_R^{P_1} \subseteq \Delta} (-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1} = \sum_{\Delta_R^{P_2} \subseteq \Delta} (-1)^{|\Delta_R^{P_1}|} = 0 \quad .$$

Returning to the lemma we replace in (1) R by P_1 and the sum over P_1 by a sum over P_2. This enables us to conclude that for $P_1 \neq P$

$$(-1)^{\dim \mathfrak{a}_{P_1}/\mathfrak{a}_P} \tau_{P_1}^P = - \sum_{P_1 \subset P_2 \subsetneq P} (-1)^{\dim \mathfrak{a}_{P_1}/\mathfrak{a}_{P_2}} \tau_{P_1}^{P_2} \hat{\tau}_{P_2}^P \quad .$$

We substitute in the sum of the lemma, obtaining the difference between

$$\varepsilon_R^P(\Lambda)\Phi_R^P(\Lambda, H) \tag{2}$$

and

$$\sum_{R \subset P_1 \subset P_2 \subsetneq P} (-1)^{\dim \mathfrak{a}_{P_2}/\mathfrak{a}_P} \varepsilon_R^{P_1}(\Lambda)\phi_R^{P_1}(\Lambda, H)\tau_{P_1}^{P_2}(H)\hat{\tau}_{P_2}^P(H) \quad .$$

We can apply induction. The sum over P_1 is 0 unless $(\alpha, \Lambda) > 0$ for all roots α in $\Delta_R^{P_2}$, thus unless $P_2 \subset P_\Lambda$ where P_Λ is defined by

$$\Delta_R^{P_\Lambda} = \{\alpha \in \Delta_{P_1}^P \,|\, (\alpha, \Lambda) > 0\} \quad .$$

If $P_2 \subset P_\Lambda$ the sum over P_1 is equal to

$$(-1)^{\dim \mathfrak{a}_{P_2}/\mathfrak{a}_P} \hat{\tau}_{P_2}^P(H) \quad .$$

Thus we obtain

$$\sum_{R \subset P_2 \subset P_\Lambda} (-1)^{\dim \mathfrak{a}_{P_2}/\mathfrak{a}_P} \hat{\tau}_{P_2}^P(H) \tag{3}$$

unless $P_\Lambda = P$ when we obtain this expression minus 1. To prove the lemma we need only show that (2) equals (3), for we are trying to show that the difference is 0 unless $P_\Lambda = P$ when it is 1.

It is however clear that (3) is equal to zero unless

$$\varpi_\alpha(H) > 0$$

for $\alpha \in \Delta_R^P - \Delta_R^{P_\Lambda}$, thus for $(\alpha, \Lambda) \leq 0$, but that if this condition is satisfied it is equal to

$$\sum_{Q \subset P_2 \subset P_\Lambda} (-1)^{\dim \mathfrak{a}_{P_2}/\mathfrak{a}_P}$$

where

$$\Delta_R^Q = \{\alpha \in \Delta_R | \varpi_\alpha(H) \leq 0\} \quad .$$

The sum is clearly 0 unless $Q = P_\Lambda$ when it is

$$(-1)^{\dim \mathfrak{a}_{P_\Lambda}/\mathfrak{a}_P} \quad ,$$

which is $\varepsilon_Q^P(\Lambda)$. The lemma follows.