Lecture 3

THE COARSE $\mathcal{O}$-EXPANSION

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3.1. Statement of the main result.

Let G be a connected reductive group and G' an extension of G, over $\mathbf{Q}$, by a finite group E generated by ε_0

$$G \longrightarrow G' \longrightarrow E \quad .$$

Notice that we do not assume that the extension is split. Choose a minimal parabolic P_0 and a Levi component M_0; there is an $\varepsilon \in G'$ projecting on ε_0 such that $\varepsilon(P_0) = \varepsilon P_0 \varepsilon^{-1} = P_0$ and $\varepsilon(M_0) = M_0$. Let A_0 be the split component of the center of M_0; the action of ε on A_0 is of finite order. All parabolics considered below will be assumed to be standard.

Any $g \in G'$ has a Jordan decomposition $g = g_u g_s$ with g_s semisimple in G' and g_u unipotent in G.

We shall use the following equivalence relation in G, which could be called ε-semisimple-conjugacy.

Two elements γ_1 and γ_2 in G will be called equivalent if $\gamma_1' = \gamma_1\varepsilon$ and $\gamma_2' = \gamma_2\varepsilon$ have conjugate semisimple parts. In particular if γ_1' and γ_2' are semisimple this means that γ_1 and γ_2 are ε-conjugate, i.e., there exist $\delta \in G$ such that $\gamma_1 = \delta^{-1}\gamma\varepsilon(\delta)$.

LEMMA 3.1.1. *Given* P *an* ε-*invariant parabolic and* $\gamma \in P$, *denote by* N^{o} *the centralizer of the semisimple part of* $\gamma' = \gamma\varepsilon$ *in* N *the unipotent radical of* P. *Let* ϕ *be a function with finite support on* P, *then*

$$\sum_{n \in N} \phi(n\gamma) = \sum_{\delta \in N^{o} \backslash N} \; \sum_{\eta \in N^{o}} \phi(\delta^{-1} \eta\gamma\varepsilon(\delta)) \quad .$$

Notice that N^{o} is normalized by γ'. Let us denote by θ the automorphism of N defined by the conjugation by $\gamma\varepsilon$. We shall prove a slightly generalized version of the above lemma. Consider a nilpotent group N_1 and an automorphism θ (over $\mathbf{Q}$), let N_2 be a subgroup θ-invariant such that N_2 contains the subgroup of θ_s-fixed points in N_1 (where θ_s is the semisimple part of θ), then given ϕ on N_1 with finite support one has

$$\sum_{n \in N_1} \phi(n) = \sum_{\delta \in N_2 \backslash N_1} \; \sum_{\eta \in N_2} \phi(\delta^{-1} \eta\theta(\delta)) \quad .$$

Better – a reference to the relevant fact on algebraic groups. In any case to be revised

We can now proceed by "dévissage" and it is enough to prove this when N_2 is invariant in N_1 and $N_2 \backslash N_1$ abelian; in such a case θ induces in the Lie algebra of $N_2 \backslash N_1$ a linear map θ' which is such that $\theta' - 1$ is invertible, and the lemma follows. □

The preceding lemma shows that if P is an ε-invariant parabolic and σ an ε-semisimple-conjugacy class then

$$P \cap \sigma = N.(P \cap \sigma) \quad .$$

We can now define

$$K_{P,\mathfrak{o}}(x, y) = \int_{\textcircled{N}} \sum_{\gamma \in P \cap \mathfrak{o}} \omega(y)\phi(x^{-1}n^{-1}\gamma\varepsilon(y))dn$$

where $\textcircled{N} = N(\mathbf{Q}) \backslash N(\mathbf{A})$. Obviously one has

$$K_p = \sum_{\mathfrak{o} \in \mathcal{O}} K_{P,\mathfrak{o}}$$

where $\mathcal{O}$ is the set of ε-semisimple-conjugacy classes. Now introduce

$$k_{\mathfrak{o}}^{T}(x) = \sum_{\varepsilon(P)=P} \sum_{\delta \in P \backslash G} (-1)^{a_P^{\varepsilon}} \hat{\tau}_P(H(\delta x) - T) K_{P,\mathfrak{o}}(\delta x, \delta x) \quad .$$

(Undefined notations are taken over from Lectures 1 and 2.)

The aim of this lecture is to prove the

THEOREM 3.1.2. Provided T is sufficiently regular, the sum

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{\textcircled{G}^1} |k_{\mathfrak{o}}^{T}(x)| dx$$

is finite. (Here $\textcircled{G}^1$ stands for $G(\mathbf{Q}) \backslash G(\mathbf{A})^1$.)

3.2. Some partitions of $G(\mathbf{A})$.

Let P be a parabolic and T_0 a vector in $\mathfrak{a}_0^-$, define $\mathfrak{G}_P(T_0)$ to be the set of $x \in G(\mathbf{A})$ such that

$$\alpha(H(x) - T_0) > 0 \qquad \forall \alpha \in \Delta_0^P$$

Not such a good defin.
See Lecture 6

According to reduction theory we know that if $-T_0$ is sufficiently regular then

$$P.\mathfrak{S}_P(T_0) = G(\mathbb{A}) \quad .$$

We shall assume that T_0 is fixed so that the above property holds for all P. Let $P_1 \subset P$ and consider $T \in \mathfrak{a}_0^+$; define $\mathfrak{S}_P^1(T_0, T)$ to be the set of $x \in \mathfrak{S}_P(T_0)$ such that

$$\varpi(H(x) - T) \leq 0 \qquad \forall \varpi \in \hat{\Delta}_0^1 \quad .$$

We shall denote $F_P^1(\cdot, T)$ the characteristic function of the set $P_1\mathfrak{S}_P^1(T_0, T)$.

PROPOSITION 3.2.1. Assume T is sufficiently regular, then given P we have

$$\sum_{\{P_1 | P_1 \subset P\}} \sum_{\delta \in P_1 \backslash P} F_P^1(\delta x, T)\tau_1^P(H(\delta x) - T) = 1$$

for all $x \in G(\mathbb{A})$.

The proof relies on the following particular case of the combinatorial Lemma 2.5 (of Lecture 2). Assume $\Lambda \in \mathfrak{a}_0^+$ then

$$\sum_{\{P_1 | P_1 \subset P\}} \phi_0^1(\Lambda, H)\tau_1^P(H) = 1$$

for all $H \in \mathfrak{a}_0$. Recall that for $\Lambda \in \mathfrak{a}_0^+$ the function $H \longmapsto \phi_0^1(\Lambda, H)$

is the characteristic function of the set of H such that $\varpi(H) \leq 0$ for all $\varpi \in \hat{\Delta}_0^1$.

Now fix $x \in G(\mathbb{A})$; thanks to reduction theory we know that there exist at least one $\delta \in P$ such that $\delta x \in \mathfrak{S}_P(T_0)$; the combinatorial lemma applied with $H = H(\delta x) - T$ provides us with exactly one parabolic $P_1 \subset P$ such that

$$F_P^1(\delta x, T)\tau_1^P(H(\delta x) - T) = 1 \quad .$$

Hence, the sum in the proposition is at least 1. To prove that it is exactly 1 consider $x \in \mathfrak{S}_P(T_0)$ and $\delta \in P$ such that $\delta x \in \mathfrak{S}_P(T_0)$. The combinatorial lemma provides us with two parabolics P_1 and P_2 such that

$$F_P^1(x, T)\tau_1^P(H(x) - T) = F_P^2(\delta x, T)\tau_2^P(H(\delta x) - T) = 1 \quad .$$

We need to show that this implies $\delta \in P_1$ (and hence $P_1 = P_2$). We need two lemmas.

LEMMA 3.2.2. <u>Given</u> $P_1 \subset P$ <u>and</u> $H \in \mathfrak{a}_0^P$ <u>such that</u>

(i) $\alpha(H) > 0 \qquad \forall \alpha \in \Delta_1^P$

(ii) $\varpi(H) \leq 0 \qquad \forall \varpi \in \hat{\Delta}_0^1$

<u>then the following holds</u>

(iii) $\alpha(H) > 0 \qquad \forall \alpha \in \Delta_0^P - \Delta_0^1$.

In fact one can write

$$H = \sum_{\varpi \in \hat{\Delta}_1^P} c_\varpi \check{\varpi} - \sum_{\alpha \in \Delta_0^1} c_\alpha \check{\alpha} .$$

The hypotheses (i) and (ii) imply that $c_\varpi > 0$ and $c_\alpha \geq 0$. Now consider $\beta \in \Delta_0^P - \Delta_0^1$, of course $\beta(\check{\varpi}) \geq 0$ but since $\beta \notin \Delta_0^1$ at least one of the $\varpi \in \hat{\Delta}_1^P$ is not orthogonal to β; moreover $\beta(\check{\alpha}) \leq 0$ for all $\alpha \in \Delta_0^1$. and hence $\beta(H) > 0$. □

LEMMA 3.2.3. _Assume that_ x and δx _are in_ $\mathfrak{S}_P(T_0)$ _with_ $\delta \in P$ _and that_

$$\alpha(H(x) - T) > 0 \qquad \forall \alpha \in \Delta_0^P - \Delta_0^1$$

then provided T _is sufficiently regular one has_ $\delta \in P_1$.

This is a standard result in reduction theory, but we should maybe recall the proof. We may assume $\delta \in M$ the Levi component of P containing M_0, and consider the Bruhat decomposition of δ in M:

$$\delta = \nu \, w_s \, \pi$$

with $\gamma \in N_0^P = N_0 \cap M$, $\pi \in P_0 \cap M$ and w_s represents $s \in \Omega^M$ the Weyl group of M. Write $x = nak$ with $n \in \mathbf{N}_0$, $a \in \mathbf{M}_0$ and $k \in K$, then

$$H(\delta x) = (s . H(a)) . H(w_s n_1)$$

for some $n_1 \in \mathbf{N}_0$. But since $\delta x \in \mathfrak{S}_P(T_0)$ we know that

$\beta(H(\delta x)) > \beta(T_0)$ for any $\beta \in \Delta_0^P$, ~~or~~ more generally for any positive root of M. The factor $\beta(H(w_s n_1))$ is negative. Now if $s \notin \Omega^1$ the Weyl group of M^1 there is an $\alpha \in \Delta_0^P - \Delta_0^1$ such that $-\beta = s^{-1}\alpha$ is a negative root of M and then $\beta(sH(a))$ cannot be bounded from below independently of T. $\square$

The proposition follows from these two lemmas, the first one being applied to $H = H(\xi x) - T$ for some $\xi \in P_1$. $\square$

Thanks to the above proposition we see that $k_{\sigma}^T(x)$ is the sum over all pairs of parabolics $P_1 \subset P$ with $\varepsilon(P) = P$ of

$$\sum_{\delta \in P_1 \backslash G} (-1)^{a_P^\varepsilon} F_P^1(\delta x, T)\tau_1^P(H(\delta x) - T)\,\hat{\tau}_P(H(\delta x) - T)K_{P,\sigma}(\delta x, \delta x) \quad .$$

Recall that

$$\sum_{\{P_2 | P_1 \subset P \subset P_2\}} \sigma_1^2(H) = \tau_1^P(H)\hat{\tau}_P(H)$$

and define $H_1^2(x)_\sigma^T$ to be the sum over all ε-invariant parabolics P such that $P_1 \subset P \subset P_2$ of

$$(-1)^{a_P^\varepsilon} F_P^1(x, T)\sigma_1^2(H(x) - T)K_{P,\sigma}(x, x) \quad .$$

Then obviously

$$k_\sigma^T(x) = \sum_{P_1 \subset P_2} \sum_{\delta \in P_1 \backslash G} H_1^2(\delta x)_\sigma^T \quad .$$

To obtain the Theorem 3.1.2 all we need to prove is the

PROPOSITION 3.2.4. Provided T is sufficiently regular

$$\sum_{\sigma \in \mathcal{O}} \int_{P_1 \backslash G(\mathbb{A})^1} |H_1^2(x)_\sigma^T| dx$$

is finite.

This will be proved in the next lecture.

Erratum to Lecture 3

The proof of the Lemma 3.2.3 in the notes is incorrect and should be replaced by the following one. We first recall the statement.

LEMMA 3.2.3. *Assume* x *and* δx *are in* $\mathfrak{S}_P(T_0)$ *with* $\delta \in P$ *and that*

$$\alpha(H(x) - T) > 0 \qquad \forall \alpha \in \Delta_0^P - \Delta_0^1$$

then provided T *is sufficiently regular one has* $\delta \in P_1$.

We are free to modify δ and x by elements in P_0, on the left, so that we need only to consider the case $\delta = w_s$ where w_s represents $s \in \Omega^M$ the Weyl group of M. We have

$$H(\delta x) = H(w_s x) = sH(x) + H(w_s n)$$

if $x = ank$ with $a \in \mathbf{M}_0$, $n \in \mathbb{N}_0$ and $k \in K$. There exists $T_1 \in \mathfrak{a}_0^+$ such that for any $n \in \mathbb{N}_0$ and any $s \in \Omega$

$$X_s = s^{-1}H(w_s n) + T_1 - s^{-1}T_1$$

is a positive linear combination of coroots $\check{\beta}$ of M such that $\beta > 0$ and $s\beta < 0$ (cf. Lemma 6.3 of Lecture 6). Let V_s^+ be the positive linear span of those roots β and V_s^{++} be the subcone of the $\lambda \in V_s^+$ such that moreover $\lambda(\check{\beta}) > 0$ for all those β. In particular $\lambda(X_s) \geq 0$

and since

$$H(x) - T_1 + X_s = s^{-1}(H(w_s x) - T_1)$$

we have

$$\lambda(H(x) - T_1) \leq s\lambda(H(w_s x) - T_1) \quad .$$

We have assumed that $w_s x \in \mathfrak{S}_P(T_0)$ and hence

$$s\lambda(H(w_s x) - T_0) \leq 0$$

since $s\lambda$ is a positive linear combination of negative roots. This yields the following inequality:

$$\lambda(H(x)) \leq \lambda(T_1) - s\lambda(T_1 - T_0) \quad .$$

By hypothesis we may write

$$H(x) = \sum_{\alpha \in \Delta_0^P} h_\alpha \check{\varpi}_\alpha + H_P$$

with $H_P \in \mathfrak{a}_P$, $h_\alpha > \alpha(T_0)$ for all $\alpha \in \Delta_0^P$ and $h_\alpha > \alpha(T)$ for all $\alpha \in \Delta_0^P - \Delta_0^1$. Since λ is a positive linear combination of positive roots $\lambda(\check{\varpi}_\alpha) \geq 0$, and we get

$$\sum_{\alpha \in \Delta_0^P - \Delta_0^1} h_\alpha \lambda(\check{\varpi}_\alpha) \leq (\lambda - s\lambda)(T_1 - T_0) \quad .$$

If T is sufficiently regular this is possible only if $\lambda(\check{\varpi}_\alpha) = 0$ for all $\alpha \in \Delta_0^P - \Delta_0^1$ and all $\lambda \in V_s^{++}$. This implies that $V_s^+ \subset (\mathfrak{n}_0^1)^*$ so that $\check{\beta} \in \mathfrak{a}_0^1$ whenever $\beta > 0$ and $s\beta < 0$. This is the case only if $s \in \Omega^1$ the Weyl group of M_1. $\square$