Lecture 8

PREPARATION FOR THE COARSE χ-EXPANSION
II: PROOF OF THE LEMMAS

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We begin with an easy one, Lemma 7.2. Since the truncation operator applied to $K_{P_1}(pabk, \gamma\varepsilon(p'abk'))$ is applied to it as a function of p it is enough to show that

$$K_{P_1}(pabk, \gamma\varepsilon(p'abk')) = \rho_{P_1}^{-2}(b)K_{P_1}(pak, \gamma\varepsilon(p'ak')) \quad .$$

We recall that

$$K_{P_1}(pabk, \gamma\varepsilon(p'abk')) = \sum_{\delta \in N_1 \backslash P_1} \int_{\mathbf{N}_1} \phi(k^{-1}b^{-1}a^{-1}p^{-1}n_1\delta\gamma\varepsilon(pa)\varepsilon(b)\varepsilon(k))dn_1 \quad ,$$

and notice that $\varepsilon(b) = b$. Since $P_1 \subset Q$,

$$b^{-1}a^{-1}p^{-1} = a^{-1}p^{-1}b^{-1}n_2 \quad ,$$

with $n_2 \in \mathbf{N}_1$, and a change of variables allows us to absorb n_2 in n_1. Since Q is ε-invariant, $\varepsilon(pa)$ still lies in $Q(\mathbb{A})$ and

$$\gamma\varepsilon(pa)b = n_3 b\gamma\varepsilon(pa) \quad ,$$

with $n_3 \in \mathbf{N}_Q$. Since $\mathbf{N}_Q \subseteq \mathbf{N}_1$ we can also absorb n_3 in n_1, and the desired equality follows from the observation that

$$\left|\frac{dn_1}{d(b^{-1}n_1b)}\right| = \rho_{P_1}^2(b) \quad .$$

We next turn to Lemma 7.7. Recall that we established during the proof of Lemma 2.1 that if

$$\hat{\tau}_{P_1}(H(\delta g) - H(h) - T_0) \neq 0$$

then we could find a representative δ' for δ such that

$$|\delta' g| \leq c(|g|e^{\|H(h)+T_0\|})^N \quad .$$

Thus Lemma 7.7 follows from Lemma 2.1 provided we note that

$$\|H(h)\| \leq c(1 + |\ell n|h\|) \quad .$$

Working our way backwards we next prove Lemma 7.6. Since $K\Omega K = \Omega$ we may suppose that g and h lie in $\mathbf{P}_1$. Then

$$K_{P_1}(mhk, gk') \neq 0$$

implies that for some $n \in \mathbf{N}_1$ and some $m' = \gamma m \in \mathbf{M}_1^1$

$$g^{-1}nm'h \in \Omega(\text{supp } \phi)^{-1}\Omega^{-1} = C \quad .$$

Thus if we choose ρ and v_α as in the second lecture, $\alpha \in \Delta_0 - \Delta_0^{P_1}$ we have

$$\|\rho(g^{-1}nm'h)v_\alpha\| \leq c \quad .$$

On the other hand we have chosen g and h to lie in $\mathbf{P}_1$ and nm' lies in $\mathbf{P}_1^1$. Thus

$$\|\rho(g^{-1}nm'h)v_\alpha\| = e^{d_\alpha \varpi_\alpha(H(h)-H(g))}\|v_\alpha\| \quad .$$

The conclusion is that

$$\varpi_\alpha(H(h) - H(g)) \leq c$$

for all $\varpi_\alpha \in \hat{\Delta}_{P_1}$. This clearly implies the statement of the lemma.

To prove Lemma 7.5 we observe that $R(Y)K_{P_1}$ is obtained by replacing ϕ by $\psi = L(Y)\phi$ and then building the kernel attached to ψ. Thus it is enough to prove the lemma for $Y = 1$.

We have, for $\gamma \in P_1$

$$\int_{\mathbf{N}_1} \phi(k^{-1}h^{-1}n_1\gamma gk') = \rho_1^2(g) \int_{\mathbf{N}_1} \phi(k^{-1}h^{-1}\gamma g n_1 k)\,dn_1$$

$$= \rho_1^2(h) \int_{\mathbf{N}_1} \phi(k^{-1}n_1 h^{-1}\gamma gk)\,dn_1 \quad .$$

Moreover, since $K\Omega K = \Omega$ we may suppose that h and g lie in $\mathbf{P}_1$, and indeed in $\mathbf{M}_1$ because, for example,

$$|m| \leq c|g|^N$$

if $g = mn$, $n \in \mathbf{N}_1$, $m \in \mathbf{M}_1$. Then the integral can be taken over a fixed compact set in $\mathbf{N}_1$ which does not depend on h and g. So we need only estimate

$$\sum_{M_1} \chi(h^{-1}\gamma g)$$

where χ is the characteristic function of a compact set C in $\mathbf{M}_1$. This is the number of $\gamma \in M_1$ for which $\gamma \in hCg^{-1}$ and is easily estimated by Lemma 2.1.

Lemma 7.3 is geometrical. Since $\Delta_1^Q \subseteq \Delta_1^{P_2}$ we have

$$\alpha(H_1^Q) = \alpha(H) > 0$$

for $\alpha \in \Delta_1^Q$. Moreover

$$\varpi(H_Q^\varepsilon) > 0 , \qquad \varpi \in \hat{\Delta}_{P_2} .$$

Since $\hat{\Delta}_Q$ is the E-orbit of $\hat{\Delta}_{P_2}$, E being $\{1, \varepsilon, \ldots, \varepsilon^{\ell-1}\}$, and since H_Q^ε is ε-invariant we obtain the same inequality for $\varpi \in \hat{\Delta}_Q$. On the other hand

$$\alpha(H_Q^\varepsilon) < -\alpha(H_1^Q)$$

for $\alpha \in \Delta_{P_2}$. Thus

$$\alpha(H_Q^\varepsilon) \leq C\|H_1^Q\|$$

for $\alpha \in \Delta_Q$, and the lemma follows.

This leaves us with Lemmas 7.1 and 7.4. The first is the easier, and we begin with it.

The assumption of the lemma implies that for some p in $\mathbf{P}_1^1$

$$\phi(k^{-1}g^{-1}p\gamma\varepsilon(gk')) \neq 0 \quad .$$

Thus

$$g^{-1}p\gamma\varepsilon(g) \in C \quad ,$$

C being a compact set depending only on Ω and the support of ϕ. Taking $\alpha \in \Delta_0 - \Delta_0^Q$ we evaluate $\|\rho(g^{-1}p\gamma\varepsilon(g))v_\alpha\|$

$$e^{-d_\alpha \varpi_\alpha (H(g)-\varepsilon H(g))} \|v_\alpha\|$$

and $\|\rho(\varepsilon(g^{-1})\gamma^{-1}p^{-1}g)v_\alpha\|$ as

$$e^{d_\alpha \varpi_\alpha (H(g)-\varepsilon H(g))} \|v_\alpha\| \quad .$$

Both expressions are bounded above. If $\chi \in X^*(G)$ we can also bound

$$|\chi(g^{-1}\varepsilon(g))| = |\chi(g^{-1}p\gamma\varepsilon(g))|$$

above and below, concluding that $\|H(g) - \varepsilon H(g)\|$ is bounded. The lemma follows.

We deal finally with Lemma 7.4. It is clear that the assumption implies that

$$(1) \qquad K_{P_1}(m'ak, \gamma\varepsilon(nmak)) \neq 0$$

for some $m' \in \mathbf{M}_1^1$. For brevity we write $H = H(a)$.

We write $\gamma = \eta'\omega\eta$ with $\eta' \in P_1$, $\eta \in N_0$, and with ω in the normalizer of A_0. It represents an element s in the Weyl group of

A_0 or $\mathfrak{a}_0$. We are free to modify ω on the left by an element of M_1, incorporating the modification in η'. So we can suppose that $s^{-1}\alpha > 0$ if $\alpha \in \Delta_0^1$. Since H lies in A_1^Q and $\alpha(H) > 0 \,\forall \alpha \in \Delta_1^Q$ we have $H \in \mathfrak{a}_1^+ \subseteq \mathfrak{a}_0^{+}$. Thus εH is also in $\mathfrak{a}_0^{+}$ and if $\alpha \in \Delta_0^{P_1}$ then

$$\alpha(H - s\varepsilon H) = -s^{-1}\alpha(\varepsilon H) < 0 \quad .$$

We shall now show in addition that, for $\alpha \in \Delta_0 - \Delta_0^{P_1}$,

$$(2) \qquad \varpi_\alpha(H - s\varepsilon H) \leq c(1 + \|H(m)\|) \quad .$$

This will allow us to infer from (iv) of Lecture 2 that

$$(3) \qquad H - s\varepsilon H \in X - {}^{+}\mathfrak{a}_0$$

where $\|X\| \leq c(1 + \|H(m)\|)$.

To prove (2) we notice that (1) implies that for some $m' \in \mathbf{M}_1^1$ and some $n_1 \in \mathbf{N}_1$

$$\varepsilon(a^{-1}m^{-1}n^{-1})\eta^{-1}\omega^{-1}n_1 m'a \in C \quad ,$$

η' having been absorbed in $n_1 m'$. Thus

$$\|\rho(\varepsilon(a^{-1}m^{-1}n^{-1})\eta^{-1}\omega^{-1}n_1 m'a)v_\alpha\| \leq c \quad .$$

We choose $\alpha \in \Delta_0 - \Delta_0^1$. Then

$$\|\rho(n_1 m'a)v_\alpha\| = |\xi_{d_\alpha \varpi_\alpha}(a)| \; \|v_\alpha\| \quad .$$

If $w = \omega^{-1}v_\alpha$ then w is a weight vector corresponding to the weight $s^{-1}\alpha : H \longrightarrow \alpha(sH)$. Moreover

$$\rho(\varepsilon(n^{-1})\eta^{-1})w = w+u$$

where u is an adelic linear combination of weight vectors for weights of the form

$$s^{-1}\alpha + \sum_{\beta \in \Delta_0} c_\beta \beta$$

with $c_\beta \geq 0$, $\sum c_\beta \neq 0$. Consequently

$$\|\rho(\varepsilon(a^{-1}m^{-1})\varepsilon(n^{-1})\eta^{-1})w\| \geq c\|\rho(\varepsilon(a^{-1}m^{-1}))w\|$$

$$= ce^{-d_\alpha \varpi_\alpha(s\varepsilon H + s\varepsilon H(m))}\|w\| \quad .$$

We deduce that

$$c \geq e^{d_\alpha \varpi_\alpha(H - s\varepsilon H - s\varepsilon H(m))} \quad .$$

Taking logarithms we obtain (2).

We write the left side of (3) as

$$H - s\varepsilon H = H - \varepsilon H + \varepsilon H - s\varepsilon H \quad .$$

Since $H \in \mathfrak{a}_0^+$, its transform εH also lies in $\mathfrak{a}_0^{Q+}$ and $\varepsilon H - s\varepsilon H \in {}^+\mathfrak{a}_0^{Q}$. We conclude that

$$H-\varepsilon H = X-Y$$

with $Y \in {}^{+}\mathfrak{a}_0$.

Applying ε^k, $0 \leq k < \ell$ to this relation and summing over k we obtain

$$0 = X' - \sum_{k=0}^{\ell-1} \varepsilon^k(Y)$$

with $\|X'\| \leq c(1+\|H(m)\|)$. Since every $\varepsilon^k(Y)$ lies in ${}^{+}\mathfrak{a}_0$ we infer that

$$\|Y\| \leq c(1+\|H(m)\|) \quad .$$

This implies first that

$$\|H-\varepsilon H\| \leq c(1+\|H(m)\|)$$

and thus that there is an ε-invariant H_0 with

$$\|H-H_0\| \leq c(1+\|H(m)\|) \quad .$$

We are reduced to showing that

$$\|H_0\| \leq c(1+\|H(m)\|)$$

knowing that

$$H_0-sH_0 = X-{}^{+}\mathfrak{a}_0 \quad ,$$

$\|X\| \leq c(1+\|H(m)\|)$, a relation which we deduce from (3). Since we may take

$$H_0 = \frac{1}{\ell} \sum_{k=0}^{\ell-1} \varepsilon^k H \quad ,$$

we may suppose that $H_0 \in \mathfrak{a}_0^+$. Then

$$H_0 - sH_0 \in {}^+\mathfrak{a}_0 \quad ,$$

and we conclude that

$$\|H_0 - sH_0\| \leq c(1+\|H(m)\|) \quad .$$

At this point we introduce the assumption that $\gamma \in F_\varepsilon(P_1, P_2)$. We know that for any $H \in \mathfrak{a}_0$,

$$H - sH = \sum_{\alpha \in \Delta_0} c_\alpha(H, s)\alpha$$

where $H \longrightarrow c_\alpha(H, s)$ is a linear form on $\mathfrak{a}_0$, non-negative on $\mathfrak{a}_0^+$. In particular $c_\alpha(\varpi_\beta, s) \geq 0$. If $c_\alpha(\varpi_\beta, s) = 0$ for all α then $s\varpi_\beta = \varpi_\beta$. Thus

$$\sup_\alpha c_\alpha(H, s) \geq c\beta(H)$$

provided $s\varpi_\beta \neq \varpi_\beta$.

Applying this to H_0 we conclude that

$$|\beta(H_0)| \leq c(1+\|H(m)\|) \tag{4}$$

unless $s\varpi_{\beta'} = \varpi_{\beta'}$ for every β' in the E-orbit of β, because $\beta(H_0)$ is constant on such orbits.

To prove the lemma we need to establish (4) for all $\beta \in \Delta_0^Q$. If the E-orbit of β does not meet $\Delta_0^{P_1}$ then it cannot happen that $s\varpi_{\beta'} = \varpi_{\beta'}$ for all β' in this ε-orbit for then $\gamma \in Q'$ if

$$\Delta_0^{Q'} = \Delta_0^Q - \{\varepsilon^k \beta\}$$

and $Q' \supset P_1$ and is ε-invariant. On the other hand if for some β' in the E-orbit of β we have $\beta' \in \Delta_0^{P_1}$ then

$$\beta'(H_0) = \beta(H_0)$$

and

$$\beta'(H) = 0 \quad .$$

Thus

$$|\beta(H_0)| = |\beta'(H-H_0)| \leq c(1+\|H(m)\|) \quad .$$