MORNING SEMINAR ON TRACE FORMULA

References

I. Truncation and basic identity:

1. J. Arthur, A trace formula for reductive groups I: terms associated to classes in $G(\mathbb{Q})$, Duke Math. J. 45 (1978).

2. ——————, A trace formula for reductive groups II: applications of a truncation operator, Comp. Math. 40 (1980).

II. Formalities of non-invariance:

3. ——————, The trace formula in invariant form, Ann. of Math. 114 (1981).

III. The σ-expansion: (1) and

4. ——————, in preparation.

IV. The χ-expansion: (2) and

5. ——————, A Paley-Wiener theorem for real reductive groups, to appear in Acta Math.

6. ——————, On the inner product of truncated Eisenstein series, Duke Math. J. 49 (1982).

7. ——————, On a family of distributions obtained from Eisenstein series I: Applications of the Paley-Wiener theorem, Amer. Jour. of Math. 104 (1982).

8. ——————, On a family of distributions obtained from Eisenstein series II: Explicit formulas, Amer. Jour. of Math. 104 (1982).

V. Expository

9. ——————, Eisenstein series and the trace formula, Proc. Symp. in Pure Math. 33.

10. ——————, The trace formula for reductive groups, Journées automorphes, Publ. Math. de l'Univ. Paris VII (15).

This seminar has two purposes: To expound the work of Arthur systematically and with any luck to extend it to the twisted case.

Lecture 1

INTRODUCTION

R. Langlands

Basic notation

G: reductive group over $\mathbf{Q}$.

P: parabolic subgroup of G with unipotent radical N and Levi factor M.

P_0: fixed minimal over $\mathbf{Q}$ parabolic subgroup of G. If $P \supseteq P_0$ then P is called standard.

ε: automorphism of G of finite order ℓ which fixes P_0.

E: group of order ℓ generated by ε.

G': $G \rtimes E$.

For simplicity I will also usually denote $G(\mathbf{Q})$, $P(\mathbf{Q})$ and so on by G, P, ... while designating $G(\mathbf{A})$, $P(\mathbf{A})$, ... by $\mathbf{G}$, $\mathbb{P}$, ...

Z: connected center of G.

$\mathbf{Z}_0$: closed ε-invariant subgroup of $\mathbf{Z}$ with $\mathbf{Z}_0 G$ closed. We fix once and for all a unitary character ξ of $\mathbf{Z}_0$ trivial on $Z_0 = \mathbf{Z}_0 \cap G$. It will always be there but will ultimately disappear from the notation. The group $\mathbf{Z}_0$ is in many applications $\{1\}$ and we urge the reader to fix his attention on this case.

L: space of measurable functions ψ on $G\backslash\mathbf{G}$ satisfying the following two conditions

(i) $$\psi(zg) = \xi(z)\psi(g) \qquad \forall z \in Z_0$$

(ii) $$\int_{\mathbf{Z}_0 G\backslash\mathbf{G}} |\psi(g)|^2 dg < \infty .$$

We define a unitary representation R of G on L by

$$R(g)\psi : h \longrightarrow \psi(hg) \quad .$$

Let ω be a unitary character of $\mathbf{G}$ trivial on G and satisfying

$$\omega(\varepsilon^{-1}(z))\xi(\varepsilon^{-1}(z)) = \xi(z) \quad .$$

Then the operator $R(\varepsilon)$ defined by

$$R(\varepsilon)\psi : h \longrightarrow \omega(\varepsilon^{-1}(h))\psi(\varepsilon^{-1}(h))$$

acts on L. This definition does not necessarily yield a representation of the group $\mathbf{G}'$. The relations satisfied are:

$$R(\varepsilon)R(g) = \omega^{-1}(g)R(\varepsilon(g))R(\varepsilon)$$

and

$$R(\varepsilon^{\ell})\psi(h) = \omega(\varepsilon^{-1}(h)\varepsilon^{-2}(h) \ \dots \ \varepsilon^{-\ell}(h))\psi(h) \quad .$$

If ϕ is a continuous compactly supported function on $\mathbf{G}$ then $R(\phi)$ is the operator defined by

$$R(\phi)\psi(h) = \int_{\mathbf{G}} \phi(g)\psi(hg)dg \quad .$$

The operator of interest is however $R(\phi)R(\varepsilon)$ and this is given by

$$\begin{aligned} R(\phi)R(\varepsilon)\psi(h) &= \int_{\mathbf{G}} \omega(\varepsilon^{-1}(hg))\phi(g)\psi(\varepsilon^{-1}(hg))dg \\ &= \int_{\mathbf{Z}_0 G\backslash\mathbf{G}} \left\{ \sum_{\gamma\in Z_0\backslash G_{\phi}} \int_{\mathbf{Z}_0} \omega(g)\xi(z)\phi(h^{-1}\gamma z\varepsilon(g))dz \right\} \psi(g)dg \ . \end{aligned}$$

Thus it is an integral operator with kernel

$$K(h, g) = \sum_{\gamma \in Z_0 \backslash G} \int_{\mathbf{Z}_0} \omega(g)\xi(z)\phi(h^{-1}\gamma z \varepsilon(g))dz \quad .$$

In order to simplify the formulas it is convenient to denote the function

$$g \longrightarrow \int_{\mathbf{Z}_0} \xi(z)\phi(zg)dz$$

by ϕ, the original function playing no further role. Then the kernel may be written

$$K(h, g) = \omega(g) \sum_{Z_0 \backslash G} \phi(h^{-1}\gamma\varepsilon(g)) \quad .$$

Recall: If the quotient $Z_0 G \backslash \mathbf{G}$ is not compact then $R(\phi)R(\varepsilon)$ is usually not of trace class even for smooth ϕ.

Truncation

This is a process for transforming sufficiently smooth slowly increasing functions on $Z_0 G \backslash \mathbf{G}$ into rapidly decreasing functions. Composition of the truncation operator with K yields an operator of trace class. For now I content myself with a formal description of the operator, postponing proofs and a precise description of its properties until later.

If $P \supseteq P_0$ let $\mathfrak{a}$ be $X_*(P) \otimes \mathbf{R}$, $X_*(P)$ being the lattice dual to the lattice $X^*(P)$ of rational characters of P.
We have $P_0 \longrightarrow P$, thus $X^*(P) \longrightarrow X^*(P_0)$, $X_*(P_0) \longrightarrow X_*(P)$,

and so $\mathfrak{a}_0 \longrightarrow \mathfrak{a}$. On the other hand if A_0 is a maximal split torus over $\mathbf{Q}$ in P_0 we can identify $\mathfrak{a}_0$ with $X_*(A_0) \otimes \mathbf{R}$ or with the Lie algebra of $A_0(\mathbf{R})$ and we can choose A in A_0 to be a maximal split torus in P. This yields $\mathfrak{a} \longrightarrow \mathfrak{a}_0$. Thus we have a natural decomposition the center of a Levi factor M of P

$$\mathfrak{a}_0 = \mathfrak{a}_P \oplus \mathfrak{a}_0^P \quad ,$$

where to emphasize its dependence on P we have written $\mathfrak{a} = \mathfrak{a}_P$. It is convenient to fix A_0 once and for all.

Let Δ_0 be the set of simple roots of $\mathfrak{a}_0$ and let Δ_0^P be the simple roots in M. Thus $\Delta_0^P \subseteq \Delta_0$. On $\mathfrak{a}_0^G$ we introduce an inner product compatible with the root system Δ_0 and let $\hat{\Delta}_0$ be the dual basis. Thus

$$\langle \varpi_\alpha, \beta \rangle = \delta_{\alpha\beta} \qquad \alpha, \beta \in \Delta_0, \varpi_\alpha \in \hat{\Delta}_0 \quad .$$

We let $\hat{\tau}_P$ be the characteristic function of

$$^+\mathfrak{a}_P = \{H \in \mathfrak{a}_P | \varpi_\alpha(H) > 0, \ \alpha \in \Delta_0 - \Delta_0^P\} \quad .$$

Observe that if $H = H_P + H^P$, $H_P \in \mathfrak{a}_P$, $H^P \in \mathfrak{a}_0^P$ then

$$\hat{\tau}_P(H) = \hat{\tau}_P(H_P) \quad .$$

We choose once and for all a maximal compact subgroup K of $\mathbf{G}$ such that $\mathbf{G} = \mathbf{P}_0 K = \mathbf{N}_0 \mathbf{M}_0 K$. It is important to observe that many operations and many formulas, including the trace formula itself, contain K implicitly. We define $H(g)$, $g \in \mathbf{G}$ by $g = pk$ and

$$|\chi(p)| = e^{\langle H(g), \chi \rangle}, \quad \chi \in X^*(P_0) \quad . \qquad H(g) \in \mathfrak{a}_0$$

In order to define the truncation operator we have to choose $T \in \mathfrak{a}_0$. This done we define $\Lambda^T \varphi$, φ a continuous function on $G\backslash\mathbf{G}$, by

$$\Lambda^T \varphi(g) = \sum_P (-1)^{\dim \mathfrak{a}_P/\mathfrak{a}_G} \sum_{\delta \in P\backslash G} \int_{N\backslash\mathbb{N}} \varphi(n\delta g)\, dn\, \hat{\tau}_P(H(\delta g) - T) \quad .$$

Facts (to be provided later)

(a) Each of the inner sums is finite.

(b) $\Lambda^T(\Lambda^T \varphi) = \Lambda^T \varphi$.

(c) Λ^T transforms sufficiently smooth slowly increasing functions into rapidly decreasing functions.

(d) Λ^T extends to an orthogonal projection on L.

If φ is a continuous function on $P_1\backslash\mathbf{G}$ we can more generally introduce a truncated function Λ^{T,P_1} given by

$$\Lambda^{T,P_1}\varphi(g) = \sum_{P_0 \subset R \subset P_1} (-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1} \sum_{\delta \in R\backslash P_1} \int_{N_R\backslash\mathbb{N}_R} \phi(n\delta g)\, \hat{\tau}_R^{P_1}(H(\delta g) - T)\, dn .$$

Here $\hat{\tau}_R^{P_1}$ is a function on $\mathfrak{a}_0 = \mathfrak{a}_{P_1} \oplus \mathfrak{a}_0^{P_1}$ and $\hat{\tau}_R^{P_1}(H) = \hat{\tau}_R^{P_1}(H^{P_1})$. On $\mathfrak{a}_R^{P_1}$ the function is defined by a dual basis $\hat{\Delta}_R^{P_1}$ to $\Delta_{0,R}^{P_1}$. It is the characteristic function of

$$ {}^{+}\mathfrak{a}_R^{P_1} = \{H \in \mathfrak{a}_R^{P_1} \mid \varpi_\alpha(H) > 0, \varpi_\alpha \in \hat{\Delta}_R^{P_1}\} . $$

Some modification of the exposition is called for.

It is often convenient to define $\Delta_R^{P_1}$, $\hat{\Delta}_R^{P_1}$, and $\mathfrak{a}_R^{P_1}$. Here $\mathfrak{a}_R^{P_1}$ is the orthogonal complement of $\mathfrak{a}_0^R$ in $\mathfrak{a}_0^{P_1}$ and $\Delta_R^{P_1}$ is the collection of restrictions of $\alpha \in \Delta_0^{P_1} - \Delta_0^R$ to $\mathfrak{a}_R^{P_1}$. The dual basis to $\Delta_R^{P_1}$ is $\hat{\Delta}_R^{P_1}$ which may be identified with $\{\varpi_\alpha \mid \alpha \in \Delta_0^{P_1} - \Delta_0^R\} \subseteq \hat{\Delta}_0^{P_1}$ for $\varpi_\alpha|\mathfrak{a}_0^R = 0$ if $\varpi_\alpha \in \hat{\Delta}_0^{P_1}$, $\alpha \notin \Delta_0^R$.

The operator Λ^{T,P_1} has properties similar to those of Λ^T.

The basic identity

If P is an ε-invariant standard parabolic subgroup we define a kernel K_P by

$$ K_P(h, g) = \sum_{\gamma \in Z_0 \backslash M} \int_{\mathbf{N}} \phi(h^{-1}\gamma n \varepsilon(g))dn . $$

Thus K_P is a function on $\mathbf{N}P\backslash\mathbf{G} \times \mathbf{N}P\backslash\mathbf{G}$ and $K_G = K$.

If $P_1 \subseteq P_2$ are two standard parabolic subgroups we let σ_1^2 be the characteristic function of the set of H in $\mathfrak{a}_1$ $(= \mathfrak{a}_{P_1})$ or $\mathfrak{a}_0$ (depending on one's point of view) for which

(i) $\varpi_\alpha(H) > 0\ \forall \varpi_\alpha \in \hat{\Delta}_1 (= \hat{\Delta}_{P_1} = \hat{\Delta}_{P_1}^G)$

(ii) $\alpha(H) > 0\ \forall \alpha \in \Delta_{P_1}^{P_2}\ (= \Delta_1^2)$

(iii) $\alpha(H) \leq 0\ \forall \alpha \in \Delta_{P_1}\ (= \Delta_{P_1}^G) - \Delta_{P_1}^{P_2}$.

Modulo $\varepsilon^{\varpi_\alpha}(H) > 0$ $\forall$ ϖ_α, $\alpha \in \Delta_0^G - \Delta_0^1$?

The basic identity is the equality

$$\sum_{P_0 \subset P} (-1)^{\dim \mathfrak{a}_P^\varepsilon / \mathfrak{a}_G^\varepsilon} \sum_{\delta \in P \backslash G} K_P(\delta g, \delta g) \hat{\tau}_P(H(\delta g) - T)$$

$$= \sum_{P_0 \subset P_1 \subset P_2} \sum_{\delta \in P_1 \backslash G} \sigma_1^2(H(\delta g) - T) \left(\sum_{P_1 \subset P \subset P_2} (-1)^{\dim \mathfrak{a}_P^\varepsilon / \mathfrak{a}_G^\varepsilon} \Lambda^{T, P_1} K_P(\delta g, \delta g) \right) .$$

The sum over P_1 and P_2 is over all standard parabolic subgroups, but the sum over P is on both sides the sum over all ε-invariant standard parabolics. The symbol $\mathfrak{a}_P^\varepsilon$ denotes the space of ε-invariants in $\mathfrak{a}_P$ and the truncation Λ^{T,P_1} is carried out on the first variable in $K_P(h, g)$ before substitution of $(\delta g, \delta g)$ for (h, g). The symbol $\hat{\tau}_P$ is an abbreviation for $\hat{\tau}_P^G$.

Let

$$\mathbf{G}^1 = \{g \in \mathbf{G} \mid |\chi(g)| = 1 \ \forall \chi \in X^*(G)\} \quad .$$

We shall expand the integral of the left side over $G \backslash \mathbf{G}^1$ as a sum over conjugacy classes $\mathfrak{o}$, obtaining finally the fine $\mathfrak{o}$-expansion and the integral of the right as a sum over automorphic representations obtaining ultimately the fine χ-expansion. The resulting equality is the (twisted) trace formula. Observe that all integrals that arise will be shown to be convergent.

Since the twisted case remains to be worked out as we go along I will confine myself on the whole in the remainder of this introduction to the ordinary trace formula. Even here it remains uncertain that the formal statements have the form given until various papers in preparation are written.

The $\mathfrak{o}$-expansions

The fine $\mathfrak{o}$-expansion that we ultimately obtain will be formed by sums over conjugacy classes, but the first step is to obtain a coarse $\mathfrak{o}$-expansion and this runs over semi-simple conjugacy classes. If $\gamma \in G' = G \rtimes E$ then γ may be written as $\gamma = \gamma_s \gamma_u$ with γ_s semi-simple and γ_u unipotent. The two elements γ and γ_s have the same projection on E and we are interested only in those γ which project on ε. Two such elements γ and γ' are in the same conjugacy class if $\gamma' = \delta^{-1}\gamma\delta$, $\delta \in G$. They are the same semi-simple classes if $\gamma_s' = \delta^{-1}\gamma_s\delta$, $\delta \in G$.

The fine $\mathfrak{o}$-expansion will have the form

$$\sum_P \sum_{\mathfrak{o} \in \mathcal{O}(M)} c_{\mathfrak{o}} (T_M^G \Phi_{\mathfrak{o}})(\phi) \quad .$$

The sum is over (ε-invariant) standard parabolics, $\mathcal{O}(M)$ is the set of conjugacy classes in M, and $c_{\mathfrak{o}}$ is a constant. It is 0 if $\mathfrak{o}$ is not elliptic, the class of γ in M being elliptic if the maximal split component of the center of the centralizer of γ_s is contained in $\mathfrak{a}_M = \mathfrak{a}_P$.

$\Phi_{\mathfrak{o}}$ is the (twisted) orbital integral over the adelic orbit of γ and $T_M^G \Phi_{\mathfrak{o}}$ is a distribution associated to $\Phi_{\mathfrak{o}}$ and is a weighted (twisted) orbital integral over the class in G induced from $\mathfrak{o}$ in the sense of Lusztig-Spaltenstein. For $M = G$ we have $T_M^G \Phi_{\mathfrak{o}} = \Phi_{\mathfrak{o}}$. Thus the distribution

$$\sum_{\mathfrak{o} \in \mathcal{O}(G)} c_{\mathfrak{o}} T_G^G \Phi_{\mathfrak{o}}$$

is (twisted-) invariant.

The χ-expansion

The χ-expansion is also obtained in two stages. The coarse expansion is derived first. It is a sum over cuspidal pairs. A cuspidal pair consists of a standard parabolic subgroup P and a cuspidal representation ρ of $\mathbf{M}^1$. Two pairs (ρ, P) and (ρ', P') are said to be equivalent if there is an $s \in \Omega(\mathfrak{a}, \mathfrak{a}')$ with representative w_s such that the representations ρ' and

$$m' \longrightarrow \rho(w_s^{-1} m' w_s)$$

are equivalent. $\Omega(\mathfrak{a}, \mathfrak{a}')$ is the set of linear transformations from $\mathfrak{a}$ to $\mathfrak{a}'$ obtained by restriction of some element $w_s \in G(\mathbf{Q})$.

The fine χ-expansion has the form

$$\sum_P \int_{\Pi(\mathbf{M})} d(\pi)(T_M^G \sigma_\pi)(\phi) d\pi \quad .$$

The sum is over all (ε-invariant?) standard parabolics and the integral is over all (ε-invariant?) unitary automorphic representations of $\mathbf{M}$, or at least a part of them which will be described later together with the measure $d\pi$. In the integrand appear a function $d(\pi)$ and $T_M^G \sigma_\pi$. Here σ_π is the (twisted) trace of σ_π but $T_M^G \sigma_\pi$ is a distribution associated to σ_π by means of derivatives of intertwining operators on $\mathrm{Ind}_P^G \pi$. Like $T_M^G \Phi_{\mathfrak{o}}$ the distribution $T_M^G \sigma_\pi$ will in general not be (twisted) invariant for $M \neq G$. But if $M = G$ then $T_G^G \sigma_\pi = \sigma_\pi$ and

$$\int_{\Pi(\mathbf{G})} d(\pi) T_G^G \sigma_\pi d\pi$$

is a (twisted) invariant distribution.

Thus apart from the explicit determination of the functions $c(\sigma)$ and $d(\pi)$, problems which have not yet been solved completely, the final form of the trace formula, from an analytic point of view and before stabilization, is

$$\sum_P \sum_{\sigma \in \mathcal{O}(M)} c_\sigma (T_M^G \Phi_\sigma)(\phi) = \sum_P \int_{\Pi(\mathbf{M})} d(\pi)(T_M^G \sigma_\pi)(\phi) d\pi \quad .$$

Any further modification, especially any transfer of terms from one side to the other to obtain an identity between invariant distributions, will probably be determined by the problem to be solved.

<u>A final remark</u>. Let L^d be the direct sum of all irreducible invariant subspaces of L. In the course of deriving the fine χ-expansion one has to show that the restriction of $R(\phi)$ to L^d is of trace class for sufficiently smooth ϕ. As I indicated the proof of this has not yet been completely worked out. So this result and its consequences remain for the moment uncertain.

Make some remarks about the trace formula over arbitrary number fields (of finite degree over $\mathbb{Q}$)

Lecture 2

THE BASIC IDENTITY PROVED

R. Langlands

Recall that the identity asserts the equality of

$$\sum_{P_0 \subset P} (1)^{\dim \mathfrak{a}_P^\varepsilon/\mathfrak{a}_G^\varepsilon} \sum_{\delta \in P\backslash G} K_P(\delta g, \delta g)\hat{\tau}_P(H(\delta g) - T)$$

and

$$\sum_{P_0 \subset P_1 \subset P_2} \sum_{P_1\backslash G} \sigma_1^2(H(\delta g)-T)\left(\sum_{P_1 \subset P \subset P_2} (-1)^{\dim \mathfrak{a}_P^\varepsilon/\mathfrak{a}_G^\varepsilon} \Lambda^{T,P_1} K_P(\delta g, \delta g)\right) .$$

Once it is shown that the sums occurring on both sides are finite, the proof will be a purely combinatorial matter.

Recall that we can define a height function on $\mathbf{A}^n$ by setting

$$\|x\|_v = \begin{cases} \sqrt{\sum_i |x_i|_v^2} & v \text{ archimedean} \\ \max |x_i| & v \text{ non-archimedean} \end{cases}$$

$$\|x\| = \prod_v \|x\|_v .$$

Then we can choose a height function on $V(\mathbf{A})$ for any vector space over $\mathbf{Q}$ simply by choosing a basis for V over $\mathbf{Q}$ and then identifying $V(\mathbf{A})$ with $\mathbf{A}^n$. It will be useful to recall briefly the properties of these height functions and other functions derived from them.

(a) If $\|\cdot\|_1$ and $\|\cdot\|_2$ are the height functions associated to different bases of $V(\mathbf{Q})$ then there is a positive constant c such that

$$\frac{1}{c}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1$$

for all x.

(b) If a is an idèle then $\|ax\| = |a|\,\|x\|$.

(c) $\|\cdot\|$ is bounded on compact sets.

(d) There is a constant c such that for all $x_1, \ldots, x_n$

$$\|x_1 \otimes \ldots \otimes x_n\| \leq c\|x_1\| \ldots \|x_n\| \quad .$$

(e) If $\varphi : V \longrightarrow W$ is linear over $\mathbf{Q}$ then there is a positive constant c such that

$$\|\varphi(x)\| \leq c\|x\| \quad .$$

(f) If $v \in V(\mathbf{Q})$ then $\|v\| \geq 1$, provided $v \neq 0$.

A basis of V defines in a natural way a basis of the space $M(V)$ of linear transformations of V which we use to introduce a height function on $M(V, \mathbf{A})$ and on $GL(V, \mathbf{A})$. We have

(g) $$\|xy\| \leq \|x\|\|y\| \qquad x, y \in M(V, \mathbf{A})$$

(h) $$\|xv\| \leq \|x\|, \|v\| \qquad x \in M(V, \mathbf{A}),\ v \in V(\mathbf{A}) \quad .$$

We introduce a set $\rho_1, \ldots, \rho_r$ of rational representations of G over $\mathbf{Q}$ with the following two properties:

(i) Every representation of G over $\mathbf{Q}$ can be obtained from $\rho_1, \ldots, \rho_r$ by the formation of tensor products and direct summands.

(ii) For each $\alpha \in \Delta_0$ there is an $i = i(\alpha)$, a vector v_α in V_i, the space of ρ_i, and a positive integer d_α such that

$$\rho(p)v_\alpha = \xi_{d_\alpha \varpi_\alpha}(p)v_\alpha$$

for all $p \in P$.

We let ρ be the direct sum of the ρ_i. It acts on $V = \oplus V_i$. We set

$$|g| = |g|_\rho = \|\rho(g)\| \quad .$$

This height function on $G(\mathbf{A})$ has several obvious properties.

(i)
$$|hg| \leq |h|\,|g|$$

(j) If $\varphi : G \longrightarrow H$ then

$$|\varphi(g)| \leq c|g|^N$$

where c and N depend upon φ alone.

(k) If $G = GL(n)$ then

$$\|g\| \leq c|g|^N$$

where c and N are independent of g.

(ℓ) There are constants c and N such that the number of elements in

$$\{g \mid |g| \leq M\}$$

is at most cM^N.

It is enough to prove this for $GL(n) \subseteq M(n)$, the space of $n \times n$ matrices. We have a morphism from $GL(n)$ to $\mathbf{P}^1 \times \mathbf{P}M(n)$ given by

$$g \longrightarrow (\deg g, \det^{-1} g) \times g$$

and the assertion is a consequence of standard properties of heights in projective spaces, the inverse image of a point in $\mathbf{P}^1 \times \mathbf{P}M(n)$ consisting of at most two in $GL(n)$.

(m) If A is a split torus we have a homomorphism $H : A \longrightarrow X_*(A) \otimes \mathbf{R}$ such that

$$e^{\lambda(H(a))} = |\xi_\lambda(a)|$$

for all $\lambda \in X^*(A)$. There are constants c and N such that

$$\{a \mid \|H(a)\| \leq M\} \subseteq A(\mathbf{Q})\{a \mid |a| \leq cM^N\} .$$

It is enough to prove this for $GL(1)$ where it is clear.

To prove that the sum occurring on the left of the basic identity if finite we need only establish the following lemma.

LEMMA 2.1. There are constants c and N such that the number of δ in $P \backslash G$ for which

$$\varpi_\alpha(H(\delta g) - T) > 0$$

for all $\alpha \in \hat{\Delta}_P$ is at most $c(|g| e^{\|T\|})^N$.

To prove the lemma we need only show that we can find a set of representatives for these δ each of which satisfies $|\delta| \leq c'(|g| e^{\|T\|})^{N'}$. According to reduction theory we can find a compact set C and a $T_0 \in \mathfrak{a}_0$ and representatives δ for which $\delta g = am$, $a \in A_0(\mathbf{A})$,

$$\text{(1)} \qquad \alpha(H(a)) - \alpha(T_0) > 0 \quad ,$$

$\alpha \in \Delta_0^P$ and $m \in C$. Since

$$|\delta| \leq |\delta g|\,|g|^{-1} = |am|\,|g|^{-1} \leq c\,|a|\,|g|^{-1} \quad ,$$

all we need show, according to (m), is that

$$\text{(2)} \qquad \|H(a)\| \leq c(1 + \ln|g| + \|T\|) \quad .$$

Observe that

$$1 \leq \|\rho(\delta^{-1})v_\alpha\| = \|\rho(g)\rho(\delta g)^{-1}v_\alpha\| \leq |g|\,\|\rho(\delta g)^{-1}v_\alpha\| \leq c|g|e^{-d_\alpha \varpi_\alpha(H(a))} \quad .$$

Consequently

$$\text{(3)} \qquad \varpi_\alpha(H(a)) \leq c(1 + |g|) \quad ,$$

the constant c varying from time to time.

Moreover by assumption

(4) $$\varpi_\alpha(H(a)) > \varpi_\alpha(T) \qquad \alpha \in \hat{\Delta}_P .$$

Also

(5) $$|\lambda(a)| = |\lambda(g)|, \quad \lambda \in X_G^*$$

for $\alpha \in \hat{\Delta}_P$. The inequality (2) is an immediate consequence of (1), (3), (4), and (5) and standard geometric properties of root systems which we now recall. They are actually valid for each of the systems $\Delta_{P_1}^{P_2}$.

The first is

(a) $$(\alpha, \beta) \leq 0 \text{ if } \alpha \neq \beta, \text{ and } \alpha, \beta \in \Delta_{P_1}^{P_2} .$$

It implies

(b) $(\varpi_\alpha, \varpi_\beta) \geq 0 \ \forall \alpha, \beta$, where $(\varpi_\alpha, \beta) = \delta_{\alpha\beta}$. We denote the set $\{\varpi_\alpha\}$ by $\hat{\Delta}_{P_1}^{P_2}$.

As a consequence

(c) $$\mathfrak{a}_1^{2+} = \{H \in \mathfrak{a}_1^2 \mid \alpha(H) > 0 \ \forall \alpha \in \Delta_{P_1}^{P_2}\} \subseteq {}^+\mathfrak{a}_1^2 = \{H \in \mathfrak{a}_1^2 \mid \varpi_\alpha(H) > 0 \ \forall \varpi_\alpha \in \hat{\Delta}_{P_1}^{P_2}\} .$$

Moreover if $P_1 \subset P \subset P_2$ and $\alpha \in \Delta_{P_1}^P$ then

$$\varpi_\alpha = \varpi'_\alpha + \sum_{\Delta_P^2} c_\beta \omega_\beta \qquad (\Delta_P^2 = \Delta_P^{P_2})$$

with $\varpi'_\alpha \in \hat{\Delta}_{P_1}^P$, $\varpi_\alpha \in \hat{\Delta}_{P_1}^{P_2}$. Applying (c) to $\hat{\Delta}_{P_1}^P$ we see that

$$\varpi'_\alpha = \sum_{\gamma \in \Delta_{P_1}^{P}} d_\gamma \gamma$$

with $d_\gamma \geq 0$. Thus

$$(\varpi'_\alpha, \beta) \leq 0$$

for $\beta \in \Delta_{P_1} - \Delta_{P_1}^{P}$ and

$$0 = (\varpi'_\alpha, \beta) + c_\beta \quad ,$$

so that $c_\beta \geq 0$.

We conclude

(d) $\{H \,|\, \alpha(H) > 0,\ \alpha \in \Delta_{P_1}^{P},\ \varpi_\alpha(H) > 0,\ \alpha \in \hat{\Delta}_P^{P_2}\} \subseteq \{H \,|\, \omega_\alpha(H) > 0 \,\forall \alpha \in \hat{\Delta}_{P_1}^{P_2}\}$.

This is all that is necessary to complete the proof of the lemma.

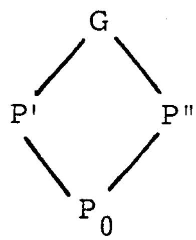
Returning to the basic identity we show that the sums on the right are finite. This will be an immediate consequence of the following lemma.

LEMMA 2.2. <u>For a given</u> P_1,

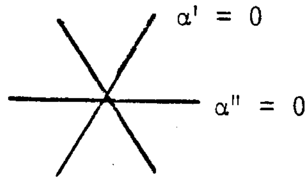
$$\sum_{P_1 \subset P_2} \sigma_1^2 = \hat{\tau}_1 \qquad (\hat{\tau}_1 = \hat{\tau}_{P_1}) \quad .$$

This lemma is clear because if $\varpi_\alpha(H) > 0 \,\forall \alpha \in \Delta_1$ then there is a unique P_2 such that $\alpha(H) > 0 \,\forall \alpha \in \Delta_{P_1}^{P_2}$ and $\alpha(H) \leq 0 \,\forall \alpha \in \Delta_{P_1} - \Delta_{P_1}^{P_2}$.

For the group $SL(3)$ we can easily describe these sets geometrically. The lattice of standard parabolic subgroups is



and the diagram of chambers is



$\{\alpha'\} = \Delta'_0 \quad \{\alpha''\} = \Delta''_0$

Then

σ_0 : 

σ'_0 : 

σ''_0 : 

$\hat{\tau}_0$: 

Moreover

$\hat{\tau}_{P'} = \sigma_{P'}^{G}$:

$\hat{\tau}_{P''} = \sigma_{P''}^{G}$:

We return to the identity and on the right side consider a fixed pair $P_1 \subset P$ and sum over all $P_2 \supset P$. This means we have to consider

$$\sum_{P_2 \supset P} \sigma_1^2(H) \quad .$$

LEMMA 2.3. <u>For fixed</u> $P_1 \subset P$,

$$\sum_{P_2 \supset P} \sigma_1^2 = \tau_{P_1}^{P} \hat{\tau}_P \quad .$$

The sum on the left is a characteristic function, namely of

$$\{H \mid \varpi_\alpha(H) > 0,\ \alpha \in \Delta_{P_1},\ \alpha(H) > 0,\ \alpha \in \Delta_{P_1}^{P}\} \quad ,$$

Modify $\varepsilon\varpi_\alpha(H) > 0$ $\alpha \in \Delta_0^{G} - \Delta_0^{P_1}$

while the function on the right is the characteristic function of

$$\{H \mid \varpi_\alpha(H) > 0,\ \alpha \in \Delta_P,\ \alpha(H) > 0,\ \alpha \in \Delta_{P_1}^{P}\} \quad .$$

Modify $\varepsilon\varpi_\alpha(H) > 0$ $\alpha \in \Delta_P$

The first conditions clearly imply the second. So we need only show that the second imply the first. This is however (d).

Second ⟹ (taking $H \in \mathfrak{a}_1$) $\alpha(H) \geq 0$ $\forall \alpha \in \Delta_0^{P_1}$. Now project on $\mathfrak{a}^\varepsilon$ and apply (d) for ε-invariant roots. $\sum c^\alpha H = H_0$. Obtain $\varpi_\alpha(H_0) \geq 0$ $\forall \alpha \in \Delta_0$ ⟹ $\varepsilon\varpi_\alpha(H) \geq 0$ $\forall \alpha$. ⟹ (by openness) $\varepsilon\varpi_\alpha(H) > 0$

This leaves us with a sum over P on the left and a sum over P_1 and P on the right. All we need do is show that for a fixed P the contribution from P on the left is equal to the sum over P_1 with the same fixed P on the right. Dropping factors which are obviously equal we see that we are reduced to showing that

$$\sum_{\delta \in P\backslash G} K_P(\delta g, \delta g)\hat{\tau}_P(H(\delta g) - T)$$

$$= \sum_{P_1} \sum_{P_1\backslash G} \tau_{P_1}^P(H(\delta g) - T)\hat{\tau}_P(H(\delta g) - T)\Lambda^{T,P_1} K_P(\delta g, \delta g) \quad .$$

The inner sum on the right may be written as a double sum, first over $P_1\backslash P$ and then over $P\backslash G$. Since

$$\hat{\tau}_P(H(\delta_1\delta g) - T) = \hat{\tau}_P(H(\delta g) - T), \quad \delta_1 \in P$$

and

$$\Lambda^{T,P_1} K(\delta_1\delta g, \delta_1\delta g) = \Lambda^{T,P_1} K(\delta_1\delta g, \delta g)$$

we need only show that

$$(1) \quad \sum_{P_1} \sum_{P_1\backslash P} \tau_{P_1}^P(H(\delta_1 g) - T)\Lambda^{T,P_1} K_P(\delta_1 g, h) = K_P(g, h) \quad .$$

LEMMA 2.4. Suppose P is a standard parabolic subgroup and ϕ a continuous function on $P\backslash \mathbf{G}$. Then

$$\sum_{P_1 \subset P} \sum_{P_1\backslash P} \Lambda^{T,P_1}\phi(\delta, g)\tau_1^P(H(\delta, g) - T) = \int_{N\backslash \mathbb{N}} \phi(ng)dn \quad .$$

Recalling the definition of Λ^{T,P_1} we see that the left side is

$$\sum_{R\subset P_1\subset P}\ \sum_{\delta\in P_1\backslash P}\ \sum_{\gamma\in R\backslash P_1} (-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1} \int_{N_R\backslash \mathbf{N}_R} \phi(n\gamma,\delta,g)\,dn \quad \hat{\tau}_R^1(H(\gamma\delta,g)-T)\ \tau_1^P(H(\delta,g)-T) \ .$$

In the double sum over R and P_1 we fix R and sum over P_1. Thus we have

$$\sum_R \sum_{R\backslash P} \left\{ \sum_{R\subset P_1\subset P} (-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1} \hat{\tau}_R^1(H)\tau_1^P(H) \right\} \int_{N_R\backslash \mathbf{N}_R} \phi(n\gamma g)\,dn$$

with

$$H = H(\gamma g) - T \ .$$

Observe that if $\gamma \in P_1$ then

$$\tau_1^P(H(\gamma\delta_1 g) - T) = \tau_1^P(H(\delta_1 g) - T) \ .$$

If $R = P$ the sum over P_1 in the parentheses is clearly 1. We need to show that it is 0 otherwise. Once this is done the left side of (1) will have been shown to equal

$$\int_{N\backslash \mathbf{N}} K_p(ng,\ h) \ .$$

Since $K_p(ng,\ h) = K_p(g,\ h)$ and

$$\int_{N\backslash \mathbf{N}} dn = 1 \ ,$$

the basic identity is proved.

We prove now a more general combinatorial statement, of which the desired identity is a special case. We fix R and P, $R \subset P$, and a Λ in $\mathfrak{a}_R^P$. Let $\varepsilon_R^{P_1}(\Lambda)$ be $+1$ or -1 according as the number of roots $\alpha \in \Delta_R^{P_1}$ such that $(\alpha, \Lambda) \leq 0$ is even or odd. Let $\phi_R^{P_1}(\Lambda, H)$ be the characteristic function of those H in $\mathfrak{a}_R^P$ such that $\varpi_\alpha(H) > 0$ if $(\alpha, \Lambda) \leq 0$ and $\varpi_\alpha(H) \leq 0$ if $(\alpha, \Lambda) > 0$.

LEMMA 2.5.

$$\sum_{R \subset P_1 \subset P} \varepsilon_R^{P_1}(\Lambda)\phi_R^{P_1}(\Lambda, H)\tau_{P_1}^P(H)$$

is 0 if $(\Lambda, \alpha) \leq 0$ for some $\alpha \in \Delta_R^P$ and is 1 otherwise.

The identity we need is the special case that $(\Lambda, \alpha) \leq 0$ for all $\alpha \in \Delta_R^P$. We observe first of all that an identity very similar to the one we need is easy to prove, namely that if $R \neq P$ then

$$(1) \qquad \sum_{R \subset P_1 \subset P} (-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1} \tau_R^1(H)\hat{\tau}_1^P(H) = 0 \quad .$$

For a given H all terms are 0 unless $\varpi_\alpha(H) > 0$ for all $\alpha \in \Delta_R^P$. If $\varpi_\alpha(H) > 0$ for all $\alpha \in \Delta_R^P$ then

$$\Delta = \{\alpha \in \Delta_R^P | \alpha(H) > 0\}$$

is not empty. For this H the sum on the left is

$$\sum_{\Delta_R^{P_1} \subseteq \Delta} (-1)^{\dim \mathfrak{a}_R/\mathfrak{a}_1} = \sum_{\Delta_R^{P_2} \subseteq \Delta} (-1)^{|\Delta_R^{P_1}|} = 0 \quad .$$

Returning to the lemma we replace in (1) R by P_1 and the sum over P_1 by a sum over P_2. This enables us to conclude that for $P_1 \neq P$

$$(-1)^{\dim \mathfrak{a}_{P_1}/\mathfrak{a}_P} \tau_{P_1}^P = - \sum_{P_1 \subset P_2 \subsetneq P} (-1)^{\dim \mathfrak{a}_{P_1}/\mathfrak{a}_{P_2}} \tau_{P_1}^{P_2} \hat{\tau}_{P_2}^P \quad .$$

We substitute in the sum of the lemma, obtaining the difference between

$$\varepsilon_R^P(\Lambda)\Phi_R^P(\Lambda, H) \tag{2}$$

and

$$\sum_{R \subset P_1 \subset P_2 \subsetneq P} (-1)^{\dim \mathfrak{a}_{P_2}/\mathfrak{a}_P} \varepsilon_R^{P_1}(\Lambda)\phi_R^{P_1}(\Lambda, H)\tau_{P_1}^{P_2}(H)\hat{\tau}_{P_2}^P(H) \quad .$$

We can apply induction. The sum over P_1 is 0 unless $(\alpha, \Lambda) > 0$ for all roots α in $\Delta_R^{P_2}$, thus unless $P_2 \subset P_\Lambda$ where P_Λ is defined by

$$\Delta_R^{P_\Lambda} = \{\alpha \in \Delta_{P_1}^P \mid (\alpha, \Lambda) > 0\} \quad .$$

If $P_2 \subset P_\Lambda$ the sum over P_1 is equal to

$$(-1)^{\dim \mathfrak{a}_{P_2}/\mathfrak{a}_P} \hat{\tau}_{P_2}^P(H) \quad .$$

Thus we obtain

$$\sum_{R \subset P_2 \subset P_\Lambda} (-1)^{\dim \mathfrak{a}_{P_2}/\mathfrak{a}_P} \hat{\tau}_{P_2}^P(H) \tag{3}$$

unless $P_\Lambda = P$ when we obtain this expression minus 1. To prove the lemma we need only show that (2) equals (3), for we are trying to show that the difference is 0 unless $P_\Lambda = P$ when it is 1.

It is however clear that (3) is equal to zero unless

$$\varpi_\alpha(H) > 0$$

for $\alpha \in \Delta_R^P - \Delta_R^{P_\Lambda}$, thus for $(\alpha, \Lambda) \leq 0$, but that if this condition is satisfied it is equal to

$$\sum_{Q \subset P_2 \subset P_\Lambda} (-1)^{\dim \mathfrak{a}_{P_2}/\mathfrak{a}_P}$$

where

$$\Delta_R^Q = \{\alpha \in \Delta_R | \varpi_\alpha(H) \leq 0\} \quad .$$

The sum is clearly 0 unless $Q = P_\Lambda$ when it is

$$(-1)^{\dim \mathfrak{a}_{P_\Lambda}/\mathfrak{a}_P} \quad ,$$

which is $\varepsilon_Q^P(\Lambda)$. The lemma follows.

Lecture 3

THE COARSE $\mathcal{O}$-EXPANSION

J.-P. Labesse

3.1. Statement of the main result.

Let G be a connected reductive group and G' an extension of G, over $\mathbf{Q}$, by a finite group E generated by ε_0

$$G \longrightarrow G' \longrightarrow E \quad .$$

Notice that we do not assume that the extension is split. Choose a minimal parabolic P_0 and a Levi component M_0; there is an $\varepsilon \in G'$ projecting on ε_0 such that $\varepsilon(P_0) = \varepsilon P_0 \varepsilon^{-1} = P_0$ and $\varepsilon(M_0) = M_0$. Let A_0 be the split component of the center of M_0; the action of ε on A_0 is of finite order. All parabolics considered below will be assumed to be standard.

Any $g \in G'$ has a Jordan decomposition $g = g_u g_s$ with g_s semisimple in G' and g_u unipotent in G.

We shall use the following equivalence relation in G, which could be called ε-semisimple-conjugacy.

Two elements γ_1 and γ_2 in G will be called equivalent if $\gamma_1' = \gamma_1 \varepsilon$ and $\gamma_2' = \gamma_2 \varepsilon$ have conjugate semisimple parts. In particular if γ_1' and γ_2' are semisimple this means that γ_1 and γ_2 are ε-conjugate, i.e., there exist $\delta \in G$ such that $\gamma_1 = \delta^{-1} \gamma \varepsilon(\delta)$.

LEMMA 3.1.1. *Given* P *an* ε-*invariant parabolic and* $\gamma \in P$, *denote by* N^o *the centralizer of the semisimple part of* $\gamma' = \gamma\varepsilon$ *in* N *the unipotent radical of* P. *Let* ϕ *be a function with finite support on* P, *then*

$$\sum_{n \in N} \phi(n\gamma) = \sum_{\delta \in N^o \backslash N} \ \sum_{\eta \in N^o} \phi(\delta^{-1} \eta\gamma\varepsilon(\delta)) \quad .$$

Notice that N^o is normalized by γ'. Let us denote by θ the automorphism of N defined by the conjugation by $\gamma\varepsilon$. We shall prove a slightly generalized version of the above lemma. Consider a nilpotent group N_1 and an automorphism θ (over $\mathbf{Q}$), let N_2 be a subgroup θ-invariant such that N_2 contains the subgroup of θ_s-fixed points in N_1 (where θ_s is the semisimple part of θ), then given ϕ on N_1 with finite support one has

$$\sum_{n \in N_1} \phi(n) = \sum_{\delta \in N_2 \backslash N_1} \ \sum_{\eta \in N_2} \phi(\delta^{-1} \eta\theta(\delta)) \quad .$$

Better - a reference to the relevant fact on algebraic groups. In any case to be revised

We can now proceed by "dévissage" and it is enough to prove this when N_2 is invariant in N_1 and $N_2 \backslash N_1$ abelian; in such a case θ induces in the Lie algebra of $N_2 \backslash N_1$ a linear map θ' which is such that $\theta' - 1$ is invertible, and the lemma follows. ☐

The preceding lemma shows that if P is an ε-invariant parabolic and $\mathfrak{o}$ an ε-semisimple-conjugacy class then

$$P \cap \mathfrak{o} = N.(P \cap \mathfrak{o}) \quad .$$

We can now define

$$K_{P,\sigma}(x, y) = \int_{(N)} \sum_{\gamma \in P \cap \sigma} \omega(y)\phi(x^{-1}n^{-1}\gamma\varepsilon(y))dn$$

where $(N) = N(\mathbb{Q}) \backslash N(\mathbb{A})$. Obviously one has

$$K_p = \sum_{\sigma \in \mathcal{O}} K_{P,\sigma}$$

where $\mathcal{O}$ is the set of ε-semisimple-conjugacy classes. Now introduce

$$k_{\sigma}^{T}(x) = \sum_{\varepsilon(P)=P} \sum_{\delta \in P \backslash G} (-1)^{a_P^{\varepsilon}} \hat{\tau}_P(H(\delta x) - T)K_{P,\sigma}(\delta x, \delta x) \quad .$$

(Undefined notations are taken over from Lectures 1 and 2.)

The aim of this lecture is to prove the

THEOREM 3.1.2. <u>Provided</u> T <u>is sufficiently regular, the sum</u>

$$\sum_{\sigma \in \mathcal{O}} \int_{(G)^1} |k_{\sigma}^{T}(x)|dx$$

<u>is finite</u>. (Here $(G)^1$ <u>stands for</u> $G(\mathbb{Q}) \backslash G(\mathbb{A})^1$.)

3.2. <u>Some partitions of</u> $G(\mathbb{A})$.

Let P be a parabolic and T_0 a vector in $\mathfrak{a}_0^-$, define $\mathfrak{G}_P(T_0)$ to be the set of $x \in G(\mathbb{A})$ such that

$$\alpha(H(x) - T_0) > 0 \qquad \forall \alpha \in \Delta_0^P$$

Not such a good defin. — See Lecture 6

According to reduction theory we know that if $-T_0$ is sufficiently regular then

$$P . \mathfrak{S}_P(T_0) = G(\mathbb{A}) \quad .$$

We shall assume that T_0 is fixed so that the above property holds for all P. Let $P_1 \subset P$ and consider $T \in \mathfrak{a}_0^+$; define $\mathfrak{S}_P^1(T_0, T)$ to be the set of $x \in \mathfrak{S}_P(T_0)$ such that

$$\varpi(H(x) - T) \leq 0 \qquad \forall \varpi \in \hat{\Delta}_0^1 \quad .$$

We shall denote $F_P^1(\cdot, T)$ the characteristic function of the set $P_1 \mathfrak{S}_P^1(T_0, T)$.

PROPOSITION 3.2.1. <u>Assume</u> T <u>is sufficiently regular, then given</u> P <u>we have</u>

$$\sum_{\{P_1 | P_1 \subset P\}} \sum_{\delta \in P_1 \backslash P} F_P^1(\delta x, T) \tau_1^P(H(\delta x) - T) = 1$$

<u>for all</u> $x \in G(\mathbb{A})$.

The proof relies on the following particular case of the combinatorial Lemma 2.5 (of Lecture 2). Assume $\Lambda \in \mathfrak{a}_0^+$ then

$$\sum_{\{P_1 | P_1 \subset P\}} \phi_0^1(\Lambda, H) \tau_1^P(H) = 1$$

for all $H \in \mathfrak{a}_0$. Recall that for $\Lambda \in \mathfrak{a}_0^+$ the function $H \longmapsto \phi_0^1(\Lambda, H)$

is the characteristic function of the set of H such that $\varpi(H) \leq 0$ for all $\varpi \in \hat{\Delta}_0^1$.

Now fix $x \in G(\mathbb{A})$; thanks to reduction theory we know that there exist at least one $\delta \in P$ such that $\delta x \in \mathfrak{S}_P(T_0)$; the combinatorial lemma applied with $H = H(\delta x) - T$ provides us with exactly one parabolic $P_1 \subset P$ such that

$$F_P^1(\delta x, T)\tau_1^P(H(\delta x) - T) = 1 \quad .$$

Hence, the sum in the proposition is at least 1. To prove that it is exactly 1 consider $x \in \mathfrak{S}_P(T_0)$ and $\delta \in P$ such that $\delta x \in \mathfrak{S}_P(T_0)$. The combinatorial lemma provides us with two parabolics P_1 and P_2 such that

$$F_P^1(x, T)\tau_1^P(H(x) - T) = F_P^2(\delta x, T)\tau_2^P(H(\delta x) - T) = 1 \quad .$$

We need to show that this implies $\delta \in P_1$ (and hence $P_1 = P_2$). We need two lemmas.

LEMMA 3.2.2. <u>Given</u> $P_1 \subset P$ <u>and</u> $H \in \mathfrak{a}_0^P$ <u>such that</u>

(i) $\alpha(H) > 0 \qquad \forall \alpha \in \Delta_1^P$

(ii) $\varpi(H) \leq 0 \qquad \forall \varpi \in \hat{\Delta}_0^1$

<u>then the following holds</u>

(iii) $\alpha(H) > 0 \qquad \forall \alpha \in \Delta_0^P - \Delta_0^1$.

In fact one can write

$$H = \sum_{\varpi \in \hat{\Delta}_1^P} c_\varpi \check{\varpi} - \sum_{\alpha \in \Delta_0^1} c_\alpha \check{\alpha} .$$

The hypotheses (i) and (ii) imply that $c_{\varpi} > 0$ and $c_\alpha \geq 0$. Now consider $\beta \in \Delta_0^P - \Delta_0^1$, of course $\beta(\check{\varpi}) \geq 0$ but since $\beta \notin \Delta_0^1$ at least one of the $\varpi \in \hat{\Delta}_1^P$ is not orthogonal to β; moreover $\beta(\check{\alpha}) \leq 0$ for all $\alpha \in \Delta_0^1$. and hence $\beta(H) > 0$. □

LEMMA 3.2.3. <u>Assume that</u> x and δx <u>are in</u> $\mathfrak{S}_P(T_0)$ <u>with</u> $\delta \in P$ <u>and that</u>

$$\alpha(H(x) - T) > 0 \qquad\qquad \forall \alpha \in \Delta_0^P - \Delta_0^1$$

<u>then provided</u> T <u>is sufficiently regular one has</u> $\delta \in P_1$.

This is a standard result in reduction theory, but we should maybe recall the proof. We may assume $\delta \in M$ the Levi component of P containing M_0, and consider the Bruhat decomposition of δ in M:

$$\delta = \nu \, w_s \, \pi$$

with $\gamma \in N_0^P = N_0 \cap M$, $\pi \in P_0 \cap M$ and w_s represents $s \in \Omega^M$ the Weyl group of M. Write $x = nak$ with $n \in \mathbf{N}_0$, $a \in \mathbf{M}_0$ and $k \in K$, then

$$H(\delta x) = (s . H(a)) . H(w_s n_1)$$

for some $n_1 \in \mathbf{N}_0$. But since $\delta x \in \mathfrak{S}_P(T_0)$ we know that

$\beta(H(\delta x)) > \beta(T_0)$ for any $\beta \in \Delta_0^P$, or more generally for any positive root of M. The factor $\beta(H(w_s n_1))$ is negative. Now if $s \notin \Omega^1$ the Weyl group of M^1 there is an $\alpha \in \Delta_0^P - \Delta_0^1$ such that $-\beta = s^{-1}\alpha$ is a negative root of M and then $\beta(sH(a))$ cannot be bounded from below independently of T. $\square$

The proposition follows from these two lemmas, the first one being applied to $H = H(\xi x) - T$ for some $\xi \in P_1$. $\square$

Thanks to the above proposition we see that $k_{\sigma}^T(x)$ is the sum over all pairs of parabolics $P_1 \subset P$ with $\varepsilon(P) = P$ of

$$\sum_{\delta \in P_1 \backslash G} (-1)^{a_P^{\varepsilon}} F_P^1(\delta x, T)\tau_1^P(H(\delta x) - T)\,\hat{\tau}_P(H(\delta x) - T)K_{P,\sigma}(\delta x, \delta x) \quad .$$

Recall that

$$\sum_{\{P_2 | P_1 \subset P \subset P_2\}} \sigma_1^2(H) = \tau_1^P(H)\hat{\tau}_P(H)$$

and define $\mathcal{H}_1^2(x)_{\sigma}^T$ to be the sum over all ε-invariant parabolics P such that $P_1 \subset P \subset P_2$ of

$$(-1)^{a_P^{\varepsilon}} F_P^1(x, T)\sigma_1^2(H(x) - T)K_{P,\sigma}(x, x) \quad .$$

Then obviously

$$k_{\sigma}^T(x) = \sum_{P_1 \subset P_2} \sum_{\delta \in P_1 \backslash G} \mathcal{H}_1^2(\delta x)_{\sigma}^T \quad .$$

To obtain the Theorem 3.1.2 all we need to prove is the

PROPOSITION 3.2.4. *Provided* T *is sufficiently regular*

$$\sum_{\sigma \in \mathcal{O}} \int_{P_1 \backslash G(\mathbb{A})^1} |H_1^2(x)_\sigma^T| dx$$

is finite.

This will be proved in the next lecture.

Erratum to Lecture 3

The proof of the Lemma 3.2.3 in the notes is incorrect and should be replaced by the following one. We first recall the statement.

LEMMA 3.2.3. Assume x and δx are in $\mathfrak{S}_P(T_0)$ with $\delta \in P$ and that

$$\alpha(H(x) - T) > 0 \qquad \forall \alpha \in \Delta_0^P - \Delta_0^1$$

then provided T is sufficiently regular one has $\delta \in P_1$.

We are free to modify δ and x by elements in P_0, on the left, so that we need only to consider the case $\delta = w_s$ where w_s represents $s \in \Omega^M$ the Weyl group of M. We have

$$H(\delta x) = H(w_s x) = sH(x) + H(w_s n)$$

if $x = ank$ with $a \in \mathbf{M}_0$, $n \in \mathbf{N}_0$ and $k \in K$. There exists $T_1 \in \mathfrak{a}_0^+$ such that for any $n \in \mathbf{N}_0$ and any $s \in \Omega$

$$X_s = s^{-1}H(w_s n) + T_1 - s^{-1}T_1$$

is a positive linear combination of coroots $\check{\beta}$ of M such that $\beta > 0$ and $s\beta < 0$ (cf. Lemma 6.3 of Lecture 6). Let V_s^+ be the positive linear span of those roots β and V_s^{++} be the subcone of the $\lambda \in V_s^+$ such that moreover $\lambda(\check{\beta}) > 0$ for all those β. In particular $\lambda(X_s) \geq 0$

and since

$$H(x) - T_1 + X_s = s^{-1}(H(w_s x) - T_1)$$

we have

$$\lambda(H(x) - T_1) \leq s\lambda(H(w_s x) - T_1) \quad .$$

We have assumed that $w_s x \in \mathfrak{S}_P(T_0)$ and hence

$$s\lambda(H(w_s x) - T_0) \leq 0$$

since $s\lambda$ is a positive linear combination of negative roots. This yields the following inequality:

$$\lambda(H(x)) \leq \lambda(T_1) - s\lambda(T_1 - T_0) \quad .$$

By hypothesis we may write

$$H(x) = \sum_{\alpha \in \Delta_0^P} h_\alpha \check{\varpi}_\alpha + H_P$$

with $H_P \in \mathfrak{a}_P$, $h_\alpha > \alpha(T_0)$ for all $\alpha \in \Delta_0^P$ and $h_\alpha > \alpha(T)$ for all $\alpha \in \Delta_0^P - \Delta_0^1$. Since λ is a positive linear combination of positive roots $\lambda(\check{\varpi}_\alpha) \geq 0$, and we get

$$\sum_{\alpha \in \Delta_0^P - \Delta_0^1} h_\alpha \lambda(\check{\varpi}_\alpha) \leq (\lambda - s\lambda)(T_1 - T_0) \quad .$$

If T is sufficiently regular this is possible only if $\lambda(\check{\varpi}_\alpha) = 0$ for all $\alpha \in \Delta_0^P - \Delta_0^1$ and all $\lambda \in V_s^{++}$. This implies that $V_s^+ \subset (\mathfrak{n}_0^1)^*$ so that $\check{\beta} \in \mathfrak{a}_0^1$ whenever $\beta > 0$ and $s\beta < 0$. This is the case only if $s \in \Omega^1$ the Weyl group of M_1. □

Lecture 4

ABSOLUTE CONVERGENCE OF THE COARSE $\mathcal{O}$-EXPANSION

J.-P. Labesse

4.1. Where the alternating sum is used.

We fix once for all a pair of parabolics $P_1 \subset P_2$. We consider P such that $P_1 \subset P \subset P_2$. We need the

LEMMA 4.1.1. *If* $\sigma_1^2(H(x) - T)F_P^1(x, T) \neq 0$ *then there exist* $\delta \in P_1$ *such that*

(i) $\alpha(H(\delta x) - T) > 0 \qquad \forall \alpha \in \Delta_0^2 - \Delta_0^1$

(ii) $\alpha(H(\delta x) - T_0) > 0 \qquad \forall \alpha \in \Delta_0^2$.

First of all, for any $\delta \in P_1$ we have

$$\alpha(H(\delta x) - T) > 0 \qquad \forall \alpha \in \Delta_1^2$$

if $\sigma_1^2(H(\delta x) - T) = \sigma_1^2(H(x) - T) \neq 0$. Now choose $\delta \in P_1$ such that $\delta x \in \mathfrak{S}_P^1(T_0, T)$ then

$$\varpi(H(\delta x) - T) \leq 0 \qquad \forall \varpi \in \hat{\Delta}_0^1 \ ;$$

the Lemma 3.2.2 above yields the assertion (i). We know moreover that

$$\alpha(H(\delta x) - T_0) > 0 \qquad \forall \alpha \in \Delta_0^P$$

but Δ_0^P contains Δ_0^1 and since $T - T_0 \in \mathfrak{a}_0^+$ assertion (ii) follows. □

COROLLARY 4.1.2. *If* $P_1 \subset P \subset P_2$

$$\sigma_1^2 F_P^1 = \sigma_1^2 F_2^1 \ . \ \square$$

Now assume that P is ε-invariant; let M be the unique Levi-component of P containing M_0 and N the unipotent radical. Let $\gamma \in P$, $n \in \mathbb{N}$, $x \in \mathbf{G}$, we need the

LEMMA 4.1.3. Provided T is sufficiently regular, and

$$\sigma_1^2(H(x) - T)F_2^1(x, T)\phi(x^{-1}n^{-1}\gamma\varepsilon(x)) \neq 0$$

$\gamma \in R$ the smallest ε-invariant parabolic containing P_1.

Our assertion is invariant by the transformation $x \longmapsto \delta x$ for $\delta \in P_1$ and hence we may assume that x satisfies the following inequalities, if the above expression is not zero:

(i) $\alpha(H(x) - T) > 0 \qquad \forall \alpha \in \Delta_0^2 - \Delta_0^1$

(ii) $\alpha(H(x) - T_0) > 0 \qquad \forall \alpha \in \Delta_0^2$.

Now write $x = n^* n_* mak$ with $n^* \in \mathbb{N}_2$, $n_* \in \mathbb{N}_0^2 = \mathbb{N}_0 \cap \mathbb{M}_2$, $m \in \mathbb{M}_0^1$, $a \in A_0(\mathbb{R})^o$ and $k \in K$. Since we are free to modify n we may assume $n^* = 1$; since we may change γ to $\delta^{-1}\gamma\varepsilon(\delta)$ with $\delta \in P_0$ we may assume that $n_* m$ remains in a compact set. Now since x verifies the inequalities (i) and (ii) we see that $a^{-1}n_*ma = a^{-1}n_*a.m$ can be assumed to remain in a fixed compact set. Since ϕ is compactly supported all we need to prove is the

LEMMA 4.1.4. Let U be a compact in $\mathbb{M}$, assume that H(a) satisfies the inequalities (i) and (ii) then provided T is sufficiently regular $a^{-1}\gamma\varepsilon(a) \in U$ and $\gamma \in M$ implies $\gamma \in P_1$.

Consider the Bruhat decomposition of γ in M:

$$\gamma = \nu w_s \pi$$

with $\gamma \in N_0^P = N_0 \cap M$, $\pi \in P_0 \cap M$ and $s \in \Omega^M$ the Weyl group of M. Let ϖ be a dominant weight of M. For some integer d there is a rational representation ρ of M, of highest weight $d\varpi$ and with highest weight vector v. We have

$$\|\rho(a^{-1}\gamma\varepsilon(a))v\| = \|\rho(a^{-1}\nu a w_s)v\| . \Lambda(a)$$

with

$$\Lambda(a) = e^{d \left\langle \varpi, \varepsilon . H(a) - s . H(a) \right\rangle} \quad .$$

But $\|\rho(a^{-1}\nu a w_s)v\|$ is bounded from below by a constant times $\|\rho(w_s)v\|$; so if $a^{-1}\gamma\varepsilon(a)$ remains in the compact U there exists a real c independent of T such that

$$\Lambda(a) \leq e^c \quad .$$

Now if $\varpi = \varepsilon . \varpi$ we simply have

$$\left\langle \varpi, H(a) - s . H(a) \right\rangle \leq c \quad .$$

We can write

$$H(a) = \sum_{\varpi_\alpha \in \hat{\Delta}_0^P} \lambda_\alpha \check{\varpi}_\alpha + H_P$$

where $H_P \in \mathfrak{a}_P$ the intersection of the kernels of the $\alpha \in \Delta_0^P$, and where the λ_α are subjected to the inequalities

$$\lambda_\alpha > \alpha(T_0) \qquad \forall \alpha \in \Delta_0^P$$

$$\lambda_\alpha > \alpha(T) \qquad \forall \alpha \in \Delta_0^P - \Delta_0^1 \quad .$$

Hence if T is sufficiently large this implies

$$\left\langle \varpi, \varpi_\alpha - s\varpi_\alpha \right\rangle = 0$$

for any $\alpha \in \Delta_0^P - \Delta_0^1$ and any ε-invariant ω. Now assume $\varpi = \sum_{r=0}^{\ell-1} \varepsilon^r.\varpi_0$; since $\varpi_\alpha - s\varpi_\alpha$ is a sum of positive roots we also have

$$\left\langle \varpi_0 , \varpi_\alpha - s\varpi_\alpha \right\rangle = 0$$

for any $\varpi_0 \in \hat{\Delta}_0^P$, and hence $\varpi_\alpha - s\varpi_\alpha = 0$ for all $\alpha \in \Delta_0^P - \Delta_0^1$. This implies $s \in \Omega^1$ the Weyl group of M_1, and hence $\gamma \in P_1$. □□

Recall that R is the minimal ε-invariant parabolic containing P_1. As usual let M_R be the Levi-component containing M_0 and N_R the unipotent radical. Corollary 4.1.2 and Lemma 4.1.3 show that

$$H_1^2(x)_{\sigma}^T = F_2^1(x, T)\sigma_1^2(H(x) - T)\omega(x)\varphi_{\sigma}(x)$$

where $\varphi_\sigma(x)$ is the sum over $\gamma \in M_R \cap \sigma$ of

$$\sum_{\substack{\varepsilon(P)=P \\ P_1 \subset P \subset P_2}} (-1)^{a_P^\varepsilon} \int_{(N)} \sum_{\eta \in N_R} \phi(x^{-1}n^{-1}\eta\gamma\varepsilon(x))\,dn \quad .$$

Notice that this expression is non-zero only if $R \subset P_2$.

The exponential mapping is an isomorphism of N_R onto its Lie algebra n_R. Let ψ be a non-trivial additive character of $\mathbf{Q}\backslash\mathbb{A}$. Using the Poisson summation formula we get that

$$\sum_{\eta \in N_R} \phi(x^{-1} n \eta \gamma \varepsilon(x)$$

equals

$$\sum_{Y \in n_R^*} \int_{n_R(\mathbb{A})} \phi(x^{-1} n \ \exp(X) \gamma \varepsilon(x)) \psi(\langle X, Y \rangle) dX$$

where n_R^* is the dual of n_R (as a $\mathbf{Q}$-vector space).

By integration over (N) the contributions of the Y that are non-trivial on $N(\mathbb{A})$ vanish, and we are left with a sum over $n_{R,P}^*$, the subspace of n_R^* orthogonal to n_P the Lie algebra of N.

To take care of the alternating sum over P we need the

LEMMA 4.1.5. <u>Let</u> Q <u>and</u> R <u>be two invariant parabolics then</u>

$$\sum_{\left\{P \,\middle|\, \begin{array}{c} R \subset P \subset Q \\ \varepsilon(P)=P \end{array}\right\}} (-1)^{a_P^\varepsilon} = \begin{array}{l} 0 \text{ if } Q \neq R \\ 1 \text{ if } Q = R \end{array}$$

An ε-invariant parabolic P between R and Q is defined by an ε-invariant subset S of Δ_R^Q. The number of orbits of E in S is $a_R^\varepsilon - a_P^\varepsilon$. The lemma is an immediate consequence of the binomial formula for $(1-1)^d$. $\square$

Let $\tilde{n}_{1,2}$ be the set of elements in n_R^* that belong to one and only one $n_{R,P}^*$ for $P_1 \subset R \subset P \subset P_2$ and $\varepsilon(P) = P$. Using the previous lemma we see that

$$\varphi_{\sigma}(x) = \sum_{\gamma \in M_R \cap \sigma} \sum_{Y \in \tilde{n}_{1,2}} \hat{\phi}(x, Y, \gamma)$$

where

$$\hat{\phi}(x, Y, z) = \int_{n_R(\mathbb{A})} \phi(x^{-1}\exp(X) z\varepsilon(x))\psi(\langle X, Y\rangle)dX \quad .$$

Let Q be the maximal ε-invariant parabolic contained in P_2. Let $p \in \mathbb{P}_1$; since $P_1 \subset R$ normalizes N_R we have

$$\hat{\phi}(p, Y, \gamma) = \hat{\phi}(1, Ad^*(p)Y, p^{-1}\gamma\varepsilon(p))\delta_R(p)$$

where $\delta_R(p)$ is the absolute value of the determinant of $Ad(p)$ on $n_R(\mathbb{A})$. Now let $p \in \mathbb{P}_1 \cap \mathbb{G}^1$. We may write $p = n^* n_* m a$ with $n^* \in \mathbb{N}_2$, $n_* \in \mathbb{N}_1^2$, $m \in \mathbb{M}_1^1$ and $a \in A_1(\mathbb{R})^{\circ}$. Since N_2 and $\varepsilon(N_2)$ are in N_Q and $Y \in \tilde{n}_{1,2}$ then $\hat{\phi}$ is independent of n^*. Choose a compact set $\omega_1 \subset \mathbb{N}_1^2$ such that $N_1^2\omega_1 = \mathbb{N}_1^2$. There exists a compact set $\omega_2 \subset \mathbb{M}_1^1$ such that if m is such that $F_2^1(ma, T) = 1$

then $m \in M_1 \cdot \omega_2$; moreover if $\sigma_1^2(H(a) - T) = 1$ we have $\alpha(H(a)) > 0$ for all $\alpha \in \Delta_1^2$; this implies that $a^{-1}\omega_1\omega_2 a = a^{-1}\omega_1 a\omega_2$ remains in a fixed compact set $\omega_3 \subset \mathbf{N}_1^2\mathbf{M}_1^1$.

From this we conclude that the integral over $P_1\backslash \mathbf{G}^1$ of

$$\sum_{\sigma \in \mathcal{O}} |H_1^2(x)^T|$$

is bounded by

$$\int_{p \in \omega_3 K} \left(\int_{A_1(\mathbb{R})^o \cap \mathbb{G}^1} \Xi(ap)\delta_1(a)^{-1} da \right) dp$$

where

$$\Xi(x) = F_2^1(x, T)\sigma_1^2(H(x) - T) \sum_{\gamma \in M_R} \sum_{Y \in \tilde{n}_{1,2}} |\hat{\phi}(x, Y, \gamma)| .$$

4.2. Final estimates.

Let $\mathfrak{a}_1$ be the Lie algebra of $A_1(\mathbf{R})^o$ and $\mathfrak{a}_1^\varepsilon$ be the set of ε-fixed vectors in $\mathfrak{a}_1$. Let $\mathfrak{b}_1$ be the orthogonal complement of $\mathfrak{a}_1^\varepsilon$ in $\mathfrak{a}_1$. Since ε is of finite order on $\mathfrak{a}_0$ we may and shall assume that the scalar product on $\mathfrak{a}_0$ is ε-invariant. Let π_1 be the orthogonal projection from $\mathfrak{a}_0$ onto $\mathfrak{a}_1$. We have the

LEMMA 4.2.1. Let $H \in \mathfrak{a}_1$; the projection on $\mathfrak{a}_1$ of $\varepsilon(H) - H$ is an injective map from $\mathfrak{b}_1$ into $\mathfrak{a}_1$.

Assume $\pi_1(\varepsilon(H) - H) = 0$, since $H \in \mathfrak{a}_1$ we have $\pi_1(\varepsilon(H)) = H$;

but ε preserves the scalar products and hence $\varepsilon(H) = H$. □

At the end of the preceding section we introduced a function Ξ on $\mathbf{G}$. Assume $\Xi(ap) \neq 0$ with $p \in \omega_3 K$ and $a \in A_1(\mathbf{R})^o$. Since we assume

$$\hat{\phi}(ap, Y, \gamma) = \hat{\phi}(p, Ad^*(a)Y, a^{-1}\gamma\varepsilon(a))\delta_R(a)$$

is not zero, this implies that $a^{-1}\gamma\varepsilon(a)$ remains in a compact set $\omega_4 \subset \mathbf{M}_R$ independent of $p \in \omega_3 K$ and $Y \in n_R^*$ (but depending on the support of ϕ). We moreover assume that $F_2^1(ap, T)\sigma_1^2(H(a) - T) \neq 0$, and then assumptions of Lemma 4.1.4 are fulfilled. This implies that for sufficiently regular T, $\gamma \in P_1 \cap M_R$.

Now for such a γ we have

$$a^{-1}\gamma\varepsilon(a) = \gamma_1 . a^{-1}\eta a . a^{-1}\varepsilon(a)$$

for some $\gamma_1 \in M_1$ and $\eta \in N_1 \cap M_R$. Since $a^{-1}\gamma\varepsilon(a)$ remains in a compact set of $\mathbf{M}_R \cap \mathbf{P}_1$, the projection on $A_1(\mathbf{R})^o$ of $a^{-1}\varepsilon(a)$ must also remain in a compact set. But if $a = a_1 b$ where $a_1 = \exp H_1$ with $H_1 \in \mathfrak{a}_1^\varepsilon$ and $b = \exp H_2$ with $H_2 \in \mathfrak{b}_1$ we have

$$a^{-1}\varepsilon(a) = b^{-1}\varepsilon(b) = \exp(\varepsilon(H_2) - H_2) \quad .$$

Using Lemma 4.2.1 we conclude that b has to remain in a compact set ω_5. Moreover since $\mathfrak{a}_1^\varepsilon \subset \mathfrak{a}_R$ we have

$$a^{-1}\gamma\varepsilon(a) = \gamma_1 . b^{-1}\eta b . b^{-1}\varepsilon(b) \in \omega_4$$

with $b \in \omega_5$. We conclude that the set of γ_1 and η_1 (and hence of γ) that may occur is finite, and in the definition of Ξ the sum over $\gamma \in M_R$ may be restricted to a sum over a finite set E. For $Y \in n_R^*(\mathbb{A})$ we define

$$\theta(Y) = \int_{p \in \omega_5 . \omega_3 . K} \sum_{\gamma \in E} |\hat{\phi}(p, Y, \gamma)| dp \quad .$$

Since $\delta_R(a) = \delta_1(a)$ on A_R all we need to prove is that, given a compact set $\omega_6 \subset \mathfrak{a}_0$, the integral

$$\int_{\mathfrak{z}^\varepsilon \setminus \mathfrak{a}_1^\varepsilon} \sigma_1^2(H-X) \sum_{Y \in \tilde{n}_{1,2}} \theta(\mathrm{Ad}^*(\exp H)Y) dH$$

is convergent, with an upper bound independent of $X \in \omega_6$. (Here $\mathfrak{z}^\varepsilon$ is the ε-fixed part of the Lie algebra of the split part of the center of G.)

The space $\mathfrak{a}_1^\varepsilon = \mathfrak{a}_R^\varepsilon$ can be further decomposed into a sum

$$\mathfrak{a}_1^\varepsilon = (\mathfrak{a}_R^Q)^\varepsilon \oplus \mathfrak{a}_Q^\varepsilon$$

where Q is the maximal ε-invariant parabolic contained in P_2. Let $H = H_1 + H_2$ be the associated decomposition of $H \in \mathfrak{a}_R^\varepsilon$; we have the

LEMMA 4.2.2. Assume that $H \in \mathfrak{a}_1^\varepsilon$ and $X \in \omega_6$ are such that $\sigma_1^2(H-X) = 1$. Then there exist a constant c independent of X such that

$$\|H_2\| \leq c(1 + \|H_1\|) \quad .$$

Any $\alpha \in \Delta_R - \Delta_R^Q$ is the restriction of some $\alpha' \in \Delta_1 - \Delta_1^2$ and hence we have

$$\alpha(H_2) = \alpha(H-X) - \alpha(H_1) + \alpha(X) < -\alpha(H_1) + \alpha(X) < -\alpha(H_1) + c_1$$

for some constant c_1. For any $\varpi \in \hat{\Delta}_2$ we have

$$\varpi(H_2) = \varpi(H) > \varpi(X)$$

since $\mathfrak{a}_R^Q$ is orthogonal to $\varpi \in \hat{\Delta}_2 \subset \hat{\Delta}_Q$. Now H and H_2 are ε-invariants, and then the same inequalities hold if we replace ϖ by $\varepsilon^r\varpi$ and X by $\varepsilon^{-r}X$. But any $\varpi_1 \in \hat{\Delta}_Q$ is of the form $\varepsilon^r\varpi$ for some integer r and some $\varpi \in \hat{\Delta}_2$ and hence

$$\varpi_1(H_2) > c_2$$

for any $\varpi_1 \in \hat{\Delta}_Q$ and some constant c_2. □

COROLLARY 4.2.3. <u>If</u> $H = H_1 + H_2$ <u>as above, the set of</u> $H_2 \in \mathfrak{a}_Q^\varepsilon$ <u>such that</u> $\sigma_1^2(H_1+H_2-X) = 1$ <u>for some</u> $X \in \omega_6$ <u>has a volume bounded by a polynomial in</u> $\|H_1\|$. □

Let V be the cone in $\mathfrak{z}^\varepsilon \setminus (\mathfrak{a}_R^Q)^\varepsilon$ defined by $\alpha(H) > 0$ for all $\alpha \in \Delta_R^Q$. If $a_2 = \exp H_2$ with $H_2 \in \mathfrak{a}_Q^\varepsilon$ then

$$Ad(a_2)^* Y = Y$$

for $Y \in \tilde{n}_{1,2} \subset n_{R,Q}^*$. Then all that is left to prove is that

$$\|Y\| \geq \sup_{\lambda \in \Lambda} \|Y_\lambda\|$$

then if $n(Y)$ is the number of λ such that $Y_\lambda \neq 0$ we have

$$\|Y\|^{n(Y)} \geq \prod_{\substack{\lambda \in \Lambda \\ Y_\lambda \neq 0}} \|Y_\lambda\| .$$

Let L be a lattice in $n^*_{R,Q} \otimes \mathbb{R}$, then there is a constant c_1 such that if $Y \in L - \{0\}$

$$\|Y\|^n \geq c_1 \|Y\|^{n(Y)}$$

with n the cardinal of Λ. Then for $Y \in \tilde{n}_{1,2} \cap L$ and $H \in V$ we have

$$\|Ad^*(\exp H) Y\|^n \geq c_1 e^{c_2 \|H\|} \prod_{\substack{\lambda \in \Lambda \\ Y_\lambda \neq 0}} \|Y_\lambda\| ,$$

for some strictly positive constant c_2. Since the function θ is obtained by integration over p in a compact set of the absolute value of a Schwartz-Bruhat function on $n^*_{R,Q} \otimes \mathbf{A}$ depending smoothly on p, the convergence is now an easy exercise left to the reader. $\square$

$$\int_V \|H_1\|^r \sum_{Y \in \tilde{n}_{1,2}} \theta(Ad^*(H_1)Y)\,dH_1$$

is finite for any positive real number r. To prove this we must recall the definition of $\tilde{n}_{1,2}$: it is the subset of the $Y \in n^*_{R,Q}$ that belong to one and only one $n^*_{R,P}$ with $R \subset P \subset Q$ and P ε-invariant; in other words

$$\tilde{n}_{1,2} = n^*_{R,Q} - \bigcup_{\substack{R \subset P \subsetneq Q \\ \varepsilon(P)=P}} n^*_{R,P} .$$

The space $n^*_{R,Q}$ can be decomposed into root subspaces under the action of $(\mathfrak{a}^Q_R)^\varepsilon$:

$$n^*_{R,Q} = \bigoplus_{\lambda \in \Lambda} n^*_\lambda .$$

The set Λ is in natural bijection with the orbits of E in the roots of $\mathfrak{a}^Q_R$ in $n^*_{R,Q}$. Let ϖ_α be a weight in $\hat{\Delta}^Q_R$ and define

$$\varpi_{\bar{\alpha}} = \frac{1}{\ell} \sum_{r=0}^{\ell-1} \varepsilon^r \varpi_\alpha$$

where $\bar{\alpha}$ represents the orbit of α under E. The $\varpi_{\bar{\alpha}}$ are a basis of $(\mathfrak{a}^Q_R)^\varepsilon$. An element $Y = \sum_{\lambda \in \Lambda} Y_\lambda$ in $n^*_{R,Q}$ is in $\tilde{n}_{1,2}$ if and only if for any $\bar{\alpha}$ there exist $\lambda \in \Lambda$ such that $Y_\lambda \neq 0$ and $\langle \lambda, \varpi_{\bar{\alpha}} \rangle \neq 0$ (and hence strictly positive).

Choose a norm on $n^*_{R,Q} \otimes \mathbb{R}$ such that

Lecture 5

$\mathcal{O}$-EXPANSION AND WEIGHTED ORBITAL INTEGRALS

J.-P. Labesse

5.1. The second form of the $\mathcal{O}$-expansion.

Let P be an ε-invariant parabolic and $\mathfrak{o}$ an ε-semisimple-conjugacy class. For $\gamma \in \mathfrak{o} \cap M$ let $\gamma' = \gamma\varepsilon$ and $N(\gamma'_s)$ the centralizer in N of the semisimple part γ'_s of γ'. We introduce

$$N(\phi, x, \gamma') = \int_{\mathbf{N}(\gamma'_s)} \omega(x)\phi(x^{-1}n^{-1}\gamma\varepsilon(x))dn$$

and

$$j_{P,\mathfrak{o}}(x) = \sum_{\gamma \in M \cap \mathfrak{o}} \sum_{\eta \in N(\gamma'_s)\backslash N} N(\phi, \eta x, \gamma') \quad .$$

Using Lemma 3.1.1 we see that the series are in fact finite sums since ϕ is compactly supported. We now define a truncated term by

$$j^T_{\mathfrak{o}}(x) = \sum_{\varepsilon(P)=P} \sum_{\delta \in P\backslash G} (-1)^{a^{\varepsilon}_P} \hat{\tau}_P(H(\delta x) - T) j_{P,\mathfrak{o}}(\delta x) \quad .$$

Here also the series are finite sums; this is a consequence of Lemma 2.1. The aim of this section is to prove the

THEOREM 5.1.1. (i) For a sufficiently regular T

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{Ⓖ^1} |j^T_{\mathfrak{o}}(x)|dx$$

is finite.

(ii) For any $\sigma \in \mathcal{O}$

$$\int_{Ⓖ^1} j_{\sigma}^{T}(x)\,dx = \int_{Ⓖ^1} k_{\sigma}^{T}(x)\,dx \quad .$$

The proof of statement (i) is, with minor modifications, the same as the proof of Theorem 3.1.2 and will not be repeated (see Lectures 3 and 4).

To prove the statement (ii) we need the

LEMMA 5.1.2.

$$K_{P,\sigma}(x, x) = \int_{Ⓝ} j_{P,\sigma}(nx)\,dn \quad .$$

Recall that

$$K_{P,\sigma}(x, x) = \sum_{\gamma \in M \cap \sigma} \omega(x) \int_{\mathbb{N}} \phi(x^{-1} n \gamma \varepsilon(x))\,dx \quad .$$

The continuous analogue of Lemma 3.1.1 shows that

$$\int_{\mathbb{N}} \omega(x)\phi(x^{-1}n^{-1}\gamma\varepsilon(x))\,dn = \int_{\mathbb{N}(\gamma_s')\backslash \mathbb{N}} N(\phi, nx, \gamma')\,dn \quad .$$

The lemma is now an immediate consequence of the definition of $j_{P,\sigma}$. □

COROLLARY 5.1.3. <u>Given</u> $P_1 \subset P$ <u>we have</u>

$$\int_{Ⓝ_1} j_{P,\sigma}(nx)\,dn = \int_{Ⓝ_1} K_{P,\sigma}(nx, nx)\,dn \quad .$$

We need only to remark that $P \supset N_1 \supset N$. □

In Lecture 3 we introduced a function $\mathcal{H}_1^2(x)_{\sigma}^T$ such that

$$\int_{\textcircled{G}^1} k_{\sigma}^T(x)\,dx = \sum_{P_1 \subset P_2} \int_{P_1\backslash \mathbf{G}^1} \mathcal{H}_1^2(x)_{\sigma}^T dx \quad .$$

If we substitute $j_{P,\sigma}(x)$ for $K_{P,\sigma}(x, x)$ in the definition of $\mathcal{H}_1^2(x)_{\sigma}^T$ we obtain a function $\mathcal{J}_1^2(x)_{\sigma}^T$. Then Corollary 5.1.3 tells us that

$$\int_{\textcircled{N}_1} \mathcal{H}_1^2(nx)_{\sigma}^T dn = \int_{\textcircled{N}_1} \mathcal{J}_1^2(nx)_{\sigma}^T dn$$

and the assertion (ii) in the above theorem follows from the fact that integration over $P_1\backslash \mathbf{G}^1$ can be seen as an integration over $\textcircled{N}_1$ followed by an integration over $P_1\mathbf{N}_1\backslash \mathbf{G}^1$. □

Another variant of the $\mathcal{O}$-expansion will be of interest. Let P be an ε-invariant parabolic subgroup, the group E of connected components of G' acts on Δ_P and to each orbit $\bar{\alpha}$ we may attach an averaged weight $\varpi_{\bar{\alpha}}$:

$$\varpi_{\bar{\alpha}} = \frac{1}{\ell} \sum_{r=0}^{\ell-1} \varepsilon^r \varpi_\alpha$$

where α is any element in $\bar{\alpha}$. We define ${}_{\varepsilon}\hat{\tau}_P$ as the characteristic function of the $X \in \mathfrak{a}_0$ such that $\varpi_{\bar{\alpha}}(X) > 0$ for any $\alpha \in \Delta_P$. If we substitute ${}_{\varepsilon}\hat{\tau}_P$ for $\hat{\tau}_P$ in the definition of k_{σ}^T and j_{σ}^T we obtain new functions which we shall denote by ${}_{\varepsilon}k_{\sigma}^T$ and ${}_{\varepsilon}j_{\sigma}^T$: their definition makes sense since the analogue of Lemma 2.1 is available. We may reproduce the proofs in Lectures 2, 3, 4 with minor changes; we simply have to replace from time to time weights by averaged weights and σ_1^2 by ${}_\varepsilon\sigma_1^2$ the characteristic functions of the H such that $\alpha(H) > 0$ if $\alpha \in \Delta_1^2$, $\alpha(H) \leq 0$ if $\alpha \in \Delta_1 - \Delta_1^2$ and $\varpi_{\bar{\alpha}}(H) > 0$ if $\alpha \in \Delta_Q$ where Q is the maximal ε-invariant parabolic subgroup contained in P_2 if $Q \supset P_1$, and ${}_\varepsilon\sigma_1^2 = 0$ if there is no ε-invariant P between P_1 and P_2. More details will be given in Lecture 9.

5.2. Conjugacy classes and parabolic subgroups.

Let P be a (not necessarily standard) parabolic subgroup and let $P' = N_{G'}(P)$ be its normalizer in G'. We shall say that P' is a parabolic subgroup in G' if its projection on E, the group of connected components of G', is surjective.

LEMMA 5.2.1. Assume P' is a parabolic subgroup in G' whose neutral connected component P is standard, then $\varepsilon \in P'$.

By assumption there is an element $\varepsilon_1 \in P'$ which projects on ε_0 the given generator of E. We have $P_0 \subset P'$, let $P_1 = \varepsilon_1(P_0)$; this is a minimal parabolic subgroup and hence there exist $\delta_1 \in P$ such that $\delta_1 P_1 \delta_1^{-1} = P_0$. Then $\delta_1\varepsilon_1$ leaves P_0 invariant; so does ε and hence $\delta = \delta_1\varepsilon_1\varepsilon^{-1}$ normalizes P_0 and is an element of G so that $\delta \in P_0$ and $\varepsilon = \delta^{-1}\delta_1\varepsilon_1 \in P'$. □

Such parabolic subgroups in G' will be called standard; P' is standard if and only if P is standard and ε-invariant; moreover $P' \supset P'_0$. Let M be the Levi component of P containing M_0, then M

and ε generate a subgroup M' in P' which will be called "the" Levi component of P'. Let A be the split component of the center of M, then A^{ε} is the split component of the center of M'. The weights of A^{ε} in G are the orbits under E of the weights of A; since E preserves positivity of weights, the centralizers of A and A^{ε} in G (which are connected) are equal to M. The centralizer of A^{ε} in G' is M'.

Consider $\gamma_1 \in G$ such that $\gamma_1' = \gamma_1 \varepsilon$ is semisimple and P_1' a standard parabolic subgroup of G' such that $\gamma_1' \in M_1'$ its Levi component and such that moreover no strictly smaller standard parabolic subgroup contains an M_1'-conjugate of γ_1' in its Levi component.

Let $A_1\varepsilon$ be the split component of the center of M_1'.

LEMMA 5.2.2. <u>The torus</u> A_1^{ε} <u>is a maximal split torus in</u> $G'(\gamma_1')$ <u>the centralizer of</u> γ_1' <u>in</u> G'.

Let B be a maximal split torus in $G'(\gamma_1')$. Since γ_1' is semi-simple $G'(\gamma_1')$ is reductive and up to conjugacy in $G'(\gamma_1')$ we may assume $A_1^{\varepsilon} \subset B$. Let M_2 (resp. M_2') be the centralizer of B in G (resp. G'), we have $M_2' \subset M_1'$. Up to conjugacy inside M_1 we may assume that M_2 is the Levi component of a standard parabolic subgroup $P_2 \subset P_1$ of G. Since γ_1' commutes with B we have $\gamma_1' \in M_2'$ and γ_1' normalizes N_2 the unipotent radical of P_2 (γ_1' fixes the weights of B) and hence $\gamma_1' \in P_2'$. This implies that P_2' projects surjectively on E. The minimality property of P_1' implies $P_1' = P_2'$; moreover $\varepsilon \in M_1' = M_2'$ so that $B = B^{\varepsilon} = A_1^{\varepsilon}$. □

COROLLARY 5.2.3. Up to conjugacy M'_1 is well defined by $c(\gamma'_1)$ the G-conjugacy class of γ'_1.

Given $\gamma'_1 \in M'_1 \subset P'_1$ and $\gamma'_2 \in M'_2 \subset P'_2$ minimal as above we know that A_1^{ε} and A_2^{ε} are maximal split tori in $G'(\gamma'_1)$ and $G'(\gamma'_2)$. If γ'_1 and γ'_2 are conjugate then A_1^{ε} and A_2^{ε} are also conjugate and the same is true for the M'_i. □

COROLLARY 5.2.4. Given P' a standard parabolic subgroup of G' with Levi component M' and $\gamma' \in M' \cap C(\gamma'_1)$ there exists a standard parabolic P'_2 of G' associated with P'_1 such that $P'_2 \subset P'$ and $m\gamma' m^{-1} \in M'_2$ for some $m \in M'$. □

Given P'_1 and P'_2 as above, let $\mathfrak{a}_i$ be the Lie algebra of $A_i(\mathbf{R})^o$. Let us denote as usual by $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ the set of restrictions to $\mathfrak{a}_1$ of elements $s \in \Omega$, the Weyl group of G, such that $s(\mathfrak{a}_1) = \mathfrak{a}_2$. Given $\sigma \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ there exist a unique element $s \in \Omega$ such that s induces σ and such that moreover $s^{-1}\alpha > 0$ for all $\alpha \in \Delta_0^2$; it is the element with minimal length in the class σ. This provides us with an injective map from $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ into Ω. We shall identify $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ with its image.

Let us denote by $\Omega(\mathfrak{a}_1^{\varepsilon}, \mathfrak{a}_2^{\varepsilon})$ the set of restrictions to $\mathfrak{a}_1^{\varepsilon}$ of elements $s \in \Omega$ such that $s(\mathfrak{a}_1^{\varepsilon}) = \mathfrak{a}_2^{\varepsilon}$. Since M_i is the centralizer of A_i^{ε} (in G) such an s defines an element in $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ and hence $\Omega(\mathfrak{a}_1^{\varepsilon}, \mathfrak{a}_2^{\varepsilon})$ may be regarded as a subset of $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ and be identified with a subset of Ω.

LEMMA 5.2.5. $\Omega(\mathfrak{a}_1^\varepsilon, \mathfrak{a}_2^\varepsilon)$ <u>is the set of</u> $s \in \Omega$ <u>such that</u>

(i) $s(\mathfrak{a}_1) = \mathfrak{a}_2$

(ii) $s^{-1}(\alpha) > 0 \qquad \forall \alpha \in \Delta_0^2$

(iii) $\varepsilon s = s\varepsilon$.

The first two conditions define $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$; if an element satisfies the three conditions it clearly defines an element in $\Omega(\mathfrak{a}_1^\varepsilon, \mathfrak{a}_2^\varepsilon)$. Conversely if $s(\mathfrak{a}_1^\varepsilon) = \mathfrak{a}_2^\varepsilon$ and $s^{-1}(\alpha) > 0$ for all $\alpha \in \Delta_0^2$ the same is true for $s_1 = \varepsilon s \varepsilon^{-1}$ since P_2 is ε-invariant. Moreover s_1 and s have equal restrictions to $\mathfrak{a}_1^\varepsilon$ and hence equal restrictions to $\mathfrak{a}_1$. This implies $s = s_1$. □

Given P' a standard parabolic subgroup of G', let us denote by $\tilde{\Omega}(\mathfrak{a}_1^\varepsilon, P')$ the set of elements $s \in \Omega$ such that there exists a parabolic subgroup $P_2' \subset P'$ standard in G' with $s(\mathfrak{a}_1^\varepsilon) = \mathfrak{a}_2^\varepsilon$. The Weyl group of M, denoted by Ω^M, acts on the left on $\tilde{\Omega}(\mathfrak{a}_1^\varepsilon, P')$ and each class $\sigma \in \Omega^M \backslash \tilde{\Omega}(\mathfrak{a}_1^\varepsilon, P')$ contains a unique element s such that $s^{-1}\alpha > 0$ for any $\alpha \in \Delta_0^P$. As usual s is the element of minimal length in σ. Thanks to Lemma 5.2.5 we see that such an s commutes with ε. We shall identify $\Omega^M \backslash \tilde{\Omega}(\mathfrak{a}_1^\varepsilon, P')$ with the set $\Omega(\mathfrak{a}_1^\varepsilon, P')$ of those s in Ω.

We can now describe rather explicitly the set $M' \cap c(\gamma_1')$. Given $\gamma' \in M' \cap c(\gamma_1')$ there exists $s \in \Omega(\mathfrak{a}_1^\varepsilon, P')$ and $m \in M$ such that

$$\gamma' = m^{-1} w_s \gamma_1' w_s^{-1} m$$

where $w_s \in G$ represents s. But s is not always uniquely defined by γ'; it defines only a double coset in Ω:

$$\Omega^M . s . \Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$$

where $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ is the subgroup of the $\sigma \in \Omega(\mathfrak{a}_1^\varepsilon, \mathfrak{a}_1^\varepsilon)$ such that

$$w_\sigma \gamma_1' w_\sigma^{-1} = m_1 \gamma_1' m_1^{-1}$$

for some $m_1 \in M_1$. The element $m \in M$ is defined by γ' and $w_s \gamma_1' w_s^{-1}$ up to an element in $M(w_s \gamma_1' w_s^{-1})$ its centralizer in M.

5.3. Tame semisimple conjugacy classes.

The aim of this section is to give a simple expression for $j_{\mathcal{O}}^T(x)$ when $\mathcal{O}$ contains only semisimple elements. Such classes will be called tame semisimple. Given such a class $\mathcal{O}$ and $\gamma \in \mathcal{O}$, then $\gamma' = \gamma\varepsilon$ is semisimple and for any parabolic subgroup P' of G' containing γ' we have $N(\gamma') = N(\gamma_1') = \{1\}$.

An element γ' defines a tame semisimple class if and only if its centralizer $G(\gamma')$, in G, contains no unipotent element. In particular, regular semisimple elements give rise to tame semisimple classes.

Let γ_1', P_1', M_1' be as in Lemma 5.2.2 with γ' conjugate to γ_1' (in G) and assume that $G(\gamma_1')$ contains no unipotent elements. Recall that A_1^ε is a maximal split torus in $G(\gamma_1')$, since $G(\gamma_1')$ contains no unipotent element the neutral component $G(\gamma_1')^o$ lies in the centralizer of A_1^ε that is M_1'. Hence $M_1(\gamma_1')$ is of finite index say $d(\gamma_1')$ in $G(\gamma_1')$.

More generally given P' standard in G' with Levi component M' such that $\gamma' \in M'$ let us denote by $d(M, \gamma')$ the index of $M(\gamma')$ in $G(\gamma')$.

Let $s \in \tilde{\Omega}(\mathfrak{a}_1^\varepsilon, P')$ be such that $\gamma' = m^{-1} w_s \gamma_1' w_s^{-1} m$ where w_s represents s and $m \in M$.

LEMMA 5.3.1. _The cardinality of the set_

$$\Omega^M \backslash \Omega^M . s . \Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$$

_is_ $d(M, \gamma)$.

Consider first the case where $\gamma' = \gamma_1'$, $s = 1$ and $P' = P_1'$, then all we have to prove is that the order of $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ is $d(\gamma_1')$ and this follows from the

LEMMA 5.3.2. _There is a natural map from_ $G(\gamma_1')$ _onto_ $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ _with kernel_ $M_1(\gamma_1')$.

An element $g \in G(\gamma_1')$ normalizes A_1^ε the center of $G(\gamma_1')^o$ and hence it normalizes M_1. Then g defines an element s_g of $\Omega(\mathfrak{a}_1^\varepsilon, \mathfrak{a}_1^\varepsilon)$ and since g commutes with γ_1' it lies in $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$. By the very definition of $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ this map is surjective and its kernel is $M_1 \cap G(\gamma_1') = M_1(\gamma_1')$. $\square$

We can now return to the general case. We need only to prove it when $\gamma' = \gamma_1'$, $w_s = 1$, $P_1' \subset P'$, in which case it amounts to saying that the index of $M_1(\gamma_1')$ in $M(\gamma_1')$ is the cardinality of $\Omega^M \cap \Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ which is clear. □

Given $\mathfrak{o}$ a tame semisimple class we have

$$j_{P,\mathfrak{o}}(x) = \sum_{\gamma \in M \cap \mathfrak{o}} \sum_{\eta \in N} \omega(x)\phi(x^{-1}\eta^{-1}\gamma\varepsilon(\eta x))$$

since $N(\gamma_s') = N(\gamma') = \{1\}$. Now since in such a case $\mathfrak{o}$ is the twisted conjugacy class of some γ_1 with $\gamma_1' = \gamma_1\varepsilon$ semisimple in M_1', minimal as above, we may use the description of $c(\gamma_1') \cap M$ obtained at the end of 5.2 to see that $j_{P,\mathfrak{o}}(x)$ is the sum over

$$s \in \Omega(\mathfrak{a}_1^\varepsilon, P', \gamma_1') = \Omega^M \backslash \tilde{\Omega}(\mathfrak{a}_1^\varepsilon, P')/\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$$

and over $\xi \in M(w_s\gamma_1'w_s^{-1}) \backslash P$, where w_s represents s of

$$\omega(x)\phi(x^{-1}\xi^{-1}w_s\gamma_1\varepsilon(w_s^{-1}\xi x)) \quad .$$

We may replace the sum over $\Omega(\mathfrak{a}_1^\varepsilon, P', \gamma_1')$ by a sum over $\Omega(\mathfrak{a}_1^\varepsilon, P')$ but we must divide each term by the integer $d(M, w_s\gamma_1'w_s^{-1})$ as follows from the Lemma 5.3.1. We may also replace the sum over $M(w_s\gamma_1'w_s^{-1}) \backslash P$ by a sum over $w_sM_1(\gamma_1')w_s^{-1} \backslash P$ but we must divide each term by the index of $w_sM_1(\gamma_1')w_s^{-1}$ in $M(w_s\gamma_1'w_s^{-1})$ which equals

$$d(\gamma_1')/d(M, w_s\gamma_1'w_s^{-1}) \quad .$$

We finally obtain $j_{P,\sigma}(x)$ as the sum over $s \in \Omega(\mathfrak{a}_1^{\varepsilon}, P')$ of the sum over $\xi \in w_s M_1(\gamma_1') w_s^{-1} \backslash P$ of

$$d(\gamma_1')^{-1} \omega(x) \phi(x^{-1} \xi^{-1} w_s \gamma_1 \varepsilon(w_s^{-1} \xi x)) \quad .$$

This yields immediately the following expression for $j_{\sigma}^{T}(x)$:

$$j_{\sigma}^{T}(x) = \sum_{\delta \in M_1(\gamma_1') \backslash G} d(\gamma_1')^{-1} \omega(x) \phi(x^{-1} \delta^{-1} \gamma_1 \varepsilon(\delta x)) e_1(\delta x, T)$$

where

$$e_1(x, T) = \sum_{\varepsilon(P)=P} \sum_{s \in \Omega(\mathfrak{a}_1^{\varepsilon}, P')} (-1)^{a_P^{\varepsilon}} \hat{\tau}_P(H(w_s x) - T)$$

depends only on the parabolic subgroup P_1'. We may get rid of the factor $d(\gamma_1')^{-1}$ if we replace $M_1(\gamma_1')$ by $G(\gamma_1')$; we obtain the

LEMMA 5.3.3. Given σ a tame semisimple class we have

$$j_{\sigma}^{T}(x) = \sum_{\delta \in G(\gamma_1') \backslash G} \omega(x) \phi(x^{-1} \delta^{-1} \gamma_1 \varepsilon(\delta x)) e_1(\delta x, T) \quad .$$

Replacing $\hat{\tau}_P$ by ${}_{\varepsilon}\hat{\tau}_P$ we define e_1^{ε} the analogue of e_1 and we have the

LEMMA 5.3.4. Given σ a tame semisimple class we have

$${}_{\varepsilon}j_{\sigma}^{T}(x) = \sum_{\delta \in G(\gamma_1') \backslash G} \omega(x) \phi(x^{-1} \delta^{-1} \gamma_1 \varepsilon(\delta x)) e_1^{\varepsilon}(\delta x, T) \quad .$$

The reason for introducing the e_1^{ε} is so that the weighted orbital integrals have a usable form, that is, can be treated along the lines suggested by Y. Flicker in "Base change for GL(3)" and used again in his preprints on GL(3) and SU(3).

Given $s \in \Omega(\mathfrak{a}_1^{\varepsilon}, \mathfrak{a}_2^{\varepsilon})$ we define Δ_0^s to be the set of $\alpha \in \Delta_0$ such that $s^{-1}\alpha > 0$. The Lemma 5.2.5 tells us that this is the set of simple roots attached to a standard parabolic subgroup P'_s of G' containing P'_2. Given s as above we introduce a function on $\mathfrak{a}_0$:

$$ {}_{\varepsilon}B_1^s(x) = \sum_{P'_2 \subset P' \subset P'_s} (-1)^{a_P^{\varepsilon}} \, {}_{\varepsilon}\hat{\tau}_P(sX) \quad . $$

This is the product of $(-1)^{a_{P_s}^{\varepsilon}}$ and of the characteristic functions of the $X \in \mathfrak{a}_0$ such that $\varpi_{\bar{\alpha}}(sX) > 0$ for any $\alpha \in \Delta_0 - \Delta_0^s$ and $\varpi_{\bar{\alpha}}(sX) \leq 0$ for any $\alpha \in \Delta_0^s - \Delta_0^2$.

Given $s \in \Omega$ we introduce

$$ H_s(x, T) = s^{-1}(T - H(w_s x)) \quad . $$

With these notations we have

$$ e_1^{\varepsilon}(x, T) = \sum_{s \in \Omega(\mathfrak{a}_1^{\varepsilon})} {}_{\varepsilon}B_1^s(-H_s(x, T)) $$

where $\Omega(\mathfrak{a}_1^{\varepsilon})$ is the (disjoint) union over the P'_2 of $\Omega(\mathfrak{a}_1^{\varepsilon}, \mathfrak{a}_2^{\varepsilon})$. Let $c_1^{\varepsilon}(x, T)$ be the set of $H \in \mathfrak{a}_0$ whose projection on $\mathfrak{z}^{\varepsilon} \backslash \mathfrak{a}_1^{\varepsilon}$ lies in the convex hull of the projections on $\mathfrak{z}^{\varepsilon} \backslash \mathfrak{a}_1^{\varepsilon}$ of the set of

$H_s(x, T)$ with $s \in \Omega(\mathfrak{a}_1^\varepsilon)$.

LEMMA 5.3.5. Assume T is sufficiently regular then

$$H \longrightarrow \sum_{s \in \Omega(\mathfrak{a}_1^\varepsilon)} {}_\varepsilon B_1^s(H - H_s(x, T))$$

is the characteristic function of $c_1^\varepsilon(x, T)$.

This lemma is essentially Lemma 3.2 in Arthur's paper [Inventiones Math. 32, 1976]. More details will be given in Lecture 9 below. □

We shall denote by $v_1^\varepsilon(x, T)$ the volume of the projection on $\mathfrak{z}^\varepsilon \backslash \mathfrak{a}_1^\varepsilon$ of $c_1^\varepsilon(x, T)$. We obtain the

PROPOSITION 5.3.6. Given σ a tame semisimple class we have

$$\int_{\textcircled{G}^1} {}_\varepsilon j_\sigma^T(x)\,dx = \int_{G(\gamma_1')\mathbf{G}(\gamma_1')^o \backslash \mathbf{G}} v(\gamma_1')\omega(x)\phi(x^{-1}\gamma_1\varepsilon(x))v_1^\varepsilon(x, t)\,dx$$

where $v(\gamma_1')$ is the volume of $A_1^\varepsilon(\mathbf{R})^o G(\gamma_1')^o \backslash \mathbf{G}(\gamma_1')^o$.

Lecture 6

PROPERTIES OF THE TRUNCATION OPERATOR

R. Langlands

The most important property of Λ^T is that it converts smooth slowly increasing functions into rapidly decreasing functions but we begin by studying its formal properties.

Recall that Λ^T is defined for T suitably regular in $\mathfrak{a}_0^+$ and that it is defined first of all for continuous or, better, bounded measurable φ by

$$\Lambda^T\varphi(g) = \sum_P (-1)^{a_P} \sum_{\delta \in P\backslash G} \int_{N\backslash \mathbb{N}} \varphi(n\delta g)\hat{\tau}_P(H(\delta g) - T) \quad ,$$

where

$$a_P = \dim \mathfrak{a}_P/\mathfrak{a}_G \quad .$$

By Lemma 2.1 the sums appearing on the right are finite.

PROPOSITION 6.1. <u>The operator</u> Λ^T <u>is an idempotent, so that</u>

$$\Lambda^T(\Lambda^T\varphi) = \Lambda^T\varphi \quad .$$

This proposition is of course an immediate consequence of the following lemma.

LEMMA 6.2. <u>If</u> φ <u>is bounded measurable then</u>

$$\int_{N_1\backslash \mathbb{N}_1} \Lambda^T\varphi(n_1 g)dn_1 = 0$$

<u>unless</u> $\varpi(H(g) - T) \le 0$ <u>for every</u> $\varpi \in \hat{\Delta}_1$.

We first consider

$$(1) \qquad \int_{N_1\backslash \mathbf{N}_1} \sum_{\delta \in P\backslash G} \int_{N\backslash \mathbf{N}} \varphi(n\delta n_1 g)\hat{\tau}_P(H(\delta n_1 g) - T)\,dn\,dn_1 \quad .$$

Let $\Omega(\mathfrak{a}_0, P)$ be the set of s in $\Omega(\mathfrak{a}_0, \mathfrak{a}_0)$ such that $s^{-1}\alpha > 0$ for all $\alpha \in \Delta_0^P$. The Bruhat decomposition assures us that $P\backslash G$ is a disjoint union

$$\bigcup_{s \in \Omega(\mathfrak{a}_0, P)} Pw_s N_0 \quad ,$$

w_s being a representative of s.

Thus the expression (1) is equal to the sum over $\Omega(\mathfrak{a}_0, P)$ of

$$(2) \quad \int_{N_1\backslash \mathbf{N}_1} \sum_{\nu \in w_s^{-1}N_0 w_s \cap N_0\backslash N_0} \int_{N\backslash \mathbf{N}} \varphi(nw_s\nu n_1 g)\hat{\tau}_P(H(w_s\nu n_1 g) - T)\,dn\,dn_1 \quad .$$

The outer integral and the sum can be fused to obtain an integral over

$$w_s^{-1}N_0 w_s \cap N_0\backslash N_0\mathbf{N}_1 \quad ,$$

which we then decompose as an iterated integral, so that (2) becomes a triple integral

$$\int_{w_s^{-1}\mathbf{N}_0 w_s \cap N_0\mathbf{N}_1\backslash N_0\mathbf{N}_1} \int_{w_s^{-1}N_0 w_s \cap N_0\backslash w_s^{-1}\mathbf{N}_0 w_s \cap N_0\mathbf{N}_1} \int_{N\backslash \mathbf{N}}$$

The domain of integration in the outer integral depends on the choice of N_1 and on s but not on P. Since it is the alternation over P that will force the vanishing we ignore the final integration and concentrate on the inner double integral. A little reflection convinces one that

$$w_s^{-1}N_0w_s \cap N_0 \backslash w_s^{-1}\mathbf{N}_0w_s \cap N_0\mathbf{N}_1 = w_s^{-1}N_0w_s \cap N_1 \backslash w_s^{-1}\mathbf{N}_0w_s \cap \mathbf{N}_1 \quad .$$

Since $s \in \Omega(\mathfrak{a}_0, P)$ the intersection $w_sN_0w_s^{-1} \cap M$ is $N_0 \cap M$. Thus $w_sP_1w_s^{-1} \cap M$ is a parabolic subgroup of M with unipotent radical $w_sN_1w_s^{-1} \cap M$. If we pass the variable in $w_s^{-1}\mathbf{N}_0w_s \cap \mathbf{N}_1$ through w_s we obtain a variable in $\mathbf{N}_0 \cap w_s\mathbf{N}_1w_s^{-1} = (\mathbf{N} \cap w_s\mathbf{N}_1w_s^{-1})(\mathbf{M} \cap w_s\mathbf{N}_1w_s^{-1})$. Thus the second integration in the double integral can be taken over the product

$$(N \cap w_sN_1w_s^{-1}\backslash \mathbf{N} \cap w_s\mathbf{N}_1w_s^{-1}) \times (M \cap w_sN_1w_s^{-1}\backslash \mathbf{M} \cap w_s\mathbf{N}_1w_s^{-1} \quad .$$

The volume of the first factor is 1 and since the first integration is taken over $N\backslash\mathbf{N}$ the integral does not depend on the first variable in the product.

Thus the double integral becomes finally

$$\int_{M \cap w_sN_1w_s^{-1}\backslash \mathbf{M} \cap w_s\mathbf{N}_1w_s^{-1}} \int_{N\backslash\mathbf{N}} \quad .$$

However

$$(M \cap w_sN_1w_s^{-1}).N$$

is the unipotent radical of a parabolic subgroup P_s of G. So the double integral becomes a single integral over $N_s \backslash \mathbf{N}_s$, which we now write out explicitly.

(3) $$\int_{N_s \backslash \mathbf{N}_s} \varphi(nw_s n_1 g)\hat{\tau}_P(H(w_s n_1 g) - T)\,dn \quad ,$$

the n_1 being the variable for the outer integration, which does not concern us at the moment.

The group P_s is contained in P. The group N_1 is fixed but s varies over $\Omega(\mathfrak{a}_0, P)$ and we are to sum over P and $\Omega(\mathfrak{a}_0, P)$. What we do is fix s and a $P^o \supseteq P_0$ and sum over all P with $s \in \Omega(\mathfrak{a}_0, P)$ and $P_s = P^o$.

The set $\{\alpha \in \Delta_0 | s^{-1}\alpha > 0\}$ is the disjoint union of two subsets, the first S^1 consisting of those α in it for which $s^{-1}\alpha$ is orthogonal to $\mathfrak{a}_1$ and the second S_1 of those for which it is not. It is clear that $\Delta_0^{P_s} \subset \Delta_0^P$ and that

$$\Delta_0^{P_s} = \Delta_0^P \cap S^1 \quad ,$$

for $\alpha \in S_1$ if and only if $s^{-1}\alpha$ is a root in N_1. Thus the freedom of P is that the intersection of Δ_0^P with S_1 can be chosen at will.

The dependence of (3) on P is through the function $\hat{\tau}_P(H(w_s n_1 g) - T)$. The sum

$$\sum (-1)^{a_P} \hat{\tau}_P(H(w_s n_1 g) - T)$$

There is considerable overlap with Clozel's 1st lecture

over the allowed P is therefore 0 unless

$$\varpi_\alpha(H(w_s n_1 g) - T) > 0$$

for $\alpha \notin \Delta_0^{P^o} \cup S_1$ and

$$\varpi_\alpha(H(w_s n_1 g) - T) \leq 0$$

for $\alpha \in S_1$.

To complete the proof of the lemma we have to show that these inequalities imply that

$$\varpi(H(g) - T) \leq 0$$

for $\varpi \in \hat{\Delta}_1$. We have

$$s^{-1}(H(w_s n_1 g) - T) = H(g) - T + s^{-1}H(w_s v) + T - s^{-1}T$$

with $v \in N_0(\mathbf{A})$.

We write, identifying $\mathfrak{a}_0$ and its dual,

$$H(w_s n_1 g) - T = \sum_{\alpha \in \Delta_0} t_\alpha \alpha$$

with $t_\alpha > 0$ for $\alpha \notin \Delta_0^{P^o} \cup S_1$ and $t_\alpha \leq 0$ for $\alpha \in S_1$. Then

$$\begin{aligned} \varpi(s^{-1}(H(w_s n_1 g) - T)) &= \sum t_\alpha \varpi(s^{-1}\alpha) \\ &= \sum_{\alpha \notin S^1} t_\alpha \varpi(s^{-1}\alpha) \end{aligned}$$

for $s^{-1}\alpha$ is orthogonal to $\mathfrak{a}_1$ if $\alpha \in S^1$. If $\alpha \notin S^1 \cup S_1$ then $t_\alpha > 0$ and $\varpi(s^{-1}\alpha) \leq 0$ and if $\alpha \in S_1$ then $t_\alpha \leq 0$ and $\varpi(s^{-1}\alpha) \geq 0$. Thus this expression is less than or equal to zero.

To complete the proof of the lemma we need only show that for sufficiently regular T

$$\varpi(s^{-1}H(w_s v)) + \varpi(T - s^{-1}T) \geq 0 \quad .$$

There is certainly no harm in replacing G by a Levi factor of the smallest standard parabolic containing s, which to simplify the notation we suppose is G itself. Then given any constant C we can take T sufficiently regular and suppose that

$$\varpi(T-s^{-1}T) \geq C \quad .$$

It therefore remains to show that there exists a constant C such that

$$\varpi(s^{-1}H(w_s v)) \geq -C \tag{4}$$

for all $v \in N_0(\mathbb{A})$. This is a statement which is easily seen to be independent of the choice of K. Indeed it is enough to prove it over a field which splits G. So we can suppose G is split and semi-simple.

Then one has the usual optimal choice of K and for this one proves by induction on the length of s the following lemma.

LEMMA 6.3. <u>If</u> v <u>lies in</u> $\mathbb{N}_0$ <u>then</u>

$$s^{-1}H(w_s v) = \sum_{\substack{\alpha>0 \\ s\alpha<0}} c_\alpha \alpha$$

with $c_\alpha \geq 0$.

This gives the relation (4) with $C = 0$. To prove the lemma one begins with $SL(2)$, taking $w_s \in K$. So, for the non-trivial s,

$$w_s v w_s^{-1} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} .$$

Moreover $H(w_s v w_s^{-1})$ is the sum of its local contributions and these are

(i) v real

$$-\tfrac{1}{2}\ln(1+|x_v|^2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(ii) v complex

$$-\ln(1+|x_v|^2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(iii) v non-archimedean

$$-\ln \max\{|1|, |x_v|\}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Thus for $SL(2)$ and hence in general the lemma is proved for an s of length one.

For a Chevalley group and an optimal choice of K we may take $w_s \in K$. If $s = s_1 s_2$ with s_1 a reflection associated to the root β

and $1 + \text{length } s_2 = \text{length } s$ then

$$s^{-1}H(w_s v) = s_2^{-1}s_1^{-1}H(w_{s_1} w_{s_2} v) = s_2^{-1}s_1^{-1}H(w_{s_1} v') + s_2^{-1}s_1^{-1}(s_1 H(w_{s_2} v)) \quad .$$

The induction assumption allows us to write this as

$$s_2^{-1} d_\beta \beta + \sum_{\substack{\alpha>0 \\ s_2\alpha<0}} c_\alpha \alpha$$

with $d_\beta \geq 0$, $c_\alpha \geq 0$. Since

$$\{\alpha > 0 | s\alpha < 0\} = \{\alpha > 0 | s_2\alpha < 0\} \cup \{s_2^{-1}\beta\}$$

the lemma follows.

PROPOSITION 6.4. <u>Suppose that</u> φ_1 <u>and</u> φ_2 <u>are continuous functions on</u> $G\backslash \mathbf{G}$ <u>and that on</u>

$$|\varphi_1(g)| \leq c|g|^N$$

<u>for some</u> N <u>and that on any Siegel domain in</u> $\mathbf{G}^1$ <u>we have an inequality</u>

$$|\varphi_2(g)| \leq c_N |g|^{-N}$$

<u>for all</u> N. <u>Then</u>

$$\int_{G\backslash \mathbf{G}^1} \Lambda^T \varphi_1(g)\varphi_2(g)dg = \int_{G\backslash \mathbf{G}^1} \varphi_1(g)\Lambda^T\varphi_2(g)dg \quad .$$

This clearly reduces to showing that

$$\int_{G\backslash \mathbf{G}^1} \left\{ \sum_{\delta \in P\backslash G} \int_{N\backslash \mathbf{N}} \varphi_1(n\delta g)\,dn\,\hat{\tau}_P(H(\delta g) - T) \right\} \varphi_2(g)\,dg$$

is equal to

$$\int_{G\backslash G^1} \varphi_1(g) \left\{ \sum_{\delta \in P\backslash G} \int_{N\backslash \mathbf{N}} \varphi_2(n\delta g)\,dn\,\hat{\tau}_P(H(\delta g) - T) \right\} dg \quad .$$

It follows readily from Lemma 7.8 of the next lecture that the second integral is absolutely convergent when φ_1 and φ_2 are replaced by their absolute values. Thus a formal proof of the equality assures us of both the equality and the convergence of the first integral.

The formal proof is of course easy, the second expression reducing to

$$\int_{P\backslash G} \varphi_1(g)\hat{\tau}_P(H(g) - T) \left\{ \int_{N\backslash \mathbf{N}} \varphi_2(ng)\,dn \right\} dg$$

which equals

$$\int_{\mathbf{N}P\backslash \mathbf{G}} \hat{\tau}_P(H(g) - T) \left\{ \int_{N\backslash \mathbf{N}} \varphi_1(ng)\,dn \right\} \left\{ \int_{N\backslash \mathbf{N}} \varphi_2(ng)\,dn \right\} dg \quad ,$$

an expression symmetric in φ_1 and φ_2.

COROLLARY 6.5. Λ^T <u>extends to an orthogonal projection on the Hilbert space</u> L.

We will not need any of these assertions in the next two lectures. What we will need is the fact that Λ^T transforms smooth slowly increasing functions into rapidly decreasing functions. For now we content ourselves with a relatively simple statement.

To any element Y of the universal enveloping algebra of the Lie algebra of G we can associate a left-invariant differential operator $R(Y)$ on $\mathbf{G}$.

LEMMA 6.6. Suppose T is sufficiently regular. Let $\mathfrak{S}$ be a Siegel domain on $\mathbf{G}^1$. For any pair of positive numbers N and N' and any open compact subgroup K_0 of $G(\mathbf{A}^f)$ we can find a finite subset $\{Y_1, \ldots, Y_r\}$ in the universal enveloping algebra such that

$$|\Lambda^T \varphi(g)|\,|g|^{N'} \le \sum_i \sup_{h \in \mathbf{G}^1 g} |R(Y_i)\varphi(h)|\,|h|^{-N}$$

for $g \in \mathfrak{S}$ provided φ is invariant on the right under K_0 and sufficiently smooth that all the operators $R(Y_i)$ can be applied to it.

This is proved by an argument similar to that used for the proof of the σ-expansion. Its structure is more transparent, many of the incidental difficulties met with the σ-expansion no longer arising. However the alternating sum is used in a slightly different way and it is best to dispose of the necessary technical lemma immediately.

For this purpose we fix $P_1 \subset P_2$ and consider a continuous function ψ on $N_1 \backslash \mathbf{N}_1$. If $P_1 \subset P \subset P_2$ then

$$\Pi_P \psi : n_1 \longrightarrow \int_{N \backslash \mathbf{N}} \psi(n n_1)\, dn_1$$

is also a function on $N_1 \backslash \mathbf{N}_1$ because N is a normal subgroup of N_1. We want to consider

$$\prod \psi = \sum_{P} (-1)^{a_P} \prod_{P} \psi \quad .$$

Let $\Delta_0^2 - \Delta_0^1 = \{\alpha_1, \ldots, \alpha_s\}$ and let $\sum_i$ be the set of positive roots α of the form

(5) $$\alpha = \sum_{\beta \in \Delta_0} b_\beta \beta$$

with $b_\beta \neq 0$ for $\beta = \alpha_i$ or $\beta \in \Delta_0 - \Delta_0^2$. There is a parabolic P^i between P_1 and P_2 such that the Lie algebra of N^i is spanned by the root vectors attached to the roots α in $\sum_i$. For any P between P_1 and P_2 there is a unique subset $\sum_P$ of $\{\alpha_1, \ldots, \alpha_r\}$ such that Δ_0^2 is the disjoint union of $\sum_P$ and Δ_0^P. Moreover

$$N = \prod_{i \in \sum_P} N^i \quad .$$

It follows easily, all the groups N being normal in N_1, that

(6) $$\prod = \prod_{i=1}^{r} \left(\prod_{P_2} - \prod_i \right) \quad ,$$

where for simplicity of notation we have set $\prod_{P^i} = \prod_i$.

Let $\sum_i^o$ be the set of positive roots which when written as in (5) have $b_{\alpha_i} \neq 0$. Let an integer $r \geq 0$ be given. For the purposes of the next lemma we define a left-invariant differential operator of type r to be a product

$$\prod_{i=1}^{r} \prod_{j=1}^{r} X_{ij} \quad ,$$

the order being immaterial and X_{ij} being a root vector of type α with $\alpha \in \sum_i^o$.

LEMMA 6.7. <u>For any integer</u> $r \geq 0$ <u>and any open compact subgroup</u> U <u>of</u> $\mathbb{N}_1^f = N_1(\mathbf{A}^f)$ <u>there is a constant</u> $c = c(r, U)$ <u>and a finite collection</u> $Y_1, \ldots, Y_m$ <u>of differential operators of type</u> r, <u>the collection depending on</u> r <u>alone and not on</u> U, <u>such that</u>

$$\|\prod\psi\|_\infty \leq c \sum_i \|R(Y_i)\psi\|_\infty$$

<u>for any function</u> ψ <u>on</u> $N_1\backslash\mathbb{N}_1/U$ <u>which has continuous derivations up to order</u> rs.

The norms in the inequality are of course L_∞-norms. A little reflection shows that we can make a number of simplifications. First of all replacing ψ by $\prod_{P_2}\psi$ we can work in the group M_2 rather than in G. In other words we may suppose that $G = P_2$. Then the formula (6) reduces to the case that P_1 is a maximal proper parabolic of G over $\mathbf{Q}$.

We choose a composition series of groups over $\mathbf{Q}$

$$N_1 = V_\ell \supseteq V_{\ell-1} \supseteq \cdots \supseteq V_0 = \{1\}$$

with V_{i+1}/V_i isomorphic to the additive group. Since

$$(1 - \prod_{P_1})\psi(n_1) = \sum_{i=0}^{\ell-1} \int_{V_i\backslash\mathbb{V}_i} \psi(vn_1)dv - \int_{V_{i+1}\backslash\mathbb{V}_{i+1}} \psi(vn_1)dv$$

it is enough to prove the following lemma.

LEMMA 6.8. *Let* $r \geq 0$ *be an integer and let* U *be an open subgroup of* $\mathbb{A}$. *There is a constant* $c = c(r, U)$ *such that for any function* ψ *on* $Q\backslash\mathbb{A}/U$ *which is continuously differentiable of order* r

$$\sup_{K} \left|\psi(x) - \int_{Q\backslash\mathbb{A}} \psi(Y)\,dy\right| \leq c\left\|\frac{\partial^r\psi}{\partial x^r}\right\|_\infty .$$

To be a function on $Q\backslash\mathbb{A}/U$ is to be a function on a quotient $L\backslash\mathbb{R}$ where $L = L(U)$ is a lattice in $\mathbb{R}$. The inequality thus follows readily from

$$\sum_{n\neq 0} |a_n| \leq \left(\sum_{n\neq 0} |n^r a_n|^2\right)^{1/2}\left(\sum_{n\neq 0} \frac{1}{n^{2r}}\right)^{1/2} ,$$

at least for $r > 0$, but the case $r = 0$ is quite trivial.

We shall apply Lemma 6.7 to a function

$$n \longrightarrow \psi(na)$$

where ψ is a function on $G\backslash\mathbb{G}$ and $a \in A_0(\mathbb{A})$. If we want to regard the Y_i as left-invariant differential operators on $\mathbb{G}$ we must write the inequality of Lemma 6.7 as

$$(7) \qquad \sup_{n_1} \left|\prod\psi(n_1 a)\right| \leq c\sum_i \sup_i \left|R(ada^{-1}(Y_i))\psi(n_1 a)\right| .$$

This will be to our advantage.

We now take up the proof of Lemma 6.8. The first step is to replace

$$\hat{\tau}_P(H(x) - T)\int_{N\backslash \mathbf{N}} \varphi(ng)\,dn$$

by

$$\sum_{P_1 \subset P \subset P_2} \sum_{P_1\backslash P} F_P^1(x, T)\sigma_1^2(H(x) - T)\int_{N\backslash \mathbf{N}} \varphi(ng)\,dn \quad ,$$

the sum being over the pairs P_1, P_2. There is then a sum over $P\backslash G$ and an alternating sum over P. The final result is a sum over pairs $P_1 \subset P_2$ of

$$\sum_{P_1\backslash G} \sum_{\{P\backslash P_1 \subset P \subset P_2\}} (-1)^{a_P} F_P^1(\delta g, T)\sigma_1^2(H(\delta g) - T)\int_{N\backslash \mathbf{N}} \psi(n\delta g)\,dn \quad .$$

However Corollary 4.1.2 allows us to replace F_P^1 by F_2^1. The upshot is that we are forced to estimate

$$\sum_{P_1\backslash G} F_2^1(\delta g, T)\sigma_1^2(H(\delta g) - T)\left|\sum_P (-1)^{a_P} \int_{N\backslash \mathbf{N}} \varphi(n\delta g)\right| \quad .$$

Lemma 2.1 shows that

$$\sum_{P_1\backslash G} F_2^1(\delta g, T)\sigma_1^2(H(\delta g) - T) \leq c|g|^M$$

for some M, T being held constant. Thus the problem is to estimate

$$\left|\sum_P (-1)^{a_P} \int_{N\backslash \mathbf{N}} \varphi(n\delta g)\right| \quad .$$

It is now best to be more precise about Siegel domains. In contrast to the previous definition the elements g of $\mathfrak{S}_P(T_0)$ will now be required to have all of the following properties:

(i) If $g = pk$ and $a = a(g)$ is the projection of p on A_0 then $g \in a\Omega$ where Ω is a fixed compact set. ← OK for G but not for arbitrary P

(ii) $\alpha(H(g) - T_0) > 0$ for all $\alpha \in \Delta_0^P$.

(iii) There are constants c_1 and c_2 so that   Consider all of N_P

$$|\ln|a|| \le c_1(1+\|H(g)\|) \le c_2(1+|\ln|a||) \quad .$$

The final condition is easily seen to force the component of a in $A_0(\mathbb{A}^f)$ to lie in a compact set. This modification entails a modification in $\mathfrak{S}_P^1(T_0, T)$ but the set

$$P_1\mathfrak{S}_P^1(T_0, T)$$

and thus the function F_P^1 is not changed; provided of course c_1, c_2 and Ω, which affect the size of $\mathfrak{S}_P(T_0)$, are all chosen large enough.

This definition has the advantage that for a given $\mathfrak{S}_G(T_0)$, for example that of Lemma 6.6, there are positive constants c_1 and ε such that

$$|\delta g|^{-1} \le c|g|^{-\varepsilon}$$

for all $\delta \in G$ and all $g \in \mathfrak{S}_G(T_0)$. (I know no reference for this fact. It can be deduced from Prop. II.1.5 of A. Borel, <u>Ensembles fondamentaux</u>

<u>pour les groupes arithmétiques et formes automorphes</u>, Cours à l'IHP (1964).)

Thus all we need do is show that if $g \in \mathfrak{G}_2^1(T_0, T)$ and $\sigma_1^2(H(g) - T) \neq 0$ then, for a suitable choice of g modulo P_1,

$$(8) \quad |g|^{M+N'} |\sum_P (-1)^{a_P} \int_{N\backslash \mathbf{N}} \varphi(ng)| \leq \sum_i \sup_{h \in \mathbf{G}^1 g} |R(Y_i)\varphi(h)|\,|h|^{-N} \quad .$$

The suitable choice of g will be an element in $\mathfrak{G}^1(T_0, T)$. Then conditions (i) and (iii) yield

$$|g|^{M+N'} \leq ce^{M'\|H\|} \quad ,$$

with $H = H(g) = H(a)$. Thus denoting the right side of (8) by A we need only show that

$$(9) \qquad ce^{M'\|H\|} \, |\sum_P (-1)^{a_P} \int_{N\backslash \mathbf{N}} \varphi(ng)| \leq A \quad .$$

Since we can readily deal with right translations by elements from a compact set in $\mathbf{G}^1$ we may suppose that $g = a(g) = a$. As in Lemma 4.2 we may write $H = H_1 + H_2$ with $H_1 \in \mathfrak{a}_1^2$ and $H_2 \in \mathfrak{a}_2$ and deduce from the fact that $\sigma_1^2(H-T) \neq 0$ that

$$\|H_2\| \leq c(1+\|H_1\|) \quad ,$$

the constant c depending of course on T, but that is of no consequence. Thus it will be enough to prove (9) with H replaced by H_1 but with a larger

M'. This is an easy consequence of (7), for the inequality (ii) applied with P_2 replacing P assures us that the coefficients of $ada^{-1}(Y)$ with respect to a fixed basis of the universal enveloping algebra are bounded by $ce^{-M''\|H_1\|}$, where $M'' \longrightarrow \infty$ with r.

Appendix

Truncation has been seen to have two essential properties. It is an idempotent and it converts slowly increasing smooth functions to rapidly decreasing functions. It may be worthwhile to see how this comes to pass in a simple case.

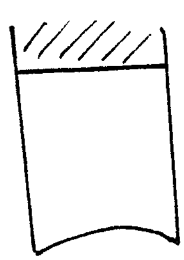
A function f on the upper half-plane which is invariant under $SL(2, \mathbf{Z})$ may also be considered as a function on $SL(2, \mathbf{Z})\backslash SL(2, \mathbf{R})$ if we set

$$\phi(g) = f\left(\frac{ai+b}{ci+d}\right), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

In particular

$$\phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = f(a^2 i+x) .$$

The function f is determined by its values on a a fundamental domain



Truncation is achieved by leaving f untouched below a certain line $y = y_0$ in the fundamental domain and by removing the constant term of its Fourier expansion above the line. So it is clearly idempotent.

The inequality

$$\sum_{n\neq 0} |a_n| \leq \sqrt{\sum_{n\neq 0} n^{2r}|a_n|^{2r}} \sqrt{\sum_{n\neq 0} \frac{1}{n^{2r}}}$$

shows that for $r \geq 1$ and $y > y_0$

(1) $$|\Lambda f(u+iy)| \leq c \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{d^r}{dx^r} f(x+iy)\right|^2 dx \quad .$$

However if X is the element

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in the Lie algebra then

$$\frac{d^r}{dx^r} f(x+iy) = \frac{d^r}{dx^r} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right)$$

with $y = a^2$ and right side of this equality is

$$a^{-2r} R(X)^r \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) \quad .$$

Thus bounds on $R(X)^r \phi$ of the form

$$\left|R(X)^r \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right)\right| \leq c(r) a^{2s} \quad ,$$

where s is a constant independent of r - and this is the kind of bound that will be available to us - yield

$$\left|\frac{d^r}{dx^r} f(x+iy)\right| \leq c(r)y^{s-r} \quad .$$

The inequality (1) then implies that

$$|\Lambda f(u+iy)| \leq cc(r)y^{s-r}$$

for any r.

Lecture 7

PREPARATION FOR THE COARSE χ-EXPANSION

I: STATEMENT OF LEMMAS

R. Langlands

Recall that the right side of the basic identity is

$$\sum_{P_1 \subset P_2} \sum_{P_1 \backslash G} \sigma_1^2(H(\delta g)) \sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_P(\delta g, \delta g) \quad ,$$

where

$$a_P^\varepsilon = \dim \mathfrak{a}_P^\varepsilon / \mathfrak{a}_G^\varepsilon \quad .$$

Since the χ-expansion can not be introduced without recalling facts from the theory of Eisenstein series, we begin by proving the absolute convergence of

$$\int_{G\backslash G^1} \sum_{P_1 \backslash G} \sigma_1^2(H(\delta g) - T) \left(\sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_P(\delta g, \delta g) \right) dg \quad .$$

This will not be a simple matter and will provide us with techniques and lemmas necessary for the proof of the absolute convergence of the χ-expansion.

We prove in fact the stronger assertion that

$$\int_{P_1 \backslash G^1} \sigma_1^2(H(g) - T) \left| \sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_P(g, g) \right| dg < \infty \quad .$$

Notice – Clozel has an even stronger assertion which is indeed proved later on.

Recall that

$$\mathbf{G}^1 = \{g \in \mathbf{G} \mid |\chi(g)| = 1 \ \forall \chi \in X^*(G)\} \quad .$$

We begin the proof with a sequence of preliminary reductions. Since

$$\mathbf{G}^1 = (\mathbf{P}_1 \cap \mathbf{G}^1)\cdot K \quad ,$$

the integral can be replaced by an integral over $P_1 \backslash (\mathbf{P}_1 \cap \mathbf{G}^1) \times K$, the measure on $\mathbf{P}_1 \cap \mathbf{G}^1$ being the left-invariant Haar measure. Let

$$\mathbf{P}_1^1 = \{p \in \mathbf{P}_1 \mid |\chi(p)| = 1 \ \forall \chi \in X^*(P_1)\}$$

and let $A_1^G(\mathbf{R})^o = \exp \mathfrak{a}_1^G$ be the connected component of $A_1^G(\mathbf{R})$. Then

$$\mathbf{P}_1 \cap \mathbf{G}^1 = \mathbf{P}_1^1 \cdot A_1^G(\mathbf{R})^o$$

and the left-invariant Haar measure on $\mathbf{P}_1 \cap \mathbf{G}^1$ is given by

$$d(pa) = \rho_{P_1}^{-2}(a)\,dp\,da \quad ,$$

with

$$\rho_{P_1}^{-2}(a) = |\mathrm{ad}(a)|_{\mathfrak{p}_1}| \quad ,$$

$\mathfrak{p}_1$ being the Lie algebra of P_1. To simplify the notation in this lecture I shall abbreviate $A_1^G(\mathbf{R})^o$ to A_1^G.

Let Ω be a compact subset of $\mathbf{G}$ satisfying $K\Omega K = \Omega$ and set

$$\Phi(g) = \sup_{k,k' \in \Omega} \Big|\sum_{P_1 \subset P \subset P_2} (-1)^{a_P^{\varepsilon}} \Lambda^{T,P_1} K_P(gk, gk')\Big| \quad .$$

It is clearly enough to show that for any real number r

$$\int_{P_1\backslash \mathbb{P}_1^1\times A_1^G} \sigma_1^2(H(p)+H(a)-T)(1+\|H(p)\|+\|H(a)\|)^r\phi(pa)(\rho_{P_1}^{-2}(a)dpda < \infty \quad .$$

Let $\mathfrak{S} = \mathfrak{S}^{P_1}(T_0, \omega) \cap \mathbb{P}_1^1$ be a Siegel domain in $\mathbb{P}_1^1$. It is certainly enough to show that for any arbitrary $\mathfrak{S}$ the integral

$$(1) \quad \int_{\mathfrak{S}\times A_1^G} \sigma_1^2(H(p)+H(a)-T)(1+\|H(p)\|+\|H(a)\|)^r\phi(pa)\rho_{P_1}^{-2}(a)dpda$$

is finite. We may suppose that on $\mathfrak{S}$

$$c'|\ell n|p\| - c_2' \leq \|H(p)\| \leq c_1|\ell n|p\| + c_2 \quad .$$

The proof has two aspects. One first shows that the integrand is zero on a large subset of the domain of integration, and then uses the estimates of the previous lecture on the set on which it does not vanish.

We begin by finding a convenient expression for

$$(2) \qquad \sum_{P_1\subset P\subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_P(h, g) \quad .$$

Recall that

$$K_P(h, g) = \omega(g)\sum_{Z_0\backslash M} \int_{\mathbf{N}} \phi(h^{-1}\gamma n\varepsilon(g))dn \quad .$$

It follows immediately from the definition of Λ^{T,P_1} that

$$\Lambda^{T,P_1}\varphi = \Lambda^{T,P_1}\psi$$

if

$$\psi(g) = \int_{N_1 \backslash \mathbf{N}_1} \phi(n_1 g)\, dn_1 \quad .$$

Thus when studying the expression (3) we may replace $K_P(h, g)$ by

$$\int_{N_1 \backslash \mathbf{N}_1} K_P(n_1 h, g)$$

and this equals

$$\omega(g) \sum_{Z_0 N \backslash P} \int_{\mathbf{N}_1 / N_1} \int_{\mathbf{N}} \phi(h^{-1} n n_1 \gamma\varepsilon(g))\, dn\, dn_1 \quad ,$$

because $\mathbf{N}$ is a normal subgroup of $\mathbf{N}_1$. This expression is in turn equal to

$$\omega(g) \sum_{\gamma \in P_1 \backslash P} \sum_{\delta \in Z_0 N_1 \backslash P_1} \int_{\mathbf{N}_1} \phi(h^{-1} n_1 \delta\gamma\varepsilon(g))\, dn_1 = \omega(g) \sum_{\gamma \in P_1 \backslash P} K_{P_1}(h, \gamma\varepsilon(g)) \quad ,$$

where K_{P_1} now denotes the kernel for the case that ε is the trivial automorphism.

The expression (3) becomes

$$\omega(g) \sum_{\gamma \in P_1 \backslash P_2} \Bigg(\sum_{\substack{P_1 \subset P \subset P_2 \\ \gamma \in P}} (-1)^{a_P^\varepsilon} \Bigg) \Lambda^{T, P_1} K_{P_1}(h, \gamma\varepsilon(g)) \quad .$$

If there is no ε-invariant parabolic subgroup between P_1 and P_2 the sum is empty, equals 0, and the convergence of (1) is trivial. Otherwise let Q be the largest such subgroup. For a given γ let P_γ

be the smallest such subgroup containing γ. Then

$$\sum_{\substack{P_1 \subset P \subset P_2 \\ \gamma \in P}} (-1)^{a_P^\varepsilon} = \sum_{P_\gamma \subset P \subset Q} (-1)^{a_P^\varepsilon} ,$$

and this is clearly 0 unless $P_\gamma = Q$, when it is 1.

Let $F_\varepsilon(P_1, P_2)$ be the set of all $\gamma \in Q$ (taken modulo P_1) for which $P_\gamma = Q$. Then (2) is equal to

$$(3) \qquad (-1)^{a_Q^\varepsilon} \omega(g) \sum_{\gamma \in F_\varepsilon(P_1,P_2)} \Lambda^{T,P_1} K_{P_1}(h, \gamma\varepsilon(g)) \quad .$$

To prove the convergence we need a number of lemmas. These we next state, explaining how the convergence follows from them, postponing the proof of the lemmas until the next lecture. It will be convenient to introduce a notational convention. We denote by C a compact set and by c a constant both depending only on Ω and the support of ϕ, and by $c(\phi)$ a semi-norm on $C_c^\infty(\mathbf{G})$. All three are allowed to vary from line to line. The first lemmas are concerned with the support of the integrand in (2).

LEMMA 7.1. <u>There is a compact set</u> C <u>in</u> $\mathfrak{a}_Q$ <u>such that</u>

$$\Lambda^{T,P_1} K_{P_1}(gk, \gamma\varepsilon(gk')) \neq 0$$

<u>for some</u> $\gamma \in Q = Q(\mathbf{Q})$ <u>and some</u> $k, k' \in \Omega$ <u>implies that</u> $H(g) \in \mathfrak{a}_0^Q + \mathfrak{a}_Q^\varepsilon + C$.

Until now Ω has needed only to contain K. Now we suppose that in addition it contains $\exp C\cdot K$, with the C of the lemma. Then, at the cost of taking a slightly different T, we can replace the integral in (1) by an integral over $\mathfrak{G} \times A_1^Q A_Q^{\varepsilon}$. Indeed T is fixed. Thus there is a $C \in \mathfrak{a}_1^Q + \mathfrak{a}_Q^{\varepsilon}$ such that

$$\sigma_1^2(H(p) + H(a) - T) = \sigma_1^2(H(a) - T) \neq 0$$

implies that $H(a) = H + X$ with $X \in C$ and $\sigma_1^2(H) \neq 0$. This allows us, again at the cost of enlarging Ω, to take $T = 0$.

LEMMA 7.2. _If_ $p, p' \in \mathbb{P}^1$, $k, k' \in \Omega$, $\gamma \in Q$, $a, a' \in A_1^Q$, $b \in A_Q^{\varepsilon}$ _then_

$$\Lambda^{T,P_1} K_{P_1}(pabk, \gamma\varepsilon(p'abk')) = \Lambda^{T,P_1} K_{P_1}(pak, \gamma\varepsilon(p'ak'))\rho_{P_1}^2(b) \quad .$$

LEMMA 7.3. _If_ $H = H_1^Q + H_Q^{\varepsilon}$ _with_ $H_1^Q \in \mathfrak{a}_1^Q$, $H_Q^{\varepsilon} \in \mathfrak{a}_Q^{\varepsilon}$ _and_ $\sigma_1^2(H) \neq 0$ _then_ $\alpha(H_1^Q) > 0$ _for all_ $\alpha \in \Delta_1^Q$ _and_

$$\|H_Q^{\varepsilon}\| \leq c(1 + \|H_1^Q\|) \quad .$$

These two lemmas and the previous reductions allow us to majorize (2) by

$$(4) \qquad c \int_{\mathfrak{G}\times A_1^Q} \tau_1^Q(H(a))(1+\|H(p)\|+\|H(a)\|)^r \Phi(pa)\rho_{P_1}^{-2}(a)\,dp\,da \quad ,$$

it being however understood that the integration over A_Q^{ε}, or to be more precise over

$$\{b \in A_Q^{\varepsilon} \mid \|H(b)\| \leq c(1+\|H(a)\|) \quad ,$$

has forced us to increase the exponent r. The function τ_1^Q is the characteristic function of

$$\{H \in \mathfrak{a}_1^Q \mid \alpha(H) > 0 \,\forall \alpha \in \Delta_1^Q\} .$$

To estimate $\Phi(pa)$ in the integral (4) we observe that it is dominated by

$$\sum_{\gamma \in F_{\varepsilon}(P_1, P_1)} |\Lambda^{T,P_1} K_{P_1}(pak, \gamma\varepsilon(pak'))|$$

for some $k, k' \in \Omega$. At this point there are three more steps left for the proof. The first is to show that if this expression does not vanish then $\|H(a)\|$ is controlled by $\|H(p)\|$ and that is the purpose of the next lemma. Then we have to show that the truncation provides us with functions rapidly decreasing at infinity in $\mathfrak{G}$, and finally that the summation over γ, although it tempers the rate of the decrease, does not destroy it.

LEMMA 7.4. *Suppose* γ *lies in* $F_{\varepsilon}(P_1, P_2)$, $nm \in \mathfrak{G}$, $a \in A_1^Q$, *and that* $\tau_1^Q(H(a)) \neq 0$. *Suppose in addition that for some* $k, k' \in \Omega$ *and some* $m' \in \mathbf{M}_1^1 = \mathbf{M}_1 \cap \mathbf{P}_1^1$ *we have*

$$\Lambda^{T,P_1} K_{P_1}(m'ak', \gamma\varepsilon(nmak)) \neq 0 \quad .$$

Then for some other m'

$$K_{P_1}(m'ak', \gamma\varepsilon(nmak)) \neq 0$$

and

$$\|H(a)\| \leq c(1+\|H(m)\|) \quad .$$

On A_1^Q we have $|a| \leq ce^{N_1\|H(a)\|}$. It therefore follows from this and the following lemma together with Lemma 6.6 of the previous lecture that for certain M_1 and N but for an arbitrary M and thus for an arbitrary M'

$$|\Lambda^{T,P_1} K_{P_1}(pak, \gamma\varepsilon(pak')| = c|p|^{-M}|a|^N|\gamma\varepsilon(p)|^{M_1} \leq c'|p|^{-M'}|\gamma\varepsilon(p)|^{M_1}$$

provided $\tau_1^Q(H(a)) \neq 0$.

If Y is an element of the universal enveloping algebra of the Lie algebra of G then we can associate to Y a left-invariant differential operator $R(Y)$ in $\mathbf{G}$ and a right-invariant differential operator $L(Y)$.

LEMMA 7.5. Let Y lie in the universal enveloping algebra of M_1 and let $R(Y)K_{P_1}(h, g)$ be the result of applying $R(Y)$ to K_{P_1} regarded as a function of the first variable, the second argument being held fixed. Then there are constants $c = c(\phi)$ and N such that for $k, k' \in \Omega$

$$|R(Y)K_{P_1}(hk, gk')| \leq c(|h||g|)^N \quad .$$

Proceeding with the proof of convergence of (5) we estimate

$$(5) \qquad \sum_{\gamma \in F_\varepsilon(P_1,P_2)}' |\gamma\varepsilon(pa)|^{M_1} \quad ,$$

where $p \in \mathbf{G}$ and the prime indicates that we sum only over those $\gamma \in F_{\varepsilon}(P_1, P_2)$ for which

$$\Lambda^{T,P_1} K_{P_1}(pak, \gamma\varepsilon(pak')$$

is non-zero for some k, k' in Ω and the given pa.

The next lemma limits the γ which appear in the sum (6).

LEMMA 7.6. <u>There exists a point</u> T_0 <u>in</u> $\mathfrak{a}_0$ <u>depending only on the support of</u> ϕ <u>and on the compact set</u> Ω <u>such that</u>

$$\hat{\tau}_1(H(g) - H(h) - T_0) = 1$$

<u>whenever</u>

$$K_{P_1}(mhk, gk') \neq 0$$

<u>for some</u> $m \in \mathbf{M}_1^1$ <u>and some</u> $k, k' \in \Omega$. <u>Here</u> h <u>and</u> g <u>lie in</u> $\mathbf{G}$.

The following lemma and Lemma 7.6 taken together allow us to estimate the sum (6).

LEMMA 7.7. <u>Suppose that</u> $T \in \mathfrak{a}_0$ <u>and</u> $\mathbf{M}_1 \geq 0$. <u>Then we can find constants</u> c <u>and</u> M_1^1 <u>and a set</u> $[P_1 \backslash G]$ <u>of representatives for</u> $P_1 \backslash G$ <u>such that for any</u> $h, g \in \mathbf{G}$

$$\sum_{\delta \in [P_1 \backslash G]} |\delta g|^{M_1} \hat{\tau}_{P_1}(H(\delta g) - H(h) - T_0) \leq c|h|^{M_1^1} |g|^{M_1^1} .$$

The upshot of these considerations is that the domain of integration in (5) can be taken to be

$$\{(m, a) \mid m \in \mathfrak{G}, a \in A_1^Q, \|H(a)\| \le c(1+\|H(m)\|)\}$$

and that the integrand is dominated on this domain by a constant times

$$(1+\|H(m)\|)^r \rho_{P_1}^{-2}(a) \, |m|^{-M'} |m|^{2M_1'} |a|^{M_1'} ,$$

where M_1' is some perhaps large but well determined number and M' is arbitrary. Thus this expression is dominated by a constant times

$$(1+\|H(m)\|)^r |m|^{-M''} ,$$

where M'' can be taken arbitrarily large. We can integrate over the variable a. Since the integration is over a ball of radius $c(1+\|H(m)\|$ we are left with

$$c\int_{\mathfrak{G}} (1+\|H(m)\|^{r'} |m|^{-M''} dm$$

and this integral is finite.

Lecture 8

PREPARATION FOR THE COARSE χ-EXPANSION

II: PROOF OF THE LEMMAS

R. Langlands

We begin with an easy one, Lemma 7.2. Since the truncation operator applied to $K_{P_1}(pabk, \gamma\varepsilon(p'abk'))$ is applied to it as a function of p it is enough to show that

$$K_{P_1}(pabk, \gamma\varepsilon(p'abk')) = \rho_{P_1}^{-2}(b)K_{P_1}(pak, \gamma\varepsilon(p'ak')) \quad .$$

We recall that

$$K_{P_1}(pabk, \gamma\varepsilon(p'abk')) = \sum_{\delta \in N_1 \backslash P_1} \int_{\mathbf{N}_1} \phi(k^{-1}b^{-1}a^{-1}p^{-1}n_1\delta\gamma\varepsilon(pa)\varepsilon(b)\varepsilon(k))dn_1 \quad ,$$

and notice that $\varepsilon(b) = b$. Since $P_1 \subset Q$,

$$b^{-1}a^{-1}p^{-1} = a^{-1}p^{-1}b^{-1}n_2 \quad ,$$

with $n_2 \in \mathbf{N}_1$, and a change of variables allows us to absorb n_2 in n_1. Since Q is ε-invariant, $\varepsilon(pa)$ still lies in $Q(\mathbb{A})$ and

$$\gamma\varepsilon(pa)b = n_3b\gamma\varepsilon(pa) \quad ,$$

with $n_3 \in \mathbf{N}_Q$. Since $\mathbf{N}_Q \subseteq \mathbf{N}_1$ we can also absorb n_3 in n_1, and the desired equality follows from the observation that

$$\left|\frac{dn_1}{d(b^{-1}n_1b)}\right| = \rho_{P_1}^2(b) \quad .$$

We next turn to Lemma 7.7. Recall that we established during the proof of Lemma 2.1 that if

$$\hat{\tau}_{P_1}(H(\delta g) - H(h) - T_0) \neq 0$$

then we could find a representative δ' for δ such that

$$|\delta' g| \leq c(|g|e^{\|H(h)+T_0\|})^N \quad .$$

Thus Lemma 7.7 follows from Lemma 2.1 provided we note that

$$\|H(h)\| \leq c(1 + |\ell n|h\|) \quad .$$

Working our way backwards we next prove Lemma 7.6. Since $K\Omega K = \Omega$ we may suppose that g and h lie in $\mathbb{P}_1$. Then

$$K_{P_1}(mhk, gk') \neq 0$$

implies that for some $n \in \mathbb{N}_1$ and some $m' = \gamma m \in \mathbb{M}_1^1$

$$g^{-1}nm'h \in \Omega(\mathrm{supp}\ \phi)^{-1}\Omega^{-1} = C \quad .$$

Thus if we choose ρ and v_α as in the second lecture, $\alpha \in \Delta_0 - \Delta_0^{P_1}$ we have

$$\|\rho(g^{-1}nm'h)v_\alpha\| \leq c \quad .$$

On the other hand we have chosen g and h to lie in $\mathbf{P}_1$ and nm' lies in $\mathbf{P}_1^1$. Thus

$$\|\rho(g^{-1}nm'h)v_\alpha\| = e^{d_\alpha \varpi_\alpha (H(h)-H(g))} \|v_\alpha\| \quad .$$

The conclusion is that

$$\varpi_\alpha(H(h) - H(g)) \leq c$$

for all $\varpi_\alpha \in \hat{\Delta}_{P_1}$. This clearly implies the statement of the lemma.

To prove Lemma 7.5 we observe that $R(Y)K_{P_1}$ is obtained by replacing ϕ by $\psi = L(Y)\phi$ and then building the kernel attached to ψ. Thus it is enough to prove the lemma for $Y = 1$.

We have, for $\gamma \in P_1$

$$\int_{\mathbf{N}_1} \phi(k^{-1}h^{-1}n_1\gamma gk') = \rho_1^2(g) \int_{\mathbf{N}_1} \phi(k^{-1}h^{-1}\gamma g n_1 k)\,dn_1$$

$$= \rho_1^2(h) \int_{\mathbf{N}_1} \phi(k^{-1}n_1 h^{-1}\gamma gk)\,dn_1 \quad .$$

Moreover, since $K\Omega K = \Omega$ we may suppose that h and g lie in $\mathbf{P}_1$, and indeed in $\mathbf{M}_1$ because, for example,

$$|m| \leq c|g|^N$$

if $g = mn$, $n \in \mathbf{N}_1$, $m \in \mathbf{M}_1$. Then the integral can be taken over a fixed compact set in $\mathbf{N}_1$ which does not depend on h and g. So we need only estimate

$$\sum_{M_1} \chi(h^{-1}\gamma g)$$

where χ is the characteristic function of a compact set C in $\mathbf{M}_1$. This is the number of $\gamma \in M_1$ for which $\gamma \in hCg^{-1}$ and is easily estimated by Lemma 2.1.

Lemma 7.3 is geometrical. Since $\Delta_1^Q \subseteq \Delta_1^{P_2}$ we have

$$\alpha(H_1^Q) = \alpha(H) > 0$$

for $\alpha \in \Delta_1^Q$. Moreover

$$\varpi(H_Q^\varepsilon) > 0 , \qquad \varpi \in \hat{\Delta}_{P_2} .$$

Since $\hat{\Delta}_Q$ is the E-orbit of $\hat{\Delta}_{P_2}$, E being $\{1, \varepsilon, \ldots, \varepsilon^{\ell-1}\}$, and since H_Q^ε is ε-invariant we obtain the same inequality for $\varpi \in \hat{\Delta}_Q$. On the other hand

$$\alpha(H_Q^\varepsilon) < -\alpha(H_1^Q)$$

for $\alpha \in \Delta_{P_2}$. Thus

$$\alpha(H_Q^\varepsilon) \leq C\|H_1^Q\|$$

for $\alpha \in \Delta_Q$, and the lemma follows.

This leaves us with Lemmas 7.1 and 7.4. The first is the easier, and we begin with it.

The assumption of the lemma implies that for some p in $\mathbf{P}_1^1$

$$\phi(k^{-1}g^{-1}p\gamma\varepsilon(gk')) \neq 0 \quad .$$

Thus

$$g^{-1}p\gamma\varepsilon(g) \in C \quad ,$$

C being a compact set depending only on Ω and the support of ϕ. Taking $\alpha \in \Delta_0 - \Delta_0^Q$ we evaluate $\|\rho(g^{-1}p\gamma\varepsilon(g))v_\alpha\|$

$$e^{-d_\alpha\varpi_\alpha(H(g)-\varepsilon H(g))}\|v_\alpha\|$$

and $\|\rho(\varepsilon(g^{-1})\gamma^{-1}p^{-1}g)v_\alpha\|$ as

$$e^{d_\alpha\varpi_\alpha(H(g)-\varepsilon H(g))}\|v_\alpha\| \quad .$$

Both expressions are bounded above. If $\chi \in X^*(G)$ we can also bound

$$|\chi(g^{-1}\varepsilon(g))| = |\chi(g^{-1}p\gamma\varepsilon(g))|$$

above and below, concluding that $\|H(g) - \varepsilon H(g)\|$ is bounded. The lemma follows.

We deal finally with Lemma 7.4. It is clear that the assumption implies that

$$(1) \qquad K_{P_1}(m'ak,\ \gamma\varepsilon(nmak)) \neq 0$$

for some $m' \in \mathbf{M}_1^1$. For brevity we write $H = H(a)$.

We write $\gamma = \eta'\omega\eta$ with $\eta' \in P_1$, $\eta \in N_0$, and with ω in the normalizer of A_0. It represents an element s in the Weyl group of

A_0 or $\mathfrak{a}_0$. We are free to modify ω on the left by an element of M_1, incorporating the modification in η'. So we can suppose that $s^{-1}\alpha > 0$ if $\alpha \in \Delta_0^1$. Since H lies in A_1^Q and $\alpha(H) > 0 \;\forall \alpha \in \Delta_1^Q$ we have $H \in \mathfrak{a}_1^+ \subseteq \mathfrak{a}_0^{+}$. Thus εH is also in $\mathfrak{a}_0^{+}$ and if $\alpha \in \Delta_0^{P_1}$ then

$$\alpha(H - s\varepsilon H) = -s^{-1}\alpha(\varepsilon H) < 0 \quad .$$

We shall now show in addition that, for $\alpha \in \Delta_0 - \Delta_0^{P_1}$,

$$(2) \qquad \varpi_\alpha(H - s\varepsilon H) \leq c(1 + \|H(m)\|) \quad .$$

This will allow us to infer from (iv) of Lecture 2 that

$$(3) \qquad H - s\varepsilon H \in X - {}^+\mathfrak{a}_0$$

where $\|X\| \leq c(1 + \|H(m)\|)$.

To prove (2) we notice that (1) implies that for some $m' \in \mathbf{M}_1^1$ and some $n_1 \in \mathbf{N}_1$

$$\varepsilon(a^{-1}m^{-1}n^{-1})\eta^{-1}\omega^{-1}n_1 m'a \in C \quad ,$$

η' having been absorbed in $n_1 m'$. Thus

$$\|\rho(\varepsilon(a^{-1}m^{-1}n^{-1})\eta^{-1}\omega^{-1}n_1 m'a)v_\alpha\| \leq c \quad .$$

We choose $\alpha \in \Delta_0 - \Delta_0^1$. Then

$$\|\rho(n_1 m'a)v_\alpha\| = |\xi_{d_\alpha \varpi_\alpha}(a)| \; \|v_\alpha\| \quad .$$

If $w = \omega^{-1}v_\alpha$ then w is a weight vector corresponding to the weight $s^{-1}\alpha : H \longrightarrow \alpha(sH)$. Moreover

$$\rho(\varepsilon(n^{-1})\eta^{-1})w = w+u$$

where u is an adelic linear combination of weight vectors for weights of the form

$$s^{-1}\alpha + \sum_{\beta \in \Delta_0} c_\beta \beta$$

with $c_\beta \geq 0$, $\sum c_\beta \neq 0$. Consequently

$$\|\rho(\varepsilon(a^{-1}m^{-1})\varepsilon(n^{-1})\eta^{-1})w\| \geq c\|\rho(\varepsilon(a^{-1}m^{-1}))w\|$$

$$= ce^{-d_\alpha \varpi_\alpha(s\varepsilon H+s\varepsilon H(m))}\|w\| \quad .$$

We deduce that

$$c \geq e^{d_\alpha \varpi_\alpha(H-s\varepsilon H-s\varepsilon H(m))} \quad .$$

Taking logarithms we obtain (2).

We write the left side of (3) as

$$H-s\varepsilon H = H-\varepsilon H+\varepsilon H-s\varepsilon H \quad .$$

Since $H \in \mathfrak{a}_0^+$, its transform εH also lies in $\mathfrak{a}_0^+$ and $\varepsilon H-s\varepsilon H \in {}^+\mathfrak{a}_0^{\Phi}$. We conclude that

$$H-\varepsilon H = X-Y$$

with $Y \in {}^{+}\mathfrak{a}_0$.

Applying ε^k, $0 \leq k < \ell$ to this relation and summing over k we obtain

$$0 = X' - \sum_{k=0}^{\ell-1} \varepsilon^k(Y)$$

with $\|X'\| \leq c(1+\|H(m)\|)$. Since every $\varepsilon^k(Y)$ lies in ${}^{+}\mathfrak{a}_0$ we infer that

$$\|Y\| \leq c(1+\|H(m)\|) \quad .$$

This implies first that

$$\|H-\varepsilon H\| \leq c(1+\|H(m)\|)$$

and thus that there is an ε-invariant H_0 with

$$\|H-H_0\| \leq c(1+\|H(m)\|) \quad .$$

We are reduced to showing that

$$\|H_0\| \leq c(1+\|H(m)\|)$$

knowing that

$$H_0-sH_0 = X-{}^{+}\mathfrak{a}_0 \quad ,$$

$\|X\| \leq c(1+\|H(m)\|)$, a relation which we deduce from (3). Since we may take

$$H_0 = \frac{1}{\ell} \sum_{k=0}^{\ell-1} \varepsilon^k H \quad ,$$

we may suppose that $H_0 \in \mathfrak{a}_0^+$. Then

$$H_0 - sH_0 \in {}^+\mathfrak{a}_0 \quad ,$$

and we conclude that

$$\|H_0 - sH_0\| \leq c(1+\|H(m)\|) \quad .$$

At this point we introduce the assumption that $\gamma \in F_\varepsilon(P_1, P_2)$. We know that for any $H \in \mathfrak{a}_0$,

$$H-sH = \sum_{\alpha \in \Delta_0} c_\alpha(H, s)\alpha$$

where $H \longrightarrow c_\alpha(H, s)$ is a linear form on $\mathfrak{a}_0$, non-negative on $\mathfrak{a}_0^+$. In particular $c_\alpha(\varpi_\beta, s) \geq 0$. If $c_\alpha(\varpi_\beta, s) = 0$ for all α then $s\varpi_\beta = \varpi_\beta$. Thus

$$\sup_\alpha c_\alpha(H, s) \geq c\beta(H)$$

provided $s\varpi_\beta \neq \varpi_\beta$.

Applying this to H_0 we conclude that

$$|\beta(H_0)| \leq c(1+\|H(m)\|) \tag{4}$$

unless $s\varpi_{\beta'} = \varpi_{\beta'}$ for every β' in the E-orbit of β, because $\beta(H_0)$ is constant on such orbits.

To prove the lemma we need to establish (4) for all $\beta \in \Delta_0^Q$. If the E-orbit of β does not meet $\Delta_0^{P_1}$ then it cannot happen that $s\varpi_{\beta'} = \varpi_{\beta'}$ for all β' in this ε-orbit for then $\gamma \in Q'$ if

$$\Delta_0^{Q'} = \Delta_0^Q - \{\varepsilon^k \beta\}$$

and $Q' \supset P_1$ and is ε-invariant. On the other hand if for some β' in the E-orbit of β we have $\beta' \in \Delta_0^{P_1}$ then

$$\beta'(H_0) = \beta(H_0)$$

and

$$\beta'(H) = 0 \quad .$$

Thus

$$|\beta(H_0)| = |\beta'(H-H_0)| \leq c(1+\|H(m)\|) \quad .$$

Lecture 9

THE MODIFIED BASIC IDENTITY AND WEIGHTED ORBITAL INTEGRALS

J.-P. Labesse

9.1. The modified basic identity.

As was pointed out in Lecture 5, it seems to be more convenient when dealing with σ-expansion to use a completely ε-invariant truncation. Let P be an ε-invariant parabolic subgroup of G; we define ${}_\varepsilon\hat{\tau}_P$ to be the characteristic function of the set of $X \in \mathfrak{a}_0$ such that

$$\varpi_{\bar{\alpha}}(X) > 0 \qquad \forall \alpha \in \Delta_0 - \Delta_0^P$$

where by definition

$$\varpi_{\bar{\alpha}} = \frac{1}{\ell} \sum_{r=0}^{\ell-1} \varepsilon^r \varpi_\alpha$$

and $\bar{\alpha}$ is the orbit of α under E.

We need the analogue of Lemma 2.1, namely the

LEMMA 9.1.1. *There are constants* c *and* N *such that the number of* $\delta \in P\backslash G$ *for which*

$$\varpi_{\bar{\alpha}}(H(\delta x) - T) > 0$$

for all $\alpha \in \Delta_0 - \Delta_0^P$ *is at most* $c|x|^N e^{N\|T\|}$.

We need only to show that we can find a set of representatives for these δ each of which satisfies

$$|\delta| < c'|x|^{N_1} e^{\|T\| N_1} .$$

According to reduction theory we may choose δ such that $\delta x \in \mathfrak{S}_P(T_0)$ and more precisely we may assume that $\delta x = nam$ with $n \in \omega_1$ a compact set in $\mathbf{N}$, $m \in \omega_2$ a compact set in $\mathbf{G}$ and $a \in A_0(\mathbb{R})^0$; moreover $H(\delta x) = H(a)$ and

$$(1) \qquad \alpha(H(\delta x)) > \alpha(T_0) \qquad \forall \alpha \in \Delta_0^P .$$

Taking over the numbering of Lecture 2, we have

$$(3) \qquad \varpi_\alpha(H(\delta x)) \le c_1(1 + \log|x|) \qquad \forall \alpha \in \Delta_0 .$$

Our assumption is not (4) as in Lecture 2 but only

$$(4') \qquad \varpi_{\bar{\alpha}}(H(\delta x)) > \varpi_{\bar{\alpha}}(T) \qquad \forall \alpha \in \Delta_0 - \Delta_0^P .$$

But (3) and (4') imply

$$(4'') \qquad \varpi_\alpha(H(\delta x)) > \varpi_{\bar{\alpha}}(T) - \frac{\ell-1}{\ell} c_1(1 + \log|x|)$$

for any $\alpha \in \Delta_0 - \Delta_0^P$.

The inequalities (1), (3), and (4) yield the inequality

$$(2) \qquad \|H(\delta x)\| < c_2(1 + \log|x| + \|T\|) .$$

Since $\delta x = nam$ with n and m in compact sets and $H(\delta x) = H(a)$ we conclude that

$$\log|\delta x| \leq c_3(1 + \log|x| + \|T\|) \quad . \qquad \square$$

Given $P_1 \subset P \subset P_2$ three parabolic subgroups with P ε-invariant we define ${}_P\sigma_1^2$ to be the characteristic function of the set of $H \in \mathfrak{a}_0$ such that

$$\begin{array}{lll} \text{(i)} & \alpha(H) > 0 & \forall \alpha \in \Delta_1^2 \\ \text{(ii)} & \alpha(H) \leq 0 & \forall \alpha \in \Delta_1 - \Delta_1^2 \\ \text{(iii)} & \varpi_{\bar{\alpha}}(H) > 0 & \forall \alpha \in \Delta_0 - \Delta_0^P \quad . \end{array}$$

We obviously have

$$\sum_{P_2 \supset P} {}_P\sigma_1^2 = \tau_{P_1}^P \, {}_\varepsilon\hat{\tau}_P \quad .$$

Given P_1 and P_2 we define ${}_\varepsilon\sigma_1^2$ to be zero if there is no ε-invariant parabolic subgroup between P_1 and P_2 and to be ${}_Q\sigma_1^2$ if Q, the maximal ε-invariant parabolic subgroup in P_2, contains P_1.

LEMMA 9.1.2. _If_ $P_1 \subset P \subset P_2$ _with_ P ε-_invariant then_

$${}_P\sigma_1^2 = {}_\varepsilon\sigma_1^2 \quad .$$

Clearly ${}_P\sigma_1^2 \leq {}_\varepsilon\sigma_1^2$; now consider H in the support of ${}_\varepsilon\sigma_1^2$, we need to prove that for any $\alpha \in (\Delta_0 - \Delta_0^P) \cap \Delta_0^Q$ we have $\varpi_{\bar{\alpha}}(H) > 0$.

We know (cf. Lecture 2, page 7) that

$$\varpi_\alpha = \varpi'_\alpha + \sum_{\beta \in \Delta_0 - \Delta_0^Q} \lambda_{\alpha\beta} \varpi_\beta$$

with $\lambda_{\alpha,\beta} \geq 0$ and $\varpi'_\alpha \in \hat{\Delta}_P^Q \subset \hat{\Delta}_1^Q$. The same equation holds for averaged weights, and by assumption $\varpi_{\bar{\beta}}(H) > 0$. All we need to prove is that $\varpi'_\alpha(H) > 0$, but $\Delta_1^Q \subset \Delta_1^2$ and hence $\gamma(H) > 0$ for any $\gamma \in \Delta_1^Q$; using assertion (c) of Lecture 2, page 6, which tells us that

$$(\mathfrak{a}_1^Q)^+ \subset {}^+\mathfrak{a}_1^Q$$

we see that $\varpi'_\alpha(H) > 0$ for any $\varpi'_\alpha \in \hat{\Delta}_P^Q$. □

We may now state the modified basic identity

PROPOSITION 9.1.3. <u>Given</u> P, <u>an</u> ε-<u>invariant parabolic subgroup we have</u>

$$\sum_{\delta \in P \backslash G} K_P(\delta x, \delta x)\, {}_\varepsilon\hat{\tau}_P\, (H(\delta x) - T)$$

$$= \sum_{P_1 \subset P \subset P_2} \; \sum_{\xi \in P_1 \backslash G} {}_\varepsilon\sigma_1^2 (H(\delta x) - T)\, \Lambda^{T,P_1} K_P(\xi x, \xi x) \quad .$$

Using Lemma 2.4 (in Lecture 2) one needs only to remark that, thanks to Lemma 9.1.2 we have

$$\sum_{P \subset P_2} {}_\varepsilon\sigma_1^2 = \tau_1^P \; {}_\varepsilon\hat{\tau}_P \quad .$$ □

In Lectures 3 and 4 we may now substitute ${}_\varepsilon\hat{\tau}_P$ for $\hat{\tau}_P$ and ${}_\varepsilon\sigma_1^2$ for σ_1^2, and no change in the proofs is needed except Lemma 4.2.2 which has to be replaced by

LEMMA 9.1.4. Assume that $H \in \mathfrak{z}^\varepsilon \setminus \mathfrak{a}_1^\varepsilon$ and $X \in \omega$ (some compact set in $\mathfrak{a}_0$) are such that ${}_\varepsilon\sigma_1^2(H-X) = 1$; then there exist a constant c independent of X such that

$$\|H_2\| < c(1 + \|H_1\|) \quad .$$

Recall that $H = H_1 + H_2$ is the decomposition associated with the direct sum

$$\mathfrak{z}^\varepsilon \setminus \mathfrak{a}_1^\varepsilon = (\mathfrak{a}_R^Q)^\varepsilon \oplus \mathfrak{z}^\varepsilon \setminus \mathfrak{a}_2^\varepsilon \quad .$$

By assumption $\alpha(H-X) \leq 0$ for any $\alpha \in \Delta_1 - \Delta_1^2$ and then

$$\alpha(H_2) \leq -\alpha(H_1) + c_1$$

for any $\alpha \in \Delta_R - \Delta_R^Q$, and some constant c_1.

For any $\alpha \in \Delta_0 - \Delta_0^Q$ we have assumed that $\varpi_{\bar{\alpha}}(H-X) > 0$ and hence

$$\varpi_\alpha(H_2) = \varpi_{\bar{\alpha}}(H) > \varpi_{\bar{\alpha}}(X) \geq c_2 \quad . \qquad \square$$

The modified truncation was already used in Lecture 5. No modification has to be made in Lecture 6, in particular the $\hat{\tau}_P$ in that lecture should not be modified. In Lectures 7 and 8 one has to substitute ${}_\varepsilon\sigma_1^2$ for σ_1^2 and the only slight change is in the proof of Lemma 7.3 (essentially the same as Lemma 4.2.2).

9.2. The convex hull of some "orthogonal sets."

The aim of this section is to prove Lemma 5.3.5. Given $s \in \Omega$ we introduced

$$H_s(\mathfrak{a}, T) = s^{-1}(T - H(w_s x)) \quad .$$

LEMMA 9.2.1. *Given* s *and* t *in* Ω *and* T *sufficiently regular*

$$H_s(x, T) - H_t(x, T)$$

is a positive linear combination of the roots γ *such that* $s\gamma > 0$ *and* $t\gamma < 0$.

Let $y = w_s x$ and $\sigma = ts^{-1}$, we have

$$\begin{aligned} H_s(x, T) - H_t(x, T) &= s^{-1}(T - H(y) - H_\sigma(y, T)) \\ &= s^{-1}(T - \sigma^{-1}T + H(w_\sigma n)) \end{aligned}$$

if $y = ank$ with $a \in \mathbf{M}_0$, $n \in \mathbf{N}_0$ and $k \in K$. Using Lemma 6.3 and the comments preceding it we see that this equals

$$s^{-1}\left(\sum_{\substack{\beta > 0 \\ \sigma\beta < 0}} h_\beta \check{\beta} \right) = \sum_{\substack{s\gamma > 0 \\ t\gamma < 0}} h_{s\gamma} \check{\gamma}$$

with $h_\beta = h_{s\gamma} > 0$ if T is sufficiently regular. □

Let P_1 be an ε-invariant parabolic subgroup of G; the real vector space $(\mathfrak{a}_1^G)^\varepsilon$ isomorphic to $\mathfrak{z}^\varepsilon \backslash \mathfrak{a}_1^\varepsilon$ will be denoted by V_1. Any root β of A_1 in N_1 defines a hyperplane $V_{\bar\beta}$ which depends only on the orbit $\bar\beta$ of β under E. The chambers are the connected components of V_1' the complement in V of the union of the $V_{\bar\beta}$. The positive chamber C_1^ε is defined by the following inequalities:

$$\bar\alpha(H) > 0 \qquad \forall \alpha \in \Delta_0 - \Delta_0^1 .$$

Given $s \in \Omega(\mathfrak{a}_1^\varepsilon)$ there is a standard ε-invariant parabolic subgroup P_2 such that $s(\mathfrak{a}_1^\varepsilon) = \mathfrak{a}_2^\varepsilon$, we define $C_1^\varepsilon(s)$ to be $s^{-1}(C_2^\varepsilon)$ where C_2^ε is the positive chamber of V_2. The chamber $C_1^\varepsilon(s)$ is the set of $H \in V_1$ such that

$$\gamma(H) > 0 \qquad \forall \gamma \in \Delta_1(s, \varepsilon)$$

where $\Delta_1(s, \varepsilon)$ is the set of restrictions to V_1 of the elements in $s^{-1}(\Delta_0 - \Delta_0^2)$.

LEMMA 9.2.2. _The map_ $s \longrightarrow C_1^\varepsilon(s)$ _is a bijection between_ $\Omega(\mathfrak{a}_1^\varepsilon)$ _and the set of chambers in_ V_1.

Given a chamber C^ε in V_1 there is a unique chamber C in $\mathfrak{a}_1^G$ which contains C^ε, and there is at least one $s \in \Omega$ such that $F = sC$ is a "facette" of C_0 the positive Weyl chamber in $\mathfrak{a}_0^G$. Since Ω acts simply transitively on the set of Weyl chambers the "facette" F depends only on C^ε. Since Ω^{M_1} acts trivially on $\mathfrak{a}_1^G$ we may choose

s so that $s\alpha > 0$ for any $\alpha \in \Delta_0^1$. Under those conditions s is uniquely determined by C^ε. Since P_1 is ε-invariant we see that s and $\varepsilon s \varepsilon^{-1}$ have the same properties and hence $s = \varepsilon s \varepsilon^{-1}$. This implies $s \in \Omega(\mathfrak{a}_1^\varepsilon)$ and we conclude that $C^\varepsilon = C_1^\varepsilon(s)$ for a unique $s \in \Omega(\mathfrak{a}_1^\varepsilon)$. □

Two chambers $C_1^\varepsilon(s)$ and $C_1^\varepsilon(t)$ are said to be adjacent if there exist a linear form λ on V_1 unique up to scalars such that λ is positive on $C_1^\varepsilon(s)$ and negative on $C_1^\varepsilon(t)$; in particular the roots γ such that $s\gamma > 0$ and $t\gamma < 0$ have restrictions to V_1 equal to $c_\gamma \lambda$ with $c_\gamma > 0$; the projection $\check{\gamma}_1$ on V_1 of $\check{\gamma}$ equals $c'_\gamma \check{\lambda}$ with $c'_\gamma > 0$.

Given $s \in \Omega(\mathfrak{a}_1^\varepsilon)$ we define H_s^ε to be the projection on V_1 of $H_s(x, T)$. $\sim s^{-1}\Gamma$

LEMMA 9.2.3. *If* s *and* t *define adjacent chambers then*

$$H_s^\varepsilon - H_t^\varepsilon = c\check{\lambda}$$

with $c > 0$ (*provided* T *is sufficiently regular*).

According to Lemma 9.2.1 we have

$$H_s^\varepsilon - H_t^\varepsilon = \sum_{\substack{s\gamma>0 \\ t\gamma<0}} h_{s\gamma} \check{\gamma}_1 = \sum h_{s\gamma} c'_\gamma \check{\lambda}$$

with $h_{s\gamma}$ and c'_γ positive. □

Given $s \in \Omega(\mathfrak{a}_1^{\varepsilon})$ we have introduced $\Delta_1(s, \varepsilon)$; let $\hat{\Delta}_1(s, \varepsilon)$ be the set of ϖ_γ the dual basis of the basis defined by the $\check{\gamma}$ with $\gamma \in \Delta_1(s, \varepsilon)$.

LEMMA 9.2.4. _If_ s _and_ t _define adjacent chambers there is a bijection_ $\theta : \Delta_1(s, \varepsilon) \longrightarrow \Delta_1(t, \varepsilon)$ _such that_ $\theta(\beta) = -\beta$ _if_ β _defines the wall between the two chambers and_ $\varpi_{\theta(\gamma)} = \varpi_\gamma$ _if_ $\gamma \neq \beta$.

Let β be the element in $\Delta_1(s, \varepsilon)$ which defines the wall between $C_1^{\varepsilon}(s)$ and $C_1^{\varepsilon}(t)$, then $-\beta \in \Delta_1(t, \varepsilon)$ and we define θ on β by $\theta(\beta) = -\beta$. Let V_β be the hyperplane defined by β and $\gamma \in \Delta_1(s, \varepsilon)$ with $\gamma \neq \beta$; there exist $\gamma_1 \in \Delta_s(t, \varepsilon)$ such that γ and γ_1 have the same projection on V_β and we define $\theta(\gamma)$ to be γ_1. Then clearly $\varpi_{\theta(\gamma)} = \varpi_\gamma$ if $\gamma \neq \beta$. □

Let $\Lambda \in V_1'$, the complement of the walls in V_1, and $s \in \Omega(\mathfrak{a}_1^{\varepsilon})$; we define φ_s^{Λ} to be the characteristic function of the set of $H \in V_1$ such that

$$\left.\begin{aligned} \varpi_\gamma(H) &\leq 0 \quad \text{if} \quad \gamma(\Lambda) > 0 \\ \varpi_\gamma(H) &> 0 \quad \text{if} \quad \gamma(\Lambda) < 0 \end{aligned}\right\} \quad \forall \gamma \in \Delta_1(s, \varepsilon) \quad .$$

Let $a(s, \Lambda)$ be the number of $\gamma \in \Delta_1(s, \varepsilon)$ such that $\gamma(\Lambda) < 0$. In Lecture 5 we have introduced functions ${}_{\varepsilon}B_1^{s}$ on $\mathfrak{a}_0$, which depend only on the projection on V_1. If $\Lambda \in C_1^{\varepsilon}$ the positive chamber in V_1 we have

$$ {}_{\varepsilon}B_1^s(H) = (-1)^{a(s,\Lambda)} \varphi_s^{\Lambda}(H) \quad . $$

We now introduce

$$ \psi(\Lambda, H) = \sum_{s \in \Omega(\mathfrak{a}_1^{\varepsilon})} (-1)^{a(s,\Lambda)} \varphi_s^{\Lambda}(H-H_s^{\varepsilon}) \quad . $$

Lemma 5.3.5 is an immediate consequence of the

PROPOSITION 9.2.5. <u>The function</u> $H \longrightarrow \psi(\Lambda, H)$ <u>is independent of</u> $\Lambda \in V_1'$ <u>and is the characteristic function of the convex hull of the</u> H_s^{ε} (<u>provided</u> T <u>is sufficiently regular</u>).

We need some more lemmas.

LEMMA 9.2.6. <u>Assume</u> $\varpi_{\gamma}(H-H_s^{\varepsilon}) \leq 0$ <u>for each</u> $\gamma \in \Delta_1(s, \varepsilon)$ <u>and each</u> $s \in \Omega(\mathfrak{a}_1^{\varepsilon})$, <u>then for any</u> $\Lambda \in V_1'$ <u>one has</u> $\psi(\Lambda, H) = 1$.

Given Λ, there is one and only one $s \in \Omega(\mathfrak{a}_1^{\varepsilon})$ such that $\Lambda \in C_1^{\varepsilon}(s)$ and by definition of φ_t^{Λ} we see that $\varphi_t^{\Lambda}(H-H_t^{\varepsilon}) = 0$ unless $t = s$ and hence

$$ \psi(\Lambda, H) = \varphi_s^{\Lambda}(H-H_s) = 1 \quad . \qquad \square $$

LEMMA 9.2.7. <u>Assume</u> $\psi(\Lambda, H) \neq 0$ <u>and</u> $\Lambda \in C_1^{\varepsilon}(s)$ <u>then provided</u> T <u>is sufficiently regular we have</u>

$$ \varpi_{\gamma}(H-H_s^{\varepsilon}) \leq 0 \quad \text{for } \underline{\text{all}} \quad \gamma \in \Delta_1(s, \varepsilon) \quad . $$

Since $\Lambda \longrightarrow \psi(\Lambda, H)$ is constant on $C_1^{\varepsilon}(s)$ it suffices to prove that $\langle \Lambda, H-H_s \rangle \leq 0$.

If $\psi(\Lambda, H) \neq 0$, then for at least one $t \in \Omega(\mathfrak{a}_1^\varepsilon)$ we have ${}^\Lambda_t(H-H_t^\varepsilon) = 1$ and hence

$$\sum \gamma(\Lambda)\varpi_\gamma(H-H_t^\varepsilon) = \langle \Lambda, H-H_t^\varepsilon \rangle \leq 0 \quad ;$$

but

$$\langle \Lambda, H-H_s^\varepsilon \rangle = \langle \Lambda, H-H_t^\varepsilon \rangle + \langle \Lambda, H_t^\varepsilon - H_s^\varepsilon \rangle$$

and Lemma 9.2.1 implies

$$\langle \Lambda, H_t^\varepsilon - H_s^\varepsilon \rangle \leq 0 \quad . \qquad \square$$

LEMMA 9.2.8. <u>The set</u> C <u>of</u> $H \in V_1$ <u>such that</u> $\varpi_\gamma(H-H_s^\varepsilon) \leq 0$ <u>for all</u> $\gamma \in \Delta_1(s, \varepsilon)$ <u>and all</u> $s \in \Omega(\mathfrak{a}_1^\varepsilon)$ <u>is the convex hull of the set of</u> H_s^ε <u>with</u> $s \in \Omega(\mathfrak{a}_1^\varepsilon)$, <u>provided</u> T <u>is sufficiently regular</u>.

The set C is an intersection of closed convex sets and hence is a closed convex set. Thanks to Lemma 9.2.1 we see that $H_s^\varepsilon \in C$ if T is sufficiently regular. Now consider $H \in V_1$ outside the convex hull of the H_s^ε, then there exist $\Lambda \in V_1$ such that

$$\langle \Lambda, H \rangle > \langle \Lambda, H_s \rangle$$

for all $s \in \Omega(\mathfrak{a}_1^\varepsilon)$ and in particular if s is such that Λ lies in the closure of $C_1^\varepsilon(s)$. This implies $H \notin C$. $\square$

The function $\Lambda \longrightarrow \psi(\Lambda, H)$ is clearly locally constant on V_1'.

To finish the proof of Proposition 9.2.5 we need only to prove that $\psi(\Lambda_\sigma, H) = \psi(\Lambda_\tau, H)$ when $\Lambda_\sigma \in C_1^\varepsilon(\sigma)$ and $\Lambda_\tau \in C_1^\varepsilon(\tau)$ are elements in two adjacent chambers. Let λ be a linear form defining the wall V_λ between the two chambers.

Given $s \in \Omega(\mathfrak{a}_1^\varepsilon)$ then $\varphi_s^{\Lambda_\sigma}(H) = \varphi_s^{\Lambda_\tau}(H)$ if $\gamma(\Lambda_\sigma)\gamma(\Lambda_\tau) > 0$ for all $\gamma \in \Delta_1(s, \varepsilon)$. If it is not so there is one and only one root $\beta \in \Delta_1(s, \varepsilon)$ proportional to λ such that $\beta(\Lambda_\sigma)\beta(\Lambda_\tau) < 0$, and there exist $t \in \Omega(\mathfrak{a}_1^\varepsilon)$ such that $C_1^\varepsilon(s)$ and $C_1^\varepsilon(t)$ are adjacent and separated by V_λ. Since $\gamma(\Lambda_\sigma)\cdot\gamma(\Lambda_\tau) > 0$ if $\gamma \neq \beta$ we see that

$$\xi_s(H) = \varphi_s^{\Lambda_\sigma}(H-H_s^\varepsilon) + \varphi_s^{\Lambda_\tau}(H-H_s^\varepsilon)$$

is the characteristic function of the set of H such that

$$\varpi_\gamma(H-H_s^\varepsilon) > 0 \quad \text{if} \quad \gamma(\Lambda_\sigma) < 0$$

$$\varpi_\gamma(H-H_s^\varepsilon) \leq \quad \text{if} \quad \gamma(\Lambda_\sigma) > 0$$

for all $\gamma \in \Delta_1(s, \varepsilon)$ and $\gamma \neq \beta$. Let θ be the bijection of Lemma 9.2.4; using Lemma 9.2.3 we see that given $\gamma \in \Delta_1(s, \varepsilon)$,

$$\varpi_\gamma(H-H_s) = \varpi_{\theta(\gamma)}(H-H_t)$$

whenever $\gamma \neq \beta$ and hence

$$\xi_s(H) = \xi_t(H)$$

since $\gamma(\Lambda_\sigma)\cdot\theta(\gamma)(\Lambda_\sigma) > 0$ if $\gamma \neq \beta$. This implies that

$$(-1)^{a(s,\Lambda_\sigma)} \varphi_s^{\Lambda_\sigma}(H-H_s) + (-1)^{a(t,\Lambda_\sigma)} \varphi_t^{\Lambda_\sigma}(H-H_t)$$

$$= (-1)^{a(s,\Lambda_\tau)} \varphi_s^{\Lambda_\tau}(H-H_s) + (-1)^{a(t,\Lambda_\tau)} \varphi_t^{\Lambda_\tau}(H-H_t)$$

and hence $\psi(\Lambda_\sigma, H) = \psi(\Lambda_\tau, H)$. □

Let Λ be in any chamber, then

$$v_1^\varepsilon(x, T) = \int_{V_1} \psi(\Lambda, H)\,dH = \lim_{\substack{t\to 0\\ t>0}} \int_{V_1} e^{t\langle\Lambda,H\rangle}\, \psi(\Lambda, H)\,dH$$

$$= \lim_{\substack{t\to 0\\ t>0}} \sum_{s\in\Omega(\mathfrak{a}_1^\varepsilon)} \int_{V_1} e^{t\langle\Lambda,H\rangle} (-1)^{a(s,\Lambda)} \varphi_s^\Lambda(H-H_s)\,dH \quad .$$

An elementary computation yields

$$v_1^\varepsilon(x, T) = \lim_{t\to 0} \sum_{s\in\Omega(\mathfrak{a}_1^\varepsilon)} c_s \frac{e^{t\langle\Lambda,H_s^\varepsilon\rangle}}{t^{a_1^\varepsilon} \prod_{\gamma\in\Delta_1(s,\varepsilon)} \gamma(\Lambda)}$$

where $c_s = |\det(\gamma_i, \gamma_j)|^{\frac{1}{2}}$, $\gamma_i \in \Delta_1(s, \varepsilon)$. Using Lemma 9.2.4 one shows that $c_s = c_1$ is independent of s and depends only on P_1. Finally we get

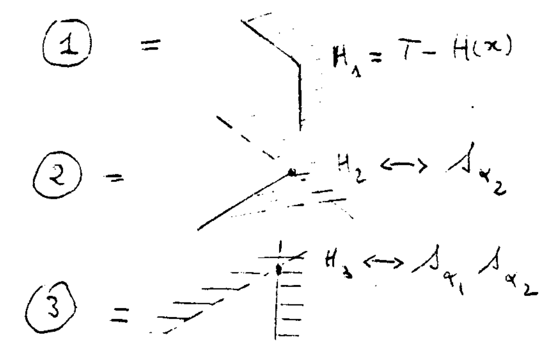
$$v_1^\varepsilon(x, T) = \frac{c_1}{(a_1^\varepsilon)!} \sum_{s\in\Omega(\mathfrak{a}_1^\varepsilon)} \frac{\langle\Lambda,H_s(x,T)\rangle^{a_1^\varepsilon}}{\prod_{\gamma\in\Delta_1(s,\varepsilon)} \gamma(\Lambda)}$$

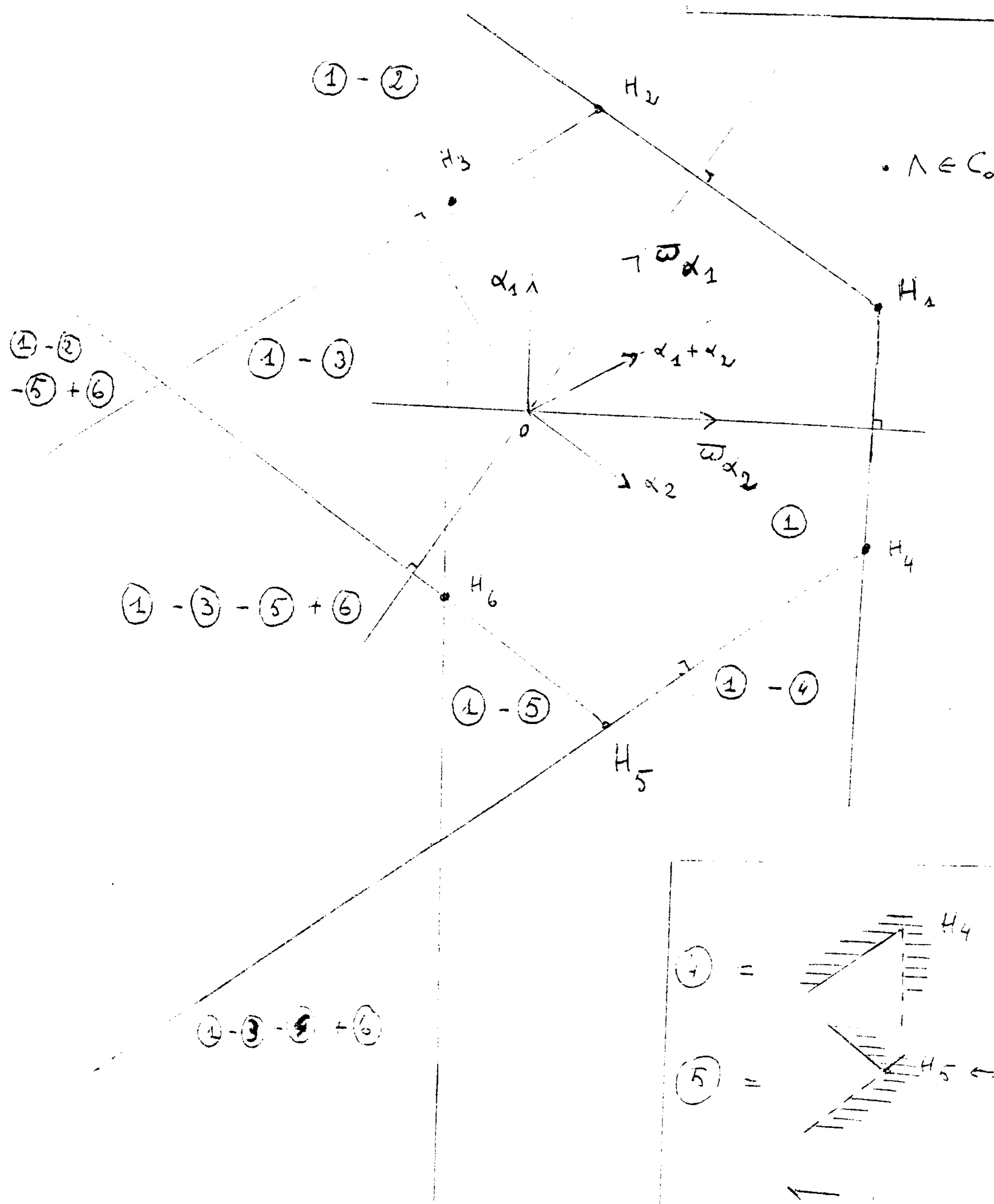
In particular it is a polynomial of T of degree a_1^ε.

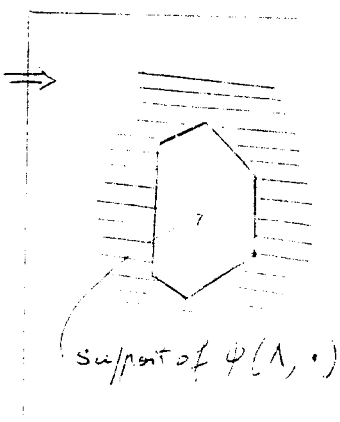
$$\Psi(\Lambda,\cdot) = \sum (-1)^{u(s,\Lambda)} \varphi_s^\Lambda(\cdot)$$

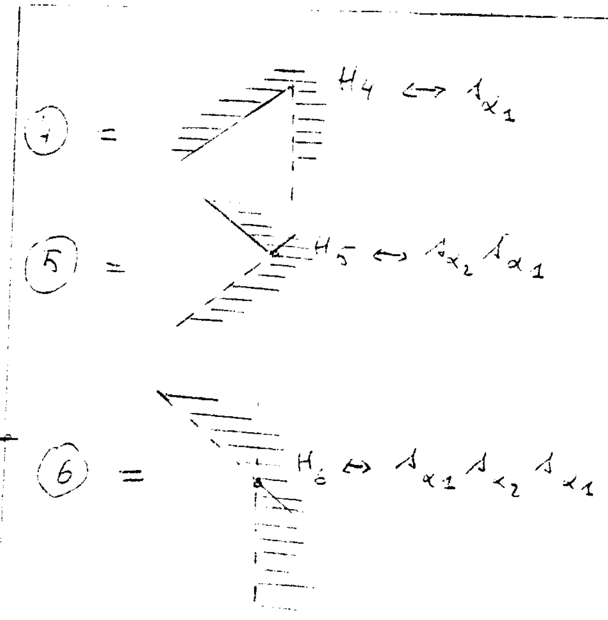
Special case $\quad G = SL_3 \quad P_1 = P_0$

$\Lambda \in C_0 \quad , \varepsilon = 1$









$$H_i = s_i^{-1}(T - H(w_{s_i} x))$$

$$\leftrightarrow s_i \leftrightarrow \{ H \to \varphi_{s_i}^\Lambda (H - H_i) \}$$

Lecture 12

THE INNER PRODUCT FORMULA

J.-P. Labesse

12.1. The constant term of Eisenstein Series.

We consider a standard parabolic subgroup $P = MN$ and a smooth function ψ on G such that

(i) $\psi(nx) = \psi(x)$ $\qquad n \in \mathbf{N},\ x \in \mathbf{G}$.

(ii) $m \longmapsto \psi(mx)$ is for all $x \in \mathbf{G}$ a square integrable automorphic form on $M\backslash\mathbf{M}^1$ which is a matrix coefficient of some unitary representation π of M with central character ω_π trivial on $A_P(\mathbb{R})^o$.

(iii) $k \longmapsto \psi(xk)$ is K-finite.

We shall say that ψ is cuspidal on P if $m \longmapsto \psi(mx)$ is cuspidal (for all x).

Given $x \in \mathbf{G}$ we have defined $H(x) \in \mathfrak{a}_0$; it may be convenient to introduce its exponential in $A_0(\mathbb{R})^o$:

$$a(x) = \exp H(x)$$

so that for $\lambda \in \mathfrak{a}_0^* \otimes \mathbf{C}$ we may write

$$a(x)^\lambda = e^{\lambda(H(x))} \quad .$$

Let ρ or ρ_P denote the half sum of positive roots of A in N; we have (with notations introduced earlier)

$$\delta_P(x) = a(x)^{2\rho} \quad .$$

Given $\lambda \in \mathfrak{a}_P^* \otimes \mathbf{C}$ and ψ as above, we define

$$\psi(x, \lambda) = \psi(x)a(x)^{\lambda}$$

and if Q is a parabolic subgroup of G containing P we introduce

$$E_Q(x, \psi, \lambda) = \sum_{\gamma \in P \backslash Q} \psi(\gamma x, \lambda+\rho)$$

a convergent series where $\mathrm{Re}(\lambda, \alpha) > (\alpha, \rho)$ for all $\alpha \in \Delta_P^Q$. The left-hand side is known to have a meromorphic analytic continuation to the whole space $\mathfrak{a}_P^* \otimes \mathbf{C}$. We need the formula giving the constant term of E_Q along a parabolic subgroup $R \subset Q$:

$$E_Q^R(x, \psi, \lambda) = \int_{\textcircled{N_R}} E_Q(nx, \psi, \lambda)\,dn \quad .$$

We assume that $\mathrm{Re}(\alpha, \lambda) > (\alpha, \rho)$ for all $\alpha \in \Delta_P^Q$ and then this equals the sum over $\overline{w} \in P\backslash Q/R$ of

$$\int_{\textcircled{N_R}} \sum_{\xi \in R(P,w)} \psi(w\xi nx, \lambda+\rho)\,dn$$

where

$$R(P, w) = R \cap w^{-1}Pw \backslash R \quad .$$

We shall assume now that ψ is cuspidal on P and hence the contribution of a double coset $\overline{w} \in P\backslash Q/R$ is zero except perhaps if $wN_Rw^{-1} \cap M$ is trivial. In such a case we may assume that the representative w of the double coset $\overline{w}$ is so chosen that $w = w_s^{-1}$ where w_s represents $s \in \Omega^Q$ (the Weyl group of M_Q) satisfying the following properties: $w_sMw_s^{-1} \subset M_R$ and $s^{-1}\alpha > 0$ for all $\alpha \in \Delta_0^R$. Then there exist a standard parabolic subgroup of R which we shall denote by $s\cdot P$ with Levi subgroup $sM = w_sMw_s^{-1}$. The set of all such s will be denoted by $\Omega^Q(\mathfrak{a}_P, R)$. Why?* Notice that sP depends on s alone and P and not on R.

Let us introduce a coset

$$N^s = N_R \cap w_sNw_s^{-1} \backslash N_R$$

$$= sN \cap w_sNw_s^{-1} \backslash sN$$

and define

$$(M(s, \lambda)\psi)(x) = a(x)^{-s\lambda-\rho_s} \int_{N^s} \psi(w_s^{-1}nx, \lambda+\rho)\,dn$$

where $\rho_s = \rho_{sP}$; the integral is absolutely convergent if $\mathrm{Re}(\lambda, \alpha) > (\rho, \alpha)$ for all $\alpha \in \Delta_P^Q$. In the contribution of $w = w_s^{-1}$ the integral over N_R may be replaced by an integral over N^s and since $R(P, w_s^{-1})/N^s = sP\backslash R$ we have obtained the

LEMMA 12.1.1. Assume that $\mathrm{Re}(\lambda, \alpha) > (\rho, \alpha)$ for all $\alpha \in \Delta_P^Q$ and ψ is cuspidal on P then

* Consider $w_sPw_s^{-1} \cap M_R$. It contains $w_sMw_s^{-1}$. It also contains $N_0 \cap M_R$ because $w_s^{-1}(N_0 \cap M_R)w_s \subseteq N_0$. Thus $w_sPw_s^{-1} \cap M_R$ is a parabolic subgroup of M_R.

$$E_Q^R(x, \psi, \lambda) = \sum_{s \in \Omega^Q(\mathfrak{a}_P, R)} \sum_{\xi \in sP \backslash R} (M(s, \lambda)\psi)(\xi x, s\lambda + \rho_s) \quad .$$

We need a formula for

$$\Lambda^{T,P_1} E_{P_1}(x, \psi, \lambda) = \sum_{R \subset P_1} (-1)^{a_R - a_1}$$

$$\sum_{\delta \in R \backslash P_1} \hat{\tau}_R^{P_1}(H(\delta x) - T) E_{P_1}^R(\delta x, \psi, \lambda) \quad .$$

We shall assume as above that $\mathrm{Re}(\lambda, \alpha) > (\rho, \alpha)$ for all $\alpha \in \Delta_P^{P_1}$ and ψ is cuspidal on P. We see that $\Lambda^{T,P_1} E_{P_1}(x, \psi, \lambda)$ is the sum over $R \subset P_1$ of the sum over $s \in \Omega^{P_1}(\mathfrak{a}_P, R)$ of

$$\sum_{\delta \in sP \backslash P_1} (-1)^{a_R - a_1} \hat{\tau}_R^{P_1}(H(\delta x) - T)(M(s, \lambda)\psi)(\delta x, s\lambda + \rho_s) \quad .$$

As in Lectures 5 and 9 it is convenient to introduce the Weyl sets $\Omega^{P_1}(\mathfrak{a}_P)$ which are the union of the $\Omega^{P_1}(\mathfrak{a}_P, \mathfrak{a}_{P_2})$ for all P_2 standard in P_1. Given $s \in \Omega^{P_1}(\mathfrak{a}_P)$ we define a function on $\mathfrak{a}_P^{P_1}$

when is $s \in \Omega^{P_1}(\mathfrak{a}_P, R)$?

$\iff$ $s^{-1}\alpha > 0$ $\alpha \in \Delta_0^R$

$s\alpha > 0$ $\alpha \in \Delta_0^P$

$$B_{P|P_1}^s(X) = \sum_{sP \subset R \subset P_1 \cap P_s} (-1)^{a_R} \hat{\tau}_R^{P_1}(sX)$$

$\iff$ $s\alpha > 0$ $\alpha \in \Delta_0^P$

$\Delta_0^R \subseteq \Delta_0^{P_s}$

where P_s is the standard parabolic subgroup such that $\alpha \in \Delta_0^{P_s}$ if and only if $s^{-1}\alpha > 0$. We obtain the

LEMMA 12.1.2. <u>Assume</u> $\mathrm{Re}(\lambda, \alpha) > (\rho, \alpha)$ <u>for all</u> $\alpha \in \Delta_P^{P_1}$ <u>and</u> ψ

cuspidal on P then $\Lambda^{T,P_1} E_{P_1}(x, \psi, \lambda)$ equals

$$(-1)^{a_1} \sum_{s \in \Omega^{P_1}(\mathfrak{a}_P)} \sum_{\delta \in s^P \backslash P_1} B^s_{P|P_1}(s^{-1}(H(\delta x)-T))(M(s, \lambda)\psi)(\delta x, s\lambda+\rho_s) \quad .$$

12.2. Computation of an inner product.

Let Q be an ε-invariant parabolic subgroup and let $P \subset P_1 \subset Q$. We consider functions ψ and θ cuspidal on P attached to unitary representations π and χ with central characters ω_π and ω_χ trivial on $A_P(\mathbf{R})^o$.

We are looking for a rather explicit formula for the function $\omega^\varepsilon_{P_1}(x, \lambda, \mu, \psi, \theta)$ defined to be the integral over $P_1 \backslash \mathbf{P}^1_1$ of

$$p \longmapsto \Lambda^{T,P_1} E_{P_1}(px, \psi, \lambda) E_Q(\varepsilon(px), \theta, \mu) \quad .$$

We shall assume that $\mathrm{Re}(\lambda)$ is sufficiently regular and in particular Lemma 12.1.2 applies. We obtain a sum over $s \in \Omega^{P_1}(\mathfrak{a}_P)$ of an integral over $P_1 \backslash \mathbf{P}^1_1$ of a sum over $sP \backslash P_1$ and we may combine the sum and the integral if the resulting expression is absolutely convergent.

This amounts to proving that the integral over $sP \backslash \mathbf{P}^1_1$ of the absolute value of

$$p \longrightarrow B^s_{P|P_1}(s^{-1}(H(px)-T)M(s, \lambda)\psi(px, s\lambda+\rho_s)E_Q(\varepsilon(px), \theta, \mu)$$

is finite. Since the Eisenstein series E_Q is known to be slowly increasing, that is, that

$$|x|^{-N}|E_Q(x, \theta, \mu)|$$

is bounded for some N, we need only to show that given N',

$$|am|^{N'}|B^s_{P|P_1}(s^{-1}(H(a)-T))a^{s\lambda+\rho_s}||\psi(mx, s\lambda+\rho_s|$$

Should be $M(s,\lambda)\varphi$ and m should lie in ${}_sM$.

is bounded for $a \in A^{P_1}_{sP}(\mathbf{R})^o$ and m in a Siegel domain of $\mathbf{M}^1$, uniformly for x in a compact set provided that $\mathrm{Re}(\lambda)$ is sufficiently regular. We have assumed that ψ is cuspidal on P and hence

$$|m|^{N'}|\psi(mx, s\lambda+\rho_s|$$

is bounded when m is in a Siegel set of $\mathbf{M}^1$ and x in a compact set. The boundedness of

$$|a|^{N'}|B^s_{P|P_1}(s^{-1}(H(a)-T))a^{s\lambda+\rho_s}|$$

$s^{-1}H \in -\mathfrak{a}$

$H \in s(-\mathfrak{a})$

$s\lambda(sX) = \lambda(X)$

is an immediate consequence of the following

LEMMA 12.2.1. <u>The support of</u> $B^s_{P|P_1}$ <u>is contained in the cone</u> $\varpi(X) \leq 0$ <u>for all</u> $\varpi \in \hat{\Delta}^{P_1}_{P}$. ... Think the s belongs here! No!

The proof of this property has already been given in the proof of Lemma 9.2.7. We repeat the argument. Up to a sign $B^s_{P|P_1}$ is the characteristic function of the set of $X \in \mathfrak{a}^{P_1}_{P}$ such that (i) $\varpi_\alpha(sX) > 0$ if $\alpha \in \Delta^{P_1}_{sP} - \Delta^{P_1\cap P_s}_{sP}$, i.e., if $s^{-1}\alpha < 0$ and (ii) $\varpi_\alpha(sX) \leq 0$ if $\alpha \in \Delta^{P_1\cap P_s}_{sP}$, i.e., if $s^{-1}\alpha > 0$; and hence

~~$\phi^s_{P|P_1}$ seems to be essentially the char. fn of $-\mathfrak{a}^{P_1}_P$~~

(i) comes from support of all $\hat{\tau}^{P_1}_R(sX)$

(ii) comes from alternation.

SIGN: Comes from terms with no alternation i.e. $R = P_1 \cap P_s$

$$(X, \Lambda) = \sum_{\alpha \in \Delta_{sP}^{P_1}} \varpi_\alpha(sX).s^{-1}\alpha(\Lambda) \le 0$$

for any regular $\Lambda \in \mathfrak{a}_{sP}^{P_1 +}$. □

Then provided that $\mathrm{Re}(\lambda)$ is sufficiently regular we see that $\omega_{P_1}^{\varepsilon}(x, \lambda, \mu, \psi, \theta)$ is the sum over $s \in \Omega^{P_1}(\mathfrak{a}_P)$ of the integral over $s\mathbf{N}sP \backslash \mathbf{P}_1^1 = sM \backslash s\mathbf{M}^1 \times \mathbf{A}_{sP}^{P_1} \times (K \cap \mathbf{P}_1)$ of the product of

$$(-1)^{a_1} B_{P|P_1}^{s}(s^{-1}(H(px)-T))M(s, \lambda)\psi(px, s\lambda+\rho_s)a(p)^{-2\rho_s}$$

times

$$\overline{E_Q^{\varepsilon sP}(\varepsilon(px), \theta, \mu)} .$$

It may be convenient to transform the last term. Let $\theta_\varepsilon(x) = \theta(\varepsilon(x))$ and $\mu_\varepsilon(H) = \mu(\varepsilon(H))$, then this term equals

$$\overline{E_Q^{sP}(px, \theta_\varepsilon, \mu_\varepsilon)}$$

which in turn equals the sum over $t \in \Omega^Q(\varepsilon^{-1}\mathfrak{a}_P, s\mathfrak{a}_P)$ of

$$\overline{M(t, \mu_\varepsilon)\theta_\varepsilon(px, t\mu_\varepsilon+\rho_s)} .$$

Summing up we get the

LEMMA 12.2.2. Assume that ψ and θ are cuspidal on P then the following equality of meromorphic function holds for x in $A_1(\mathbb{R})^0K$:

$$\omega^{\varepsilon}_{P_1}(x, \lambda, \mu, \psi, \theta) = \sum_{s \in \Omega^{P_1}(\mathfrak{a}_P)} \sum_{t \in \Omega^{Q}(\varepsilon^{-1}\mathfrak{a}_P, s\mathfrak{a}_P)} a(x)^{s\lambda + t\bar{\mu}_\varepsilon}$$

$$(-1)^{a_1} \hat{B}^{s}_{P|P_1}(s^{-1}T, (\lambda + s^{-1}t\bar{\mu}_\varepsilon))(M(s, \lambda)\psi^x, M(t, \mu_\varepsilon)\theta^x_\varepsilon)_{sM^1}$$

<u>where</u> $\psi^x(p) = \psi(px)$, <u>the scalar product</u> $(\ ,\ )_{sM^1}$ <u>is the scalar product in</u> $L^2(sM \backslash sM^1)$ <u>and</u>

$$\hat{B}^{s}_{P|P_1}(x, \lambda) = \int_{\mathfrak{a}^{P_1}_P} B^{s}_{P|P_1}(H - X)e^{\lambda(H)}\, dH \quad .$$

The two members are well defined and equal when $\mathrm{Re}(\lambda)$ and $\mathrm{Re}(\mu)$ are sufficiently regular; they have meromorphic analytic continuation in $(\lambda, \bar{\mu})$ to the whole space and hence are equal everywhere. □

12.3. <u>Some application</u>.

According to Lectures 10 and 11, the "right-hand side" of the trace formula is the sum over pairs of parabolic subgroups $P_1 \subset P_2$ such that there exists one and only one ε-invariant parabolic subgroup Q between P_1 and P_2 of terms $J^{P_2}_{P_1}$ which are equal to the sum over $w \in P_1 \backslash Q/\varepsilon(P_1)$ of the integral over $P_1 \cap \varepsilon^{-1}w^{-1}(P_1) \backslash \mathbb{G}'_\varepsilon$ of

$$_\varepsilon\sigma^2_1(H(x) - T)K_{P_1}(x, w\varepsilon(x)) \quad .$$

Using the spectral decomposition it was shown that $K_{P_1}(x, y)$ is equal to the sum over $P \subset P_1$ and over $\pi \in \prod_2(M_P)$ of not defined but meaning fairly obvious

$$\frac{1}{n_1(A_P)} \int_{i(\mathfrak{a}_P^{P_1})^*} \Xi_\pi^{P_1}(x, y, \lambda, \lambda)\, d\lambda$$

where $\Xi_\pi^{P_1}(x, y, \lambda, \nu)$ is given by

$$\int_{i\mathfrak{a}_{P_1}^* - \Lambda} \sum_{\psi \in B_\pi} \Lambda^{T_1} {}^{P_1}E_{P_1}(x, I_{\lambda+\lambda_1}(\phi)\psi, \lambda+\lambda_1)$$

$$\overline{E_{P_1}(y, \psi, -\bar{\nu}-\bar{\lambda}_1)}\, d\lambda_1 \ .$$

In this expression λ and ν belong to $(\mathfrak{a}_P^{P_1})^* \otimes \mathbf{C}$ and $\Lambda \in \mathfrak{a}_{P_1}$. We assume ϕ to be K-finite and the sum over $\psi \in B_\pi$ may be assumed to reduce to a finite sum. We have not included Λ in the notation $\Xi_\pi^{P_1}$ since it is in fact independent of Λ. To see this one should remark that $\lambda_1 \longrightarrow I_{\lambda+\lambda_1}(\phi)\psi$ is of Paley-Wiener type since ϕ is compactly supported and hence we are free to shift the integration domain. As a function of (λ, ν) it is meromorphic and we have tacitly assumed that (λ, ν) is not a singular value.

The main result of Lecture 11 may be stated in the following way: the sum over π and w and the integral over $i(\mathfrak{a}_P^{P_1})^*$ and $P_1 \cap \varepsilon^{-1}w^{-1}(P_1)\backslash \mathbf{G}_\varepsilon^1$ of $\Xi_\pi^{P_1}(x, w\varepsilon(x), \lambda, \lambda)$ is absolutely convergent, so that we are free to interchange the order of those sums and integrals. Before using this we need some preparation.

LEMMA 12.3.1. <u>The series</u>

$$\sum_{w \in P_1\backslash Q/\varepsilon(P_1)} \ \sum_{\xi \in P_1 \cap \varepsilon^{-1}w^{-1}(P_1)\backslash P_1} \Xi_\pi^{P_1}(x, w\varepsilon(\xi x), \lambda, \nu)$$

<u>is absolutely convergent and defines a meromorphic function of</u> (λ, ν) <u>which we shall denote by</u> $Z_\pi^{P_1}(x, \lambda, \nu)$.

We first remark that the series may be written

$$\sum_{\xi \in P_1 \backslash Q} \Xi_\pi^{P_1}(x, \xi\varepsilon(x), \lambda, \nu) \quad .$$

Now consider $a_1 \in A_1(\mathbb{R})^o$ and $y \in \mathbb{N}_1 \mathbb{M}_1^1 K$ then

$$E_{P_1}(a_1 y, \psi, -\bar{\nu}-\bar{\lambda}_1) = a_1^{-\bar{\lambda}_1} E_{P_1}(y, \psi, -\bar{\nu}) \quad .$$

Recall that

$$\lambda_1 \longrightarrow \Lambda^{T,P_1} E_{P_1}(x, I_{\lambda+\lambda_1}(\phi)\psi, \lambda+\lambda_1)$$

is of Paley-Wiener type on $i\mathfrak{a}_1^* \otimes \mathbf{C}$; this implies that

$$a_1 \longrightarrow \Xi_\pi^{P_1}(x, a_1 y, \lambda, \nu)$$

is compactly supported in some compact set $\omega \subset A_1(\mathbb{R})^o$ independent of y. From this we deduce that the series reduces to a finite sum uniformly when x, λ, ν are in compact set in the holomorphy domain for (λ, ν). □

LEMMA 12.3.2. <u>Assume that</u> $\mathrm{Re}(-\nu) + \Lambda$ <u>is sufficiently regular in</u> $\mathfrak{a}_Q^*$ <u>then</u> $Z_\pi^{P_1}(x, \lambda, \nu)$ <u>equals</u>

$$\int_{i\mathfrak{a}_{P_1}^* - \Lambda} \Lambda^{T,P_1} E_{P_1}(x, I_{\lambda+\lambda_1}(\phi), \lambda+\lambda_1) \overline{E_Q(\varepsilon(x), \psi, -\bar{\nu}-\bar{\lambda}_1)}\, d\lambda_1 \quad .$$

This is an immediate consequence of the fact that when $\mathrm{Re}(-\nu-\lambda_1)$ is sufficiently regular, then

$$E_Q(y, \psi, -\overline{\nu}-\overline{\lambda}_1) = \sum_{\xi \in P_1\backslash Q} E_{P_1}(\xi y, \psi, -\overline{\nu}-\overline{\lambda}_1) \quad . \quad \square$$

LEMMA 12.3.3. <u>The function</u> $p \longrightarrow Z_\pi^{P_1}(px, \lambda, \nu)$ <u>is integrable over</u> $P_1\backslash \mathbf{P}_1^1$ <u>and its integral defines a meromorphic function</u> $S_\pi^{P_1}(x, \lambda, \nu)$.

Lemma 6.6 shows that

$$p \longrightarrow \Lambda^{T_1 P_1} E_{P_1}(px, I_{\lambda+\lambda_1}(\phi)\psi, \lambda+\lambda_1)$$

is a "rapidly decreasing" function in a Siegel domain $\mathfrak{S}$ in $\mathbf{P}_1^1$, uniformly in λ_1 since λ_1 is trivial on $\mathbf{P}_1^1$, and hence all we need to prove is that

$$p \longrightarrow \sum_{\substack{\xi \in P_1\backslash Q \\ a_1(\xi\varepsilon(px)) \in \omega}} E_{P_1}(\xi\varepsilon(px), \psi, -\overline{\nu})$$

is slowly increasing in $\mathfrak{S}$. To see this we consider $w \in P_1\backslash Q/P_0$ and N_0^w a subgroup of N_0 isomorphic to $N_0 \cap w^{-1}P_1 w\backslash N_0$; using the slow increase property of Eisenstein series, that is, that

$$|E(y, \psi, -\overline{\nu})| \leq c|y|^N$$

for some N and some constant c, all we need to remark is that given a compact set $\omega' \subset A_1(\mathbb{R})^0$

$$a \longrightarrow \sum_{\substack{\eta \in N_0^w \\ a_1(w\eta a) \in \omega'}} |\eta|^N$$

is "slowly increasing" for $a \in A_0(\mathbb{R})^o$. All those evaluations are uniform for x, λ, ν in compact sets outside the singular set. □

The main result of Lecture 11 may be restated using the functions $S_\pi^{P_1}$: the $J_{P_1}^{P_2}$ are equal to the sum over $P \subset P_1$ and over $\pi \in \prod_2(M_P)$ of

$$\frac{1}{n_{\mathfrak{G}_1}(A_P)} \int_{i(\mathfrak{a}_P^{P_1})^*} \int_{\mathbf{P}_1^1\backslash \mathbf{G}_\varepsilon'} {}_\varepsilon\sigma_1^2(H(x)-T)\, S_\pi^{P_1}(x, \lambda, \lambda)\, d\lambda\, dx .$$

Using Lemma 12.2.2 a more concrete formula for $S_\pi^{P_1}(x, \lambda, \nu)$ can be given for some π and some values of λ and ν; this is the aim of the next

LEMMA 12.3.4. <u>Assume that</u> $\mathrm{Re}(\lambda)$ <u>and</u> $\mathrm{Re}(-\nu+\Lambda)$ <u>are suitably regular,</u> <u>then if</u> $x \in A_1(\mathbb{R})^o K$ <u>and</u> π <u>is cuspidal on</u> M:

$$S_\pi^{P_1}(x, \lambda, \nu) = \sum_{\substack{s \in \Omega^{P_1}(\mathfrak{a}_P) \\ t \in \Omega_\varepsilon^Q(\mathfrak{a}_P, \mathfrak{a}_P)}} \int_{i\mathfrak{a}_{P_1}^* - \Lambda} d\lambda_1\, a(x)^{\lambda_1 - \hat{t}(\nu+\lambda_1)}\, \hat{B}_{P|P_1}^s(s^{-1}T, \lambda - \hat{t}(\nu+\lambda_1)) \sum_{\psi \in B_\pi} (M(s, \lambda) I_{\lambda+\lambda_1}(\phi)\psi^x, M(st\varepsilon, -\bar{\nu}-\bar{\lambda}_1)\psi_\varepsilon^x) .$$

$\subseteq \Omega^Q(\quad) \rtimes E$; ~~$+t(\bar\nu+\bar\lambda_1)$~~ ? ; $: \mathfrak{a}_P \to \mathfrak{a}_P$; $: \to \varepsilon^{-1}$; : minimal length ; Can replacing $\lambda+\lambda_1$; $(\lambda+\lambda_1) - t(\nu+\lambda_1)$

Under the regularity assumptions all integrals and series are absolutely convergent and we may appeal to Lemma 12.2.2 since π is cuspidal. □

Lecture 13

SOME FORMAL PROPERTIES OF THE TERMS IN THE TRACE FORMULA

J.-P. Labesse

13.1. Some combinatorics.

Given two parabolic subgroups P and Q such that $P \subset Q$ we have defined τ_P^Q (resp. $\hat{\tau}_P^Q$) to be the characteristic function of the set of $H \in \mathfrak{a}_P^Q$ such that $\alpha(H) > 0$ for all $\alpha \in \Delta_P^Q$ (resp. $\varpi(H) > 0$ for all $\varpi \in \hat{\Delta}_P^Q$). By abuse of notation we also consider them as functions on $\mathfrak{a}_0$ depending only on the projection on $\mathfrak{a}_P^Q$.

When P and Q are ε-invariant we define ${}_\varepsilon\tau_P^Q$ (resp. ${}_\varepsilon\hat{\tau}_P^Q$) to be the restriction to $(\mathfrak{a}_P^Q)^\varepsilon$ the subset of ε-invariant vectors. They will also be considered as functions on $\mathfrak{a}_0^\varepsilon$ and even on $\mathfrak{a}_0$. We introduce a new functions on $(\mathfrak{a}_P^Q)^\varepsilon \times (\mathfrak{a}_P^Q)^\varepsilon$:

$$ {}_\varepsilon\Gamma_P^Q(H, X) = \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^\varepsilon - a_Q^\varepsilon} \, {}_\varepsilon\tau_P^R(H) \, {}_\varepsilon\hat{\tau}_R^Q(H-X) \quad . $$

The key observation for all that follows is the

LEMMA 13.1.1.

(i) Assume that X remains in a compact subset ω then

$$ H \longrightarrow {}_\varepsilon\Gamma_P^Q(H, X) $$

is supported in a compact subset of $(\mathfrak{a}_P^Q)^\varepsilon$ independent of $X \in \omega$.

(ii) <u>If</u> X <u>is regular then</u>

$$H \longrightarrow {}_{\varepsilon}\Gamma_P^Q(H,\ X)$$

<u>is the characteristic function of the set of</u> $H \in (\mathfrak{a}_P^Q)^{\varepsilon}$ <u>such that</u>

$$\alpha(H) > 0 \quad \underline{\text{for all}} \quad \alpha \in \Delta_P^Q$$
$$\varpi(H) \leq \omega(X) \quad \underline{\text{for all}} \quad \varpi \in \hat{\Delta}_P^Q \quad .$$

(iii) ${}_{\varepsilon}\Gamma_P^Q(H,\ 0) = \delta_P^Q$ (<u>the Kronecker symbol</u>).

Given H we define $S = S_H$ to be the ε-invariant parabolic subgroup S between P and Q such that

$$\Delta_P^S = \{\alpha \in \Delta_P^Q \mid \alpha(H) > 0\} \quad .$$

We have

$$ {}_{\varepsilon}\Gamma_P^Q(H,\ X) = \sum_{\substack{P \subset R \subset S \\ \varepsilon(R)=R}} (-1)^{a_R^{\varepsilon}-a_Q^{\varepsilon}} \ {}_{\varepsilon}\hat{\tau}_R^Q(H-X) \quad .$$

This is non zero only if $\varpi(H-X) > 0$ for all $\varpi \in \hat{\Delta}_S^Q$ and $\varpi(H-X) \leq 0$ for all $\varpi \in \hat{\Delta}_P^Q - \hat{\Delta}_S^Q$. Choose $X_1 \in (\mathfrak{a}_P^Q)^{\varepsilon}$ such that

$$\alpha(X_1) \leq \inf_{X \in \omega \cup \{0\}} \alpha(X)$$

for all $\alpha \in \Delta_P^Q$. Since $\alpha(H) > 0 \geq \alpha(X_1)$ for $\alpha \in \Delta_P^S$ and $\varpi(H) > \varpi(X) \geq \varpi(X_1)$ for $\varpi \in \hat{\Delta}_S^Q$ we have $\varpi(H) > \varpi(X_1)$ for all $\varpi \in \hat{\Delta}_P^Q$. In the same way, replacing Inf by max and changing the sense of inequalities we define X_2; then for all $\varpi \in \hat{\Delta}_P^Q$ we have

$$\varpi(X_1) < \varpi(H) \leq \varpi(X_2)$$

whenever ${}_\varepsilon\Gamma_P^Q(H, X) \neq 0$ and $X \in \omega$. Assertion (i) follows.

Consider now a fixed X such that $\alpha(X) \geq 0$ for all $\alpha \in \Delta_P^Q$, then we may take $X_1 = 0$ and $X_2 = X$; this implies $S_H = Q$ if ${}_\varepsilon\Gamma_P^Q(H, X) \neq 0$ and assertion (ii) follows.

If $X = 0$ we may take $X_1 = X_2 = 0$ and this implies $S_H = Q$ and $S_H = P$ if ${}_\varepsilon\Gamma_P^Q(H, 0) \neq 0$. This yields assertion (iii). □

Remark: Assertion (iii) above has already been proved, with other notations, in Lecture 2; see 13.1.2. below.

We now introduce matrices of function on $\mathfrak{a}_0^\varepsilon$ whose entries are indexed by pairs of ε-invariant parabolic subgroups: let ${}_\varepsilon\tau = ({}_\varepsilon\tau_{P,Q})$ be such that

$$\begin{aligned} {}_\varepsilon\tau_{P,Q} &= 0 \quad \text{if } P \not\subset Q \\ {}_\varepsilon\tau_{P,Q} &= (-1)^{a_P^\varepsilon} \, {}_\varepsilon\tau_P^Q \quad \text{if } P \subset Q \end{aligned}$$

considered as functions on $\mathfrak{a}_0^\varepsilon$. In the same way we define ${}_\varepsilon\hat{\tau}$. Assertion (iii) in the above lemma yields the

COROLLARY 13.1.2. ${}_\varepsilon\tau \, {}_\varepsilon\hat{\tau} = 1$. □

We introduce a matrix ${}_{\varepsilon}\Gamma = ({}_{\varepsilon}\Gamma_{P,Q})$ whose entries are such that

$$ {}_{\varepsilon}\Gamma_{P,Q} = 0 \quad \text{if} \quad P \not\subset Q $$
$$ {}_{\varepsilon}\Gamma_{P,Q} = (-1)^{a_P^{\varepsilon} - a_Q^{\varepsilon}} \, {}_{\varepsilon}\Gamma_P^Q \quad \text{if} \quad P \subset Q \quad . $$

Using the definition of ${}_{\varepsilon}\Gamma_P^Q$ we see that

$$ {}_{\varepsilon}\Gamma(H, X) = {}_{\varepsilon}\tau(H)\, {}_{\varepsilon}\hat{\tau}(H-X) \quad . $$

LEMMA 13.1.3.

$$ {}_{\varepsilon}\tau_P^Q(H-X) = \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^{\varepsilon} - a_Q^{\varepsilon}} \, {}_{\varepsilon}\tau_P^R(H)\, {}_{\varepsilon}\Gamma_R^Q(H, X) \quad . $$

Using Corollary 13.1.2 we see that

$$ {}_{\varepsilon}\hat{\tau}(H-X) = {}_{\varepsilon}\tau(H)^{-1}\, {}_{\varepsilon}\Gamma(H, X) = {}_{\varepsilon}\hat{\tau}(H)\, {}_{\varepsilon}\Gamma(H, X) \quad . \quad \square $$

Since $H \longrightarrow {}_{\varepsilon}\Gamma_P^Q(H, X)$ is compactly supported on $(\mathfrak{a}_P^Q)^{\varepsilon}$ the integral

$$ {}_{\varepsilon}\gamma_P^Q(\lambda, X) = \int_{(\mathfrak{a}_P^Q)^{\varepsilon}} {}_{\varepsilon}\Gamma_P^Q(H, X) e^{\lambda(H)} dH $$

is convergent for all $\lambda \in \mathfrak{a}_0^* \otimes \mathbf{C}$ and defines an analytic function. We want to compute ${}_{\varepsilon}\gamma_P^Q$. We define ${}_{\varepsilon}\Delta_P^Q$ to be the set of restrictions to

$(\mathfrak{a}_P^Q)^\varepsilon$ of ε-orbits of elements in Δ_P^Q. Given $\alpha \in {}_\varepsilon\Delta_P^Q$ the coroot $\check{\alpha}$ lies in $(\mathfrak{a}_P^Q)^\varepsilon$. We define

$$ {}_\varepsilon c_P^Q = |\det(\check{\alpha}, \check{\beta})|^{\frac{1}{2}} \qquad \alpha, \beta \in {}_\varepsilon\Delta_P^Q $$

and

$$ {}_\varepsilon\theta_P^Q(\lambda) = ({}_\varepsilon c_P^Q)^{-1} \prod_{\alpha \in {}_\varepsilon\Delta_P^Q} \lambda(\check{\alpha}) \quad . $$

Now assume that $\mathrm{Re}(\lambda(\check{\alpha})) < 0$ for all $\alpha \in {}_\varepsilon\Delta_P^Q$, then

$$ \int_{(\mathfrak{a}_P^Q)^\varepsilon} {}_\varepsilon\hat{\tau}_P^Q(H) e^{\lambda(H)} dH = \theta_P^Q(\lambda)^{-1} \quad . $$

Replacing roots by weights we define ${}_\varepsilon\hat{\Delta}_P^Q$, ${}_\varepsilon\hat{c}_P^Q$, and ${}_\varepsilon\hat{\theta}_P^Q$ is the Laplace transform of ${}_\varepsilon\tau_P^Q$. This yields the following expression for ${}_\varepsilon\gamma_P^Q$:

LEMMA 13.1.4.

$$ {}_\varepsilon\gamma_P^Q(\lambda, X) = \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^\varepsilon - a_Q^\varepsilon} e^{\lambda({}_\varepsilon X_R^Q)} \hat{\theta}_P^R(\lambda)^{-1} \theta_R^Q(\lambda)^{-1} $$

<u>where</u> ${}_\varepsilon X_R^Q$ <u>is the projection of</u> X on $(\mathfrak{a}_R^Q)^\varepsilon$.

The left-hand side is analytic, the right-hand side is meromorphic and hence they are equal everywhere and the singularities of the right-hand side cancel. □

Letting ${}_{\varepsilon}\gamma_P^Q(X) = {}_{\varepsilon}\gamma_P^Q(0, X)$ we have the

LEMMA 13.1.5. *The function*

$$X \longrightarrow {}_{\varepsilon}\gamma_P^Q(X)$$

is a homogeneous polynomial of degree $k = a_P^{\varepsilon} - a_Q^{\varepsilon}$ *given by*

$$\frac{1}{k!} \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^{\varepsilon}-a_Q^{\varepsilon}} \lambda({}_{\varepsilon}X_R^Q)^k \, {}_{\varepsilon}\hat{\theta}_P^R(\lambda)^{-1} \, {}_{\varepsilon}\theta_R^Q(\lambda)^{-1}$$

well defined if λ *is not a singular value of* $\hat{\theta}(\lambda)^{-1}$ or $\theta(\lambda)^{-1}$, *and independent of* λ.

It is clear that $X \longrightarrow {}_{\varepsilon}\gamma_P^Q(X)$ is analytic and homogeneous of degree $k = a_P^{\varepsilon} - a_Q^{\varepsilon}$ and it is easy to compute the limit

$${}_{\varepsilon}\gamma_P^Q(0, X) = \lim_{\substack{t \to 0 \\ t \in \mathbf{R}}} {}_{\varepsilon}\gamma_P^Q(t\lambda, X)$$

when λ is not a singular value for ${}_{\varepsilon}\theta(\lambda)^{-1}$ or ${}_{\varepsilon}\hat{\theta}(\lambda)^{-1}$. □

13.2. *The trace formula as a polynomial*.

The left-hand side of the trace formula for the group G and the function ϕ is a sum over $\mathfrak{o} \in \mathcal{O}$ of terms ${}_{\varepsilon}J_{\mathfrak{o}}^{G,T}(\phi)$ which are the integral over $G \backslash \mathbf{G}_{\varepsilon}^{1}$ of ${}_{\varepsilon}J_{\mathfrak{o}}^{G,T}(\phi, x)$ which in turn are the sums over ε-invariant parabolic subgroups $P \subset G$ (standard) of

$$(-1)^{a_P^{\varepsilon}-a_Q^{\varepsilon}} \sum_{\delta \in P \backslash G} {}_{\varepsilon}\hat{\tau}_P(H(\delta x)-T) K_{P,\mathfrak{o}}^{\varepsilon,\phi}(\delta x, \delta x)$$

where

$$K^{\varepsilon,\phi}_{P,\mathfrak{o}}(x,\ y) = \sum_{\gamma \in M_P \cap \mathfrak{o}} \int_{\mathbb{N}_P} \phi(x^{-1}\gamma n\varepsilon(y))\,dn \quad .$$

It was proved in Lecture 4 that the integral over $G\backslash\mathbf{G}'_\varepsilon$ is convergent provided T is suitably regular uniformly if ϕ varies in some compact set of functions.

We want to compute $J^{G,T+X}$ in terms of $J^{Q,T}$ where Q runs over ε-invariant parabolic subgroups. Using 13.1.3 we see that

$$_{\varepsilon}J^{T+X}_{\mathfrak{o}}(\phi,\ x) = \sum_{\substack{P\subset Q\\ \varepsilon(P)=P\\ \varepsilon(Q)=Q}} (-1)^{a^\varepsilon_P - a^\varepsilon_Q}$$

$$\sum_{\xi\in Q\backslash G}\ \sum_{\delta\in P\backslash Q}\ {}_{\varepsilon}\Gamma^G_Q(H(\xi x),\ X)\hat{\tau}^Q_P(H(\delta\xi x)-T)$$

$$K^{\varepsilon,\phi}_{P,\mathfrak{o}}(\delta\xi x,\ \delta\xi x) \quad .$$

But if $x = nmk$ with $n\in\mathbb{N}_Q$, $m\in\mathbb{M}_Q$ and $k\in K$ we have (if $P\subset Q$)

$$K^{\varepsilon,\phi}_{P,\mathfrak{o}}(x,\ x) = K^{\varepsilon,\phi^k_Q}_{P,\mathfrak{o}\cap Q}(m,\ m)$$

where

$$\phi^k_Q(m) = \delta_Q(m)^{\frac{1}{2}}\int_{\mathbb{N}_Q} \phi(k^{-1}mn\varepsilon(k))\,dn \quad .$$

Using the fact that the left-hand side of the trace formula is convergent for (Q, ϕ_Q^k) uniformly for $k \in K$ provided T is suitably regular we get when T and X are suitably regular

$$ {}_\varepsilon J_{\mathfrak{o}}^{G,T+X}(\phi) = \sum_{\varepsilon(Q)=Q} {}_\varepsilon \gamma_Q^G(X)\, {}_\varepsilon J_{\mathfrak{o}\cap Q}^{Q,T}(\phi_Q) $$

where

$$ \phi_Q = \int_K \phi_Q^k dk \quad . $$

The right-hand side is a polynomial in X and this allows one to define ${}_\varepsilon J_{\mathfrak{o}}^{G,T}(\phi)$ for all T as a polynomial in T of degree $a_R^\varepsilon - a_G^\varepsilon$ where R is any ε-invariant parabolic subgroup whose rank is minimal for the property $K_{R,\mathfrak{o}}^{\varepsilon,\phi} \neq 0$.

A cuspidal datum χ is a conjugacy class of pairs (π, M_P) where π is a cuspidal automorphic representation for M_P the Levi subgroup of a standard parabolic subgroup. If one considers the partial spectral decomposition indexed by cuspidal data one is led to introduce partial kernels $K_{P,\chi}(x, y)$ and one can show, using a refinement of the results in Lectures 7 and 8, that provided T is sufficiently regular

$$ {}_\varepsilon J_\chi^{G,T}(\phi, x) = \sum_{\varepsilon(P)=P} (-1)^{a_P^\varepsilon - a_G^\varepsilon} \sum_{\delta \in P\backslash G} {}_\varepsilon \hat{\tau}_P(H(\delta x)-T) K_{P,\chi}^\varepsilon(\delta x, \delta x) $$

is integrable over $G\backslash \mathbf{G}_\varepsilon^1$; we shall denote by ${}_\varepsilon J_\chi^{G,T}$ its integral. As above we get

$$ {}_{\varepsilon}J_{\chi}^{G,T+X}(\phi) = \sum_{\varepsilon(Q)=Q} {}_{\varepsilon}\gamma_Q^G(x)\, {}_{\varepsilon}J_{\chi}^{Q,T}(\phi_Q) $$

provided T and X are suitably regular. The right-hand side is a polynomial in X of degree $a_R^{\varepsilon} - a_G^{\varepsilon}$ where R is any ε-invariant parabolic subgroup whose rank is minimal for the property $K_{R,\chi} \neq 0$.

13.3. Changing the minimal parabolic.

Let $\Omega^{G,\varepsilon}$ be the subgroup of ε-invariant elements in the Weyl group; let $w \in G$ be an element which represents $s \in \Omega^{G,\varepsilon}$. Simple changes of variable yield

$$ {}_{\varepsilon}J^T(\phi) = \int_{G\backslash \mathbf{G}'_{\varepsilon}} \sum_{\varepsilon(P)=P} (-1)^{a_P^{\varepsilon}} \sum_{\delta \in w^{-1}(P)\backslash G} $$

$$ {}_{\varepsilon}\hat{\tau}_P(H(w\delta x)-T)K^{\varepsilon}_{w^{-1}(P)}(\delta x,\ \delta x) $$

where $w^{-1}(P) = w^{-1}Pw$ and where $K_{w^{-1}(P)}$ is defined in an obvious way. It is natural to define ${}_{\varepsilon}\hat{\tau}_{w^{-1}(P)}$ such that

$$ {}_{\varepsilon}\hat{\tau}_P(H) = {}_{\varepsilon}\hat{\tau}_{w^{-1}(P)}(w^{-1}(H)) \quad . $$

If $y = nmak$ is a Langlands-Iwasawa decomposition corresponding to $Q = w^{-1}(P_0)$ we define H_Q such that $H_Q(y) = H(a)$ and hence

$$ w^{-1}H(wy) = H_Q(y) + w^{-1}H(w) $$

and

$$ {}_{\varepsilon}\hat{\tau}_P(H(wy)-T) = {}_{\varepsilon}\hat{\tau}_{w^{-1}(P)}(H_Q(y)-T_Q) $$

where $T_Q = w^{-1}(T-H(w))$. With these notations we get

$$_{\varepsilon}J^T(\phi) = \int_{G\backslash \mathbf{G}'_{\varepsilon}} \sum_{\substack{\varepsilon(R)=R \\ R\supset Q}} \sum_{\delta\in R\backslash G} {}_{\varepsilon}\hat{\tau}_R(H_Q(\delta x)-T_Q)K_R^{\varepsilon}(\delta x,\ \delta x)$$

which can be written

$$_{\varepsilon}J^T(\phi) = {}_{\varepsilon}J_Q^{T_Q}(\phi)$$

where ${}_{\varepsilon}J_Q$ is the trace formula computed using the minimal ε-invariant parabolic subgroup Q in place of P_0.

13.4. Action of conjugacy.

We now want to compare $J^T(\phi)$ with $J^T(\phi^y)$ where

$$\phi^y(x) = \phi(yx\varepsilon(y)^{-1}) \quad .$$

We have

$$J^T(\phi^y) = \int_{G\backslash \mathbf{G}'_{\varepsilon}} \sum_{\substack{\varepsilon(P)=P \\ P\supset P_0}} \sum_{\delta\in P\backslash G} {}_{\varepsilon}\hat{\tau}_P(H(\delta xy)-T)K_P^{\varepsilon}(\delta x,\ \delta x)$$

but

$$H(\delta xy) = H(\delta x) + H(k(\delta x)y)$$

where $k(\delta x)$ is the K-component of an Iwasawa decomposition of (δx). Using 13.1.3 we are led to introduce

$$_{\varepsilon}u_P^Q(x, y) = \int_{{}^{\varepsilon}_Q \setminus \mathfrak{a}^{\varepsilon}_P} {}_{\varepsilon}\Gamma_P^Q(H, -H(k(x)y))\,dH$$

and

$$\phi_{Q,y}(m) = \delta_Q(m)^{\frac{1}{2}} \int_K \int_{\mathbf{N}_Q} \phi(k^{-1}mn\varepsilon(k))\,{}_{\varepsilon}u_Q^G(k, y)\,dk\,dn$$

with these notations we obtain as in 13.2

$$_{\varepsilon}J^{G,T}(\phi^y) = \sum_{\varepsilon(Q)=Q} {}_{\varepsilon}J^{Q,T}(\phi_{Q,y}) \quad .$$

13.5. <u>On some regularity property</u>.

In 13.1 we introduced

$$_{\varepsilon}\gamma_P^Q(\lambda, X) = \int_{(\mathfrak{a}_P^Q)^{\varepsilon}} {}_{\varepsilon}\Gamma_P^Q(H, X)e^{\lambda(H)}\,dH \quad .$$

We shall now study this function when λ is imaginary. Consider D a differential operator with constant coefficients on $i(\mathfrak{a}_P^Q)^{\varepsilon *}$ then if $\lambda \in i(\mathfrak{a}_P^Q)^{\varepsilon *}$ we have

$$|D\,{}_{\varepsilon}\gamma_P^Q(\lambda, X)| \leq \int_{(\mathfrak{a}_P^Q)^{\varepsilon}} |P_D(H)\,{}_{\varepsilon}\Gamma_P^Q(H, X)|\,dH$$

where P_D is the polynomial associated to D. Using that

$$\Gamma(tH, tX) = \Gamma(H, X)$$

for $t \in \mathbf{R}_+^\times$ and Lemma 13.1.1(i) it is not difficult to see that

LEMMA 13.5.1.

$$|D_\varepsilon \gamma_P^Q(\lambda, X)| < c(1 + \|X\|)^N$$

<u>for some</u> N <u>independent of</u> λ <u>when</u> λ <u>is imaginary</u>. □

In other words, $X \longrightarrow \gamma(\lambda, X)$ is a "slowly increasing" function.

Now consider φ a Schwartz-Bruhat function on $i(\mathfrak{a}_P^Q)^{*\varepsilon}$, let $\hat{\varphi}$ be its Fourier transform so that

$$\varphi(\lambda) = \int_{(\mathfrak{a}_P^Q)^\varepsilon} \hat{\varphi}(H) e^{\lambda(H)} dH \quad .$$

We define

$$_\varepsilon\gamma_P^Q(\lambda, \varphi) = \int_{(\mathfrak{a}_P^Q)^\varepsilon} \hat{\varphi}(X)\, _\varepsilon\gamma_P^Q(\lambda, X) dX \quad .$$

This makes sense also when $\hat{\varphi}$ is a "rapidly decreasing" distribution. Lemma 13.5.1 above shows that on $i(\mathfrak{a}_P^Q)^{\varepsilon *}$ the function

$$\lambda \longrightarrow {}_\varepsilon\gamma_P^Q(\lambda, \varphi)$$

is smooth and by 13.1.4 we obtain the following expression

$$_\varepsilon\gamma_P^Q(\lambda, \varphi) = \sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^\varepsilon - a_Q^\varepsilon} \varphi({}_\varepsilon\lambda_R^Q)$$

$$_\varepsilon\hat{\theta}_P^R(\lambda)^{-1}\, {}_\varepsilon\theta_R^Q(\lambda)^{-1}$$

which is valid at least when λ is imaginary and not a singular value of ${}_{\varepsilon}\hat{\theta}(\lambda)^{-1}$ or ${}_{\varepsilon}\theta(\lambda)^{-1}$ and where ${}_{\varepsilon}\lambda_R^Q$ is the projection of λ on $(\mathfrak{a}_R^Q)^{\varepsilon *} \otimes \mathbf{C}$.

The left-hand side is smooth and hence the singularities of the right-hand side cancel when φ is any Schwartz-Bruhat function. This implies that more generally we have the

LEMMA 13.5.2. *Given any smooth function* φ

$$\sum_{\substack{P \subset R \subset Q \\ \varepsilon(R)=R}} (-1)^{a_R^{\varepsilon} - a_Q^{\varepsilon}} \varphi({}_{\varepsilon}\lambda_R^Q)\, {}_{\varepsilon}\hat{\theta}_P^R(\lambda)^{-1}\, {}_{\varepsilon}\theta_R^Q(\lambda)^{-1}$$

extends to a smooth function of $\lambda \in i(\mathfrak{a}_P^Q)^{\varepsilon *}$. □

Lecture 15

(provisional text)

THE FINE χ-EXPANSION

R. Langlands

1. The operators $M_{P'|P}(s, \lambda)$. As usual M_0 is fixed and we consider only Levi factors $M \in \mathcal{L}(M_0)$ and parabolic subgroups $P \in \mathcal{F}(M_0)$. For such an M the Lie algebra $\mathfrak{a}_M$ is well-defined and so is $\Omega(\mathfrak{a}_M, \mathfrak{a}_{M'})$.

Let $s \in \Omega(\mathfrak{a}_M, \mathfrak{a}_{M'})$, $P \in \mathcal{P}(M)$, $P' \in \mathcal{P}(M')$. We define the operator $M_{P'|P}(s, \lambda)$ taking ϕ to be the function $M_{P'|P}(s, \lambda)\phi$:

$$g \longrightarrow \int_{\mathbf{N}_{P'} \cap w\mathbf{N}_P w^{-1} \backslash \mathbf{N}_{P'}} \phi(w^{-1}ng)e^{(\lambda+\rho_P)(H_P(w^{-1}ng)-(s\lambda+\rho_{P'})(H_{P'}(g))} dn \quad .$$

Some explanation is in order.

Fix a class χ in X, thus a pair (ρ_χ, M_χ) given up to association. It is referred to as a cuspidal datum. Two cuspidal data (ρ_χ, M_χ) and $(\rho_{\chi'}, M_{\chi'})$ will be said to be equivalent if after conjugation $M_{\chi'} = M_\chi$ and $\rho_{\chi'} = \alpha \otimes \rho_\chi$, α being a character of $\mathbf{G}$ trivial on $\mathbf{G}^1$. To χ is associated a closed subspace $L^2_\chi(M \backslash \mathbf{M})$ of $L^2_\omega(M \backslash \mathbf{M})$, ω being a certain central character of $\mathbf{M}$ determined by ρ_χ. If ϖ is an irreducible unitary representation of $\mathbf{M}$ let $\mathfrak{A}_{\chi,\varpi}(P)$ be the space of measurable functions ϕ on $\mathbf{G}$ satisfying:

(a) $\phi(ng) = \phi(g)$, $n \in \mathbf{N}_P$;

(b) $\phi(\gamma g) = \phi(\gamma)$ $\quad \gamma \in P$;

(c) $m \longrightarrow \phi(mg)$ is a function in $L^2_\chi(M\backslash \mathbf{M})$ for all $g \in \mathbf{G}$ transforming according to the representation σ;

(d)

$$\|\phi\|^2 = \int_{\mathbf{Z}_M M\backslash \mathbf{M}\times K} |\phi(mk)|^2 dn dk < \infty \quad .$$

The intertwining operator is defined by analytic continuation on K-finite functions and is unitary for $\mathrm{Re}\,\lambda = 0$. Notice that

$$(\phi,\ \psi) = \int_{\mathbf{Z}_M M\backslash \mathbf{M}\times K} \phi(mk)\overline{\psi}(mk)\,dm dk$$

defines an inner product on $\mathfrak{a}_{\chi,\sigma}(P)$.

The forms ρ_P and $\rho_{P'}$ have the usual meaning and w is a representative of s. Since the Iwasawa decomposition $\mathbf{G} = \mathbf{P}K$ is valid we can define $H_P(g)$.

The operator $M_{P'|P}(s, \lambda)$ is certainly an intertwining operator from $\mathfrak{a}_{\chi,\sigma}(P)$ to $\mathfrak{a}_{\chi,s(\sigma)}(P)$, the representation on the first space being $\rho_{\sigma\otimes\lambda}$ and that on the second being $\rho_{s(\sigma)\otimes s(\lambda)}$. Since χ is fixed in the present lectures we may drop it from the notation.

We shall make use of a number of relations which are either elementary or a part of the theory of Eisenstein series.

(a) If $s \in \Omega(\mathfrak{a}_M, \mathfrak{a}_{M'})$, $s' \in \Omega(\mathfrak{a}_{M'}, \mathfrak{a}_{M''})$ then

$$M_{P''|P}(s's,\ \lambda) = M_{P''|P'}(s',\ s\lambda)M_{P'|P}(s\lambda) \quad .$$

Of course $P'' \in \mathcal{P}(M'')$.

(b) Suppose L is a Levi subgroup containing both M and M' and s fixes the points of $\mathfrak{a}_L$. Associated to every pair R, Q, $R \in \mathcal{P}^L(M)$, $Q \in \mathcal{P}(L)$ is a unique parabolic subgroup $Q(R) \in \mathcal{P}(M)$ satisfying $Q(R) \subseteq Q$, $Q(R) \cap L = R$. Moreover if $\phi \in \mathfrak{A}_\sigma(Q(R))$ then for each k the function $\phi_k : m \longrightarrow \phi(mk)$ lies in $\mathfrak{A}_\sigma(R)$ and

$$(M_{Q(R')|Q(R)}(s, \lambda)\phi)_k = M_{R'|R}(s, \lambda)\phi_k \quad .$$

Notice that $M_{R'|R}(s, \lambda)$ depends only on the projection of λ on $\mathfrak{a}_M^L$.

(c) Suppose $M' = wMw^{-1}$, $P' = wPw^{-1}$ and w is a representation of s in $\Omega(\mathfrak{a}_M, \mathfrak{a}_{M'})$. Then by the definition

$$M_{P'|P}(s, \lambda)\phi : g \longrightarrow \phi(w^{-1}g)e^{(\lambda+\rho_P)H_P(w^{-1}g)-(s\lambda+\rho_{P'})(H_{P'}(g))} \quad .$$

Now if $g = p'h$ then $w^{-1}g = w^{-1}p'w\, w^{-1}k = pw^{-1}k$. Thus

$$H_P(w^{-1}g) = w^{-1}H_{P'}(g) + H_P(w^{-1}) \quad .$$

Since $\rho_{P'} = s\rho_P$ we conclude that

$$M_{P'|P}(s, \lambda)\phi = s\phi \cdot e^{(\lambda+\rho_P)(T_0-s^{-1}T_0)} \quad ,$$

for as Arthur shows in Lemma 1.1 of the Annals paper there exists a T_0 such that $H_P(w^{-1}) = T_0-s^{-1}T_0$ for all w. We define $s\phi$ by $s\phi : g \longrightarrow \phi(w^{-1}g)$.

(d) Combining (a) and (c) we obtain

$$te^{(s\lambda+\rho_{P'})(T_0-t^{-1}T_0)} M_{P'|P}(s, \lambda) = M_{t(P')|P}(ts, \lambda)$$

$$M_{P'|P}(s,\lambda) = M_{P'|t(P)}(st^{-1}, t\lambda)t\cdot e^{(\lambda+\rho_P)(T_0-s^{-1}T_0)} .$$

2. (G, M) <u>families</u>. For the moment fix M. A (G, M)-family is a set of smooth functions $c_P(\Lambda)$, $\Lambda \in i\mathfrak{a}_M^*$, indexed by the parabolic subgroups in $\mathcal{P}(M)$. These functions are to satisfy a compatibility condition. Recall that each P in $\mathcal{P}(M)$ is associated to a Weyl chamber W_P in $\mathfrak{a}_M$. This chamber is defined as the set of H such that $\alpha(H) > 0$ for all roots α in P. Thus $s(W_P) = W_{w(P)}$ if w represents s. If P and P' are adjacent, that is, if W_P and $W_{P'}$ have a wall in common, then the condition is that $c_P(\lambda) = c_{P'}(\lambda)$ on the hyperplane containing this wall.

A family of points $\{X_P | P \in \mathcal{P}(M)\}$ is said to be A_M-orthogonal if $X_P - X_{P'}$ is perpendicular to the wall separating W_P from $W_{P'}$ whenever P and P' are adjacent. Then the collection of functions $\{e^{\lambda(X_P)}\}$ is a (G, M)-family.

The set A_M-orthogonal of all families is a closed subset of $\prod_{\mathcal{P}(M)}$, $\mathfrak{a}_M$ and if ω is any rapidly decreasing measure on $\mathfrak{a}$ then

(1) $$c_P : \lambda \longrightarrow \int e^{\lambda(X_P)} d\omega .$$

is a (G, M)-family. It is likely that all compactly supported (G, M) families are of this form. Since these initial lectures on the second American Journal paper have as their sole purpose to discover a modification of the method of Arthur which <u>may</u> work in the twisted case, the rigorous treatment to be given later, I shall assume that the compactly supported families that arise are associated to a measure. Otherwise the combinatorics become unmanageable. (This turned out fortunately to be unwarranted pessimism.)

I now recall some constructions and some facts from the Inventiones and the Annals papers, many of which have already appeared in Lectures 9 and 13. First of all if $Q \supseteq P$ then $i\mathfrak{a}_Q^* \subseteq i\mathfrak{a}_P^*$ and we can project $\lambda \in i\mathfrak{a}_P^*$ onto $i\mathfrak{a}_Q^*$ obtaining λ_Q. If c_P is defined we set

$$c_Q(\lambda) = c_P(\lambda_Q) \quad .$$

Then we define c'_Q by

$$c'_Q(\lambda) = \sum_{R \supset Q} (-1)^{a_Q - a_R} \hat{\theta}_Q^R(\lambda)^{-1} c_R(\lambda) \theta_R(\lambda)^{-1} \quad .$$

Recall that

$$\theta_R^S(\lambda) = \frac{1}{c_R^S} \prod_{\alpha \in \Delta_R^S} \langle \lambda, \alpha \rangle$$

$$\hat{\theta}_Q^R(\lambda) = \frac{1}{\hat{c}_Q^R} \prod_{\varpi \in \hat{\Delta}_Q^R} \langle \lambda, \varpi \rangle \quad .$$

Here c_R^S is the volume of the parallelepiped spanned by Δ_R^S and $\hat{c}_Q^R$ the volume of that spanned by $\hat{\Delta}_Q^R$.

The functions c'_Q, $Q \supseteq P$, $P \in \mathcal{P}(M)$, depend on c_P alone and not on the entire (G, M) family. If $c_P(\lambda) = e^{\lambda(X_P)}$ then c'_Q is the Fourier transform of the function $\Gamma'_Q(\cdot, X_Q)$, where X_Q is the projection of X_P onto $\mathfrak{a}_Q$. Thus

$$c'_Q(\lambda) = \int e^{\lambda(H)}\, \Gamma'_Q(H, X_Q)\, dH \quad .$$

Recall that $\Gamma'_Q(\cdot, X_Q)$ is a function with support in a ball of radius $e\|X_Q\|$. More generally, if the family is attached to a measure ω then

$$c'_Q(\lambda) = \int_{\mathfrak{a}} \int_{\mathfrak{a}_Q} e^{\lambda(H)} \Gamma'_Q(H, X_Q)\, dH\, d\omega \quad .$$

Observe that X_Q is independent of the choice of $P \subset Q$ used to define it because the collection $\{X_P | P \in \mathcal{P}(M)\}$ is an A_M-orthogonal family.

Arthur also introduces a function $c_M(\lambda)$. It is at first defined by

$$c_M(\lambda) = \sum_{P \in \mathcal{P}(M)} c_P(\lambda) \theta_P(\lambda)^{-1} \quad ,$$

but he then shows that

$$c_M(\lambda) = \sum_{P \in \mathcal{P}(M)} c'_P(\lambda) \quad .$$

Thus if

$$\Gamma_M(H, \{X_P\}) = \sum_{P \subset \mathcal{P}(M)} \Gamma'_P(H, X_P)$$

then

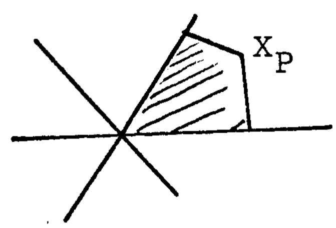
$$c_M(\lambda) = \int_{\mathfrak{a}} \int_{\mathfrak{a}} e^{\lambda(H)} \Gamma_M(H, \{X_P\})\, dH\, d\omega \quad .$$

Since the prime in $\Gamma'_P(H, X_P)$ serves no useful purpose I drop it. The measure ω being rapidly decreasing and the function $\Gamma_M(\cdot, \{X_P\})$ being supported in a ball of radius $c \sup_P \|X_P\|$ the function $c_M(\lambda)$

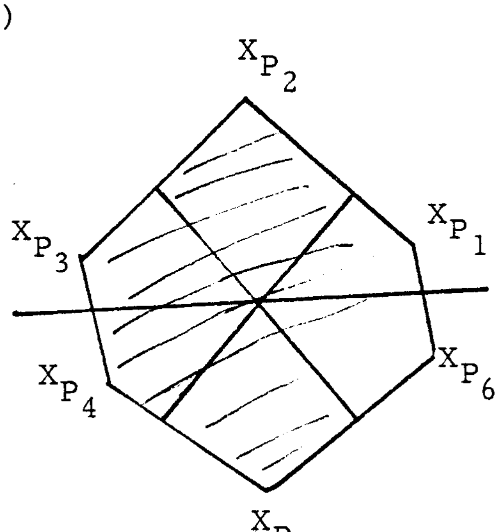
is smooth. Recall that $c_M(0)$ is usually denoted c_M. Of course if L is any Levi factor containing M then the family of functions $\{c_Q(\lambda) \mid Q \in \mathcal{P}(L)\}$ is also defined, as is $c_L(\lambda)$.

Recall that if each X_P lies in the chamber associated to P then the functions $\Gamma_P(\cdot, X_P)$ and $\Gamma_M(\cdot, \{X_P\})$ are characteristic functions. A typical pair is given by the following diagrams:

$\Gamma_P(\cdot, X_P)$



$\Gamma_M(\cdot, \{X_P\})$



Suppose M is the Levi factor of an ε-stable standard parabolic. Then the ε-roots divide $\mathfrak{a}_M^\varepsilon$ into chambers. Moreover, as we know, every root of G not lying in M has a non-zero restriction to $\mathfrak{a}_M^\varepsilon$. Thus if

W is a chamber in $\mathfrak{a}_M^\varepsilon$ we can let P_W be the group in $\mathcal{P}(M)$ defined by the condition that α is a root of N_{P_W} if and only if α is positive on W. The collection of P_W will be denoted $\mathcal{P}_\varepsilon(M)$. In a similar way, using the faces of the chambers, we define the collection $\mathcal{F}_\varepsilon(M)$ of parabolics.

Suppose more generally that M contains M_0 and that $\mathfrak{a} \subseteq \mathfrak{a}_M$. We say that the pair $(M, \mathfrak{a})$ is ε-special if it is conjugate to $(M', \mathfrak{a}_{M'}^\varepsilon)$, M' being the Levi factor of an ε-stable parabolic. Recall that if w normalizes M_0 and $\mathfrak{a}_{M'}^\varepsilon$ then $w\varepsilon w^{-1}\varepsilon^{-1}$ fixes each point of $\mathfrak{a}_{M'}^\varepsilon$ and thus normalizes the standard parabolic with M' as Levi factor. Consequently w represents an element $\Omega^\varepsilon(\mathfrak{a}_0, \mathfrak{a}_0)$ and maps the sets $\mathcal{P}_\varepsilon(M')$ and $\mathcal{F}_\varepsilon(M')$ onto themselves. So we can transport $\mathcal{P}_\varepsilon(M')$ and $\mathcal{F}_\varepsilon(M')$ from M' to M, thereby obtaining $\mathcal{P}_\varepsilon(M, \mathfrak{a})$ and $\mathcal{F}_\varepsilon(M, \mathfrak{a})$. Once we have these sets we can introduce the notion of a $(G, M, \mathfrak{a})$ family. It consists of a collection of functions c_P on $\mathfrak{a}$, one for each $P \in \mathcal{P}_\varepsilon(M, \mathfrak{a})$ which satisfy the obvious compatibility condition.

LEMMA 1. _If_ $\{c_P | P \in \mathcal{P}(M)\}$ _is a_ (G, M) _family then the collection_ $\{\bar{c}_P | P \in \mathcal{P}_\varepsilon(M, \mathfrak{a})\}$ _is a_ $(G, M, \mathfrak{a})$ _family_, $\bar{c}_P$ _being the restriction of_ c_P _to_ $\mathfrak{a}$.

Suppose Q and Q' are in $\mathcal{P}_\varepsilon(M, \mathfrak{a})$ and adjacent. The chambers W_Q^ε, $W_{Q'}^\varepsilon$ in $\mathfrak{a}$ associated to Q and Q' are then separated by a wall defined by an ε-root α. Let $\alpha_1, \ldots, \alpha_r$ be the roots whose restriction to $\mathfrak{a}$ is α. These are the only roots (up to sign) separating W_Q from $W_{Q'}$. Thus, after renumbering, we can find a sequence

$Q_0 = Q, Q_1, \ldots, Q_r = Q'$ such that Q_i is separated from Q_{i-1} by a single wall, that defined by α_i. If $\lambda \in \mathfrak{a}$ and $(\alpha, \lambda) = 0$ then $(\alpha_i, \lambda) = 0$ for all i and

$$\bar{c}_Q(\lambda) = c_{Q_0}(\lambda) = \ldots = c_{Q_r}(\lambda) = \bar{c}_{Q'}(\lambda) \quad .$$

Thus once $\mathfrak{a}$ is specified we can introduce the function $\bar{c}_M$ as well as functions $\bar{c}'_Q$, $Q \in \mathcal{F}_\varepsilon(M, \mathfrak{a})$. If $c_P(\lambda) = e^{\lambda(X_P)}$ and $\bar{X}_P$ is the projection of X_P on $\mathfrak{a}$ then

$$\bar{c}_P(\lambda) = e^{\lambda(X_P)} = e^{\lambda(\bar{X}_P)}, \qquad \lambda \in \mathfrak{a} \quad .$$

Thus if $\{c_P\}$ is associated to the measure ω then

$$\bar{c}_M(\lambda) = \int_{\mathfrak{A}} \int_{\mathfrak{a}} e^{\lambda(H)} \Gamma_{M,\mathfrak{a}}(H, \{\bar{X}_P\}) d\omega$$

where $\Gamma_{M,\mathfrak{a}}(\cdot, \{\bar{X}_P\})$ is the function on $\mathfrak{a}$ associated to the ε-roots and the family $\{\bar{X}_P\}$.

(G, M) - <u>families defined by intertwining operators</u>. Fix a standard P and let $M = M_P$. For brevity I shall denote the representation of $\mathbf{G}$ on the space of functions $g \longrightarrow \phi(g)e^{\lambda(H(g))}$, $\phi \in \mathfrak{A}_{\chi,\sigma}(P)$ by $\rho_{\sigma,\lambda}$. Thus $\rho_{\sigma,\lambda}(h)\phi = \phi'$ means $\phi(gh)e^{\lambda(H(gh))} = \phi'(g)^{\lambda(H(g))}$. There are several (G, M)-families to be introduced. The first is simple to define.

If $Q \in \mathcal{P}(M)$ then $Q = t^{-1}(P_1)$, where t is an element of the Weyl group and P_1 is standard. Let $Y_Q(T)$ be the projection onto $\mathfrak{a}_M$ of $t^{-1}(T-T_0) + T_0$ and set

$$c_Q(\Lambda) = e^{\Lambda(Y_Q(T))}, \qquad \Lambda \in \mathfrak{a}_M .$$

To define the second we choose an $s \in \Omega(\mathfrak{a}_{\varepsilon(P)}, \mathfrak{a}_P)$ and, fixing λ and Λ, we define μ by

$$\Lambda = s\varepsilon\mu - \lambda .$$

Suppose that $s\varepsilon(\sigma)$ and σ are equivalent up to tensoring with a character of $\mathbb{M}^1\backslash\mathbb{M}$. Then we set, suppressing s and σ from the notation and assuming ϕ to be K-finite,

$$d_Q(\Lambda) = \mathrm{tr}(M_{Q|P}(1, \lambda)^{-1} M_{Q|\varepsilon(P)}(s, \varepsilon(\mu))\varepsilon\rho_{\sigma,\lambda}(\phi)) .$$

LEMMA 2. <u>For each</u> λ the collection $\{d_Q\}$ <u>is a</u> (G, M)-<u>family</u>.

Suppose Q' and Q are adjacent. By one of the functional equations

$$M_{Q'|P}(1,\lambda)^{-1}M_{Q'|\varepsilon(P)}(s,\varepsilon(\mu)) = M_{Q|P}(1,\lambda)^{-1}M_{Q'|Q}(1,\lambda)^{-1}M_{Q'|Q}(1,s\varepsilon(\mu))M_{Q|\varepsilon(P)}(s,\varepsilon(\mu))$$

If the wall separating Q and Q' is defined by α and if $(\lambda, \alpha) = 0$ then λ and $s\varepsilon(\mu)$ have the same projection on $\mathbf{C}\alpha$. Thus by the functional equations (b)

$$M_{Q'|Q}(1, s\varepsilon(\mu)) = M_{Q'|Q}(1, \lambda) .$$

The lemma follows.

LEMMA 3. <u>Let</u> $\mathfrak{a}$ <u>be the set of points in</u> $\mathfrak{a}_P$ <u>fixed by</u> $s\varepsilon$. <u>Then</u> $\mathfrak{a}$

<u>is the second term of a unique</u> ε-<u>special pair</u> $(L, \mathfrak{a})$ <u>and</u> $L \supseteq M$.

The uniqueness of L is clear for it must be the centralizer of $\mathfrak{a}$. So it certainly contains M. To prove its existence we argue by induction on the semi-simple rank of G.

The lemma is clear if $\mathfrak{a}$ is central for the $L = G$. Otherwise let $X \in \mathfrak{a}$ be a point fixed by $s\varepsilon$ on which not all roots vanish. Then $\{\alpha | \alpha(X) \geq 0\}$ defines a proper parabolic subgroup of G which is fixed by $s\varepsilon$. Conjugating M, $\mathfrak{a}$ and $s\varepsilon$ we may suppose that it is standard. Since two standard parabolics which are conjugate are equal, the parabolic subgroup is invariant under both s and ε. In particular s lies in the Weyl group of its Levi factor, to which we then apply the induction assumption.

Since s is fixed we have not included the dependence of $\mathfrak{a}$ on it in the notation. If λ and Λ lie in $\mathfrak{a}$ then μ lies in $\mathfrak{a}$ and $\mu = \Lambda+\lambda$. Therefore

$$M_{Q|P}(1,\lambda)^{-1}M_{Q|\varepsilon(P)}(s,\varepsilon(\mu)) = M_{Q|P}(1,\lambda)^{-1}M_{Q|P}(1,\lambda+\Lambda)M_{P|\varepsilon(P)}(s,\varepsilon(\lambda+\Lambda)) \quad .$$

Recall that to each $Q \in \mathcal{P}(L)$ and to each $R \in \mathcal{P}^L(M)$ there is associated a unique group $Q(R) \in \mathcal{P}(M)$ such that $Q(R) \subset Q$ and $Q(R) \cap M = R$.

LEMMA 4. <u>If</u> λ <u>and</u> Λ <u>lie in</u> $\mathfrak{a}_L$ <u>then the operators</u>

$$\mathcal{M}_Q(P,\lambda,\ \Lambda) = M_{Q(R)|P}(1,\ \lambda)^{-1}M_{Q(R)|P}(1,\ \lambda+\Lambda)$$

<u>are independent of</u> R <u>and for fixed</u> λ <u>define an operator-valued</u> (G, L)-<u>family</u>.

The product

$$M_{Q(R')|P}(1, \lambda)^{-1} M_{Q(R')|P}(1, \lambda+\Lambda)$$

is equal to

$$M_{Q(R)|P}(1, \lambda)^{-1} M_{Q(R')|Q(R)}(1, \lambda)^{-1} M_{Q(R')|Q(R)}(1, \lambda+\Lambda) M_{Q(R)|P}(1, \lambda+\Lambda) \quad .$$

Since λ and $\lambda+\Lambda$ lie in $\mathfrak{a}_L$, the functional equation (b) yields

$$M_{Q(R')|Q(R)}(1, \lambda)^{-1} M_{Q(R')|Q(R)}(1, \lambda+\Lambda) = 1 \quad .$$

To prove that we have a (G, L)-family we imitate the proof of Lemma 2. It is only necessary to observe that if Q and Q' in $\mathcal{P}(L)$ are adjacent then we can find R and R' in $\mathcal{P}^L(M)$ such that $Q(R)$ and $Q'(R')$ are adjacent. Indeed if $\bar{\alpha} \in \Delta_Q$ defines the wall separating Q and Q' and if $\alpha \in \Delta_P$ restricts to $\bar{\alpha}$ then we may so choose R and R' that $Q(R)$ and $Q'(R')$ are separated by a wall lying in the hyperplane defined by α.

When $\lambda \in \mathfrak{a}$ let $\{{}_\varepsilon\mathcal{M}_Q(P, \lambda, \Lambda)\}$ be the $(G, L, \mathfrak{a})$-family attached to $\{\mathcal{M}_Q(P, \lambda, \Lambda)\}$. Finally set

$${}_\varepsilon\mathcal{M}_Q^T(P, \lambda, \Lambda) = \bar{c}_Q(\Lambda)\, {}_\varepsilon\mathcal{M}_Q(P, \lambda, \Lambda) = e^{\Lambda(Y_Q(T))}\, {}_\varepsilon\mathcal{M}_Q^T(P, \lambda, \Lambda) \quad .$$

In the statement of the following lemma and in its proof P need not be standard.

LEMMA 5. Set ${}_{\varepsilon}T_0 = 1-\varepsilon^{-1}T_0$ and suppose r is so chosen that the ε-special pair $(L, \mathfrak{a})$ associated to $rs\varepsilon(r^{-1})$ is $(M_Q, \mathfrak{a}_Q^{\varepsilon})$ where Q is an ε-invariant standard parabolic. If $\lambda \in \mathfrak{a}$ then

$$M_{P|\varepsilon(P)}(s, \varepsilon(\lambda)) = e^{\langle\lambda, T_{s\varepsilon}\rangle}{}_{\varepsilon}M(P, s) \quad ,$$

where ${}_{\varepsilon}M(P, s)$ is independent of λ and

$$T_{s\varepsilon} = (r^{-1}-1)\,{}_{\varepsilon}T_0 \quad .$$

To prove the lemma we observe that

$$M_{r(P)|\varepsilon(rP)}(rs\varepsilon(r^{-1}), \varepsilon(r\lambda))$$

is equal to

$$M_{r(P)|P}(r, s\varepsilon\lambda)M_{P|\varepsilon(P)}(s, \varepsilon\lambda)M_{\varepsilon(P)|\varepsilon(rP)}(\varepsilon(r^{-1}), \varepsilon(r\lambda)) \quad .$$

Using the functional equations (c) we see that this in turn is equal to

$$e^{(s\varepsilon\lambda+\rho_P)(T_0-r^{-1}T_0)}\, rM_{P|\varepsilon(P)}(s\ \varepsilon\lambda)\,\varepsilon(r^{-1})e^{(\varepsilon(r\lambda)+\rho_{\varepsilon(rP)})(T_0-\varepsilon(r)T_0)} \quad .$$

The dependence of the product of the two exponentials on λ is through

$$e^{\lambda(T_0-r^{-1}T_0+r^{-1}\varepsilon^{-1}T_0-\varepsilon^{-1}T_0)} = e^{\langle\lambda, T_{s\varepsilon}\rangle}$$

because $s\varepsilon\lambda = \lambda$.

The upshot is that to prove the lemma we may replace L, $\mathfrak{a}$, P and s by conjugate data. So we take L to be M_Q^ε and $\mathfrak{a}$ to be $\mathfrak{a}_L^\varepsilon$ for we have seen that this is possible. But then every point in $\mathfrak{a}$ is actually s-invariant and the equality is a consequence of the functional equations (b).

As an aside I observe that if K is ε-invariant, so that $H(\varepsilon(g)) = \varepsilon H(g)$, then

$$\varepsilon^{-1}T_o \equiv T_0 \pmod{\mathfrak{a}_G} \tag{2}$$

and consequently

$$T_{s\varepsilon} = 0 \quad .$$

To verify this we recall (Lemma 1.1 of the Annals paper) that T_0 is uniquely determined modulo $\mathfrak{a}_G$ by the condition

$$H_{P_0}(w^{-1}) = T_0 - s^{-1}T_0$$

for all $s \in \Omega(\mathfrak{a}_0, \mathfrak{a}_0)$ and all w in G representing s. However

$$\varepsilon T_0 - \varepsilon s^{-1}\varepsilon^{-1}(\varepsilon T_0) = \varepsilon(T_0 - s^{-1}T_0) = H(\varepsilon(w^{-1})) \quad .$$

Thus εT_0 is another candidate for T_0 and (2) follows.

<u>The fine</u> χ-<u>expansion</u>. The term $J_\chi^T(\phi)$ has been introduced in previous lectures, where it has been shown that it is a polynomial. Our purpose

in this (long) lecture is to prove a formula for it which we now describe.

If $M \in \mathcal{L}(M_0)$ and $\lambda \in \mathfrak{a}_M \otimes \mathbf{C}$ let χ_λ be the character of $\mathbf{M}$ defined by

$$\chi_\lambda(m) = e^{\lambda(H(m))} \quad .$$

If σ and σ' are two representations of $\mathbf{M}$ we write $\sigma' \sim \sigma$ if σ' is equivalent to $\sigma \otimes \chi_\lambda$ for some $\lambda \in \mathfrak{a}_M \otimes \mathbf{C}$. Each class has a distinguished representative, that which is trivial on $\{\exp H | H \in \mathfrak{a}_M\}$. We usually work with it.

The formula expresses $J_\chi^T(\phi)$ as a sum over quintuples $(M, L, \mathfrak{a}, \{\sigma\}, s)$ satisfying the following conditions:

(i) $M_0 \subseteq M \subseteq L$ and $\mathfrak{a} \subseteq \mathfrak{a}_M$.

(ii) $\{\sigma\}$ is a class of unitary automorphic representations of $\mathbf{M}$, the equivalence being that just defined, and σ is the distinguished representative.

(iii) $s \in \Omega(\mathfrak{a}_{\varepsilon(M)}, \mathfrak{a}_M)$ and $s\varepsilon(\sigma) \sim \sigma$.

(iv) $(L, \mathfrak{a})$ is ε-special and $\mathfrak{a}$ is the set of fixed points of $s\varepsilon$ in $\mathfrak{a}_M$.

(v) If $P \in \mathcal{P}(M)$ the space $\mathfrak{a}_{\chi,\sigma}(P)$ is not reduced to zero.

We now describe the term corresponding to a given quintuple. Let $\Omega_0 = \Omega(\mathfrak{a}_0, \mathfrak{a}_0)$ and $\Omega_0^M = \Omega^M(\mathfrak{a}_0, \mathfrak{a}_0)$. The linear transformation $s\varepsilon - 1$ is invertible on $\mathfrak{a}_M/\mathfrak{a}$. Let $\Delta = \Delta(s, \varepsilon)$ be the absolute value of its determinant. The term is

$$\frac{|\Omega_0^M|}{|\Omega_0|} \frac{1}{|\mathcal{P}(M)|\Delta} \sum_{P \in \mathcal{P}(M)} \frac{1}{(2\pi)^{a_P}} \int_{i\mathfrak{a}^*} \mathrm{tr}\left({}_\varepsilon M_L^T(P,\lambda) e^{\langle \lambda, T_{s\varepsilon} \rangle} {}_\varepsilon M(P,s)\varepsilon\rho_{\sigma,\lambda}(\phi)\right) d\lambda \quad .$$

[illegible]?

(Notice that the group $\mathbf{Z}$ and the character ω of the first lecture have unnoticed become trivial. The general case will have to await the revised lecture notes.) For P which are not standard the integrand is defined by symmetry.

Loose strands. The purpose of this section is to recapitulate results from earlier lectures, but with some minor changes and with a notation convenient for our present purposes. We begin by discussing the coarse χ-expansion more fully than in Lecture 13.

Recall that this is an expression for $J^T(\phi)$ as a sum,

$$J^T(\phi) = \sum_\chi J^T_\chi(\phi) \quad , \tag{3}$$

the index of summation running over equivalence classes of pairs of cuspidal data. To obtain it one first expresses the kernel as a sum,

$$K(h, g) = \sum_\chi K_\chi(h, g) \quad .$$

For each standard P and each distinguished σ let $\{\Phi_i\}$ be an orthonormal base of $\mathfrak{A}_{\chi,\sigma}(P)$ and set

$$K_{\chi,P,\sigma}(h, g) = \sum_i E(h, \rho_\sigma(\phi)\Phi_i)\overline{E(g, \Phi_j)} \quad .$$

The collection of unitary automorphic representations of $\mathbf{M}$ is the union of affine spaces of the form $\{\sigma_0 \otimes \chi_\lambda | \sigma_0 \text{ distinguished}, \ \lambda \in i\mathfrak{a}_M\}$. Let $d\sigma$ be the measure which on each component is $|d\lambda|$. Then

$$K_\chi(h, g) = \sum_P \frac{1}{(2\pi)^{a_{\nu}} n(P)} \int_{\{\sigma\}} K_{\chi,P,\sigma}(h, g)\,d\sigma$$

$$= \sum_P \frac{1}{(2\pi)^{a_P} n(P)} \int_{\{\sigma\}} \sum_i E(h, \rho_\sigma(\phi)\Phi_i)\overline{E(g, \Phi_j)}\,d\sigma \quad .$$

Moreover

$$(4) \qquad \sum_\chi \sum_P \frac{1}{(2\pi)^{a_P} n(P)} \int_\sigma \sum_i |E(h, \rho_\sigma(\phi)\Phi_i)\overline{E}(g, \Phi_j)|\,d\sigma < \infty \quad .$$

The integer a_P is equal to the dimension of the split component of P and $n(P)$ is the number of parabolic subgroups with the same Levi factor as P. The absolute convergence of (4) was proven in Lecture 10.

We can also introduce

$$K_\chi^\varepsilon(h, g) = K_\chi(h, \varepsilon(g))$$

and, when P is ε-invariant,

$$K_{P,\chi}^\varepsilon(h, g) = K_{P,\chi}(h, \varepsilon(g)) \quad .$$

Of course

$$K_P^\varepsilon(h, g) = \sum_\chi K_{P,\chi}^\varepsilon(h, g) \quad .$$

The basic identity, viz. the equality of

$$\sum_{P_0 \subset P} (-1)^{a_P^\varepsilon - a_G^\varepsilon} \sum_{\delta \in P\backslash G} K_{P,\chi}^\varepsilon(\delta g, \delta g)\,{}_\varepsilon\hat{\tau}_P(H(\delta g)-T)$$

and

$$\sum_{P_0 \subset P_1 \subset P_2} \sum_{\delta \in P_1 \backslash G} {}_{\varepsilon}\sigma_1^2(H(\delta g)-T)\Big(\sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon - a_G^\varepsilon} \Lambda^{T,P_1} K_{P,\chi}(\delta g, \delta g)$$

remains valid.

The expansion (3) is obtained from the χ-expansion of the left or the right side of the basic identity by integration over $G \backslash \mathbf{G}_\varepsilon^1$. It is however necessary to verify absolute convergence in order to justify the interchange of summation and integration. For this we use the right side. The left side is used only for the purpose of showing that $J_\chi^T(\phi)$ is a polynomial in T, the argument imitating that in §2 of the Annals paper.

To prove the convergence of the coarse χ-expansion we show that for each pair of standard parabolic subgroups $P_1 \subset P_2$ the sum

$$\sum_\chi \int_{P_1 \backslash \mathbf{G}_\varepsilon^1} {}_{\varepsilon}\sigma_1^2(H(g)-T) \Big| \sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_{P,\chi}^\varepsilon(g, g) \Big| dg < \infty \quad .$$

This is a stronger assertion than that treated in Lectures 7 and 8 but the proof proceeds along similar lines. We indicate the necessary modifications including those entailed by the replacement of σ_1^2 by ${}_{\varepsilon}\sigma_1^2$. The critical observation is Lemma 2.3 of the Compositio paper.

The first step is to find a substitute for the argument on pp. 3-5 in order to replace (we are taking $\omega \equiv 1$)

$$(5) \qquad \sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_{P,\chi}^\varepsilon(h, g)$$

by

$$(6) \qquad (-1)^{a_Q^\varepsilon} \sum_{\gamma \in F_\varepsilon(P_1,P_2)} \Lambda^{T,P_1} K^\varepsilon_{P_1,\chi}(h, \gamma\varepsilon(g))$$

if there is an ε-invariant parabolic Q between P_1 and P_2 and by zero otherwise.

Lemma 2.3 allows one first to extend Lemma 7.6 in Lecture 7 to $K_{P_1,\chi}$. Then following §2 of the Compositio paper we deduce the equality of (5) with (6) or with 0 from the corresponding equality for the original kernels.

Variants of Lemma 2.3 can be obtained from a simple observation, which was drawn to my attention by Clozel. The kernel $K_{P,\chi}$ is the kernel of an operator on $L^2(\mathbf{N}_P P\backslash \mathbf{G})$ and is equal to

$$\prod_\chi^1 K_P(h, g) = \prod_\chi^2 K_P(h, g)$$

where the superscript indicates whether we operate on the first or the second variable and $\prod_\chi$ is the projection on the space attached to χ. The operator $\prod_\chi$ acts of course on a function f according to

$$(\prod_\chi f)_g = \prod_\chi(\mathbf{M}^1) f_g$$

where $\prod_\chi(\mathbf{M}^1)$ is an operator on functions on $\mathbf{M}^1$ and $f_g(m) = f(mg)$, $m \in \mathbf{M}^1$.

Since

$$\Lambda^{T,P_1} K_{P_1,\chi}(h, g) = \Lambda^{T,P_1} \prod_\chi^2 K_{P_1}(h, g)$$

$$= \prod_\chi^2 \Lambda^{T,P_1} K_{P_1}(h, g)$$

we conclude that when

$$\Lambda^{T,P_1} K_{P_1,\chi}(h, mg) \not\equiv 0$$

as a function of $m \in \mathbf{M}^1$ then

$$\Lambda^{T,P_1} K_{P_1}(h, mg) \not\equiv 0 \quad .$$

The modified Lemmas 7.2, 7.4, and 7.6 follow immediately and the changes in the proof of Lemma 7.1 are minimal, for $\Lambda^{T,P_1} \prod_\chi^2 K_{P_1}(h, g) \neq 0$ implies that $K_{P_1}(h, mg) \neq 0$ for some $m \in \mathbf{M}^1$.

The proof of Lemma 7.3 for ${}_\varepsilon\sigma_1^2$ is the same as its proof for σ_1^2 and the modified Lemma 7.5 is implied by Lemma 4.4 of the first Duke Jour. paper and was proved by Clozel.

The coarse χ-expansion is the expansion

$$J^T(\phi) = \sum_\chi J_\chi^T(\phi)$$

and $J_\chi^T(\phi)$ is the sum over pairs $P_1 \subset P_2$ of standard parabolic subgroups of

$$\int_{P_1 \backslash \mathbf{G}_\varepsilon^1} {}_\varepsilon\sigma_1^2(H(g)-T)\{\sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_{P,\chi}^\varepsilon(g, g)\}dg \quad .$$

Following the arguments of Lectures 8, 10, and 11 one shows that this may be written as

$$(7)\quad (-1)^{a_Q^\varepsilon} \sum_{\gamma \in P_1 \backslash F_\varepsilon(P_1,P_2)/\varepsilon^{-1}(P_1)} \int_{P_1 \cap \varepsilon^{-1}\gamma^{-1}(P_1)\backslash \mathbf{G}_\varepsilon^1} {}_\varepsilon\sigma_1^2(H(g)-T)\Lambda^{T,P_1} K_{P_1,\chi}(g,\gamma\varepsilon(g))dg \quad .$$

It was shown in Lectures 10 and 11 that each term is zero unless there is a unique ε-invariant parabolic Q between P_1 and P_2. However more can be squeezed out of the arguments given there, namely that even when the ε-invariant parabolic between P_1 and P_2 is unique the only contribution which perhaps does not vanish is that attached to the class of $\gamma = 1$. Since this leads to an indispensable simplification, we give the necessary supplementary argument.

The element γ may of course be taken to normalize M_0. Let it represent the element s in the Weyl group. We may assume that $s\varepsilon(\alpha) > 0$ for $\alpha \in \Delta_0^1$, for we are free to modify s on the right by an element of $\Omega^{\varepsilon^{-1}(M_1)}$. Recall that there is a unique standard parabolic subgroup $P_{s\varepsilon}$ such that

$$P_{s\varepsilon} \cap M_1 = \varepsilon^{-1}s^{-1}(P_1) \cap M_1$$

and that if $H = H(g)$ the following conditions must be satisfied if the term of (7) corresponding to γ is not to vanish:

(i) $\varpi(H-T) \leq 0$, $\varpi \in \hat{\Delta}_{s\varepsilon}^1$.

(ii) ${}_\varepsilon\sigma_1^2(H-T) \neq 0$.

(iii) $\varpi(H-s\varepsilon H) \leq C$, $\varpi \in \hat{\Delta}_1^Q$. (For this it is necessary to apply the original arguments within Q.)

The conditions (i) and (ii) allow us to write the projection of $H-T$ on $\mathfrak{a}_s^Q$ as

$$-\sum_{\alpha \in \Delta_{s\varepsilon}^1} c_\alpha \alpha + \sum_{\varpi \in \Delta_1^Q} c_\varpi \varpi \quad ,$$

with all coefficients non-negative. Thus $\varpi_0(H-s\varepsilon H)$ is equal to

$$(8) \qquad \varpi_0(T-s\varepsilon T) + \sum_{\alpha \in \Delta_{s\varepsilon}^1} c_\alpha \varpi_0(s\varepsilon(\alpha)) + \sum_{\varpi \in \hat{\Delta}_1^Q} c_\varpi \varpi_0(\varpi - s\varepsilon\varpi)$$

if $\varpi_0 \in \hat{\Delta}_1^Q$. Notice that $\varpi_0(\alpha) = 0$ if $\alpha \in \Delta_{s\varepsilon}^1$.

The expression (8) must be bounded by a constant independent of T and H. The space $\mathfrak{a}_0^{s\varepsilon}$ is spanned by roots of $M_{s\varepsilon}$. Thus $s\varepsilon(\mathfrak{a}_0^{s\varepsilon})$ is orthogonal to $\mathfrak{a}_1$ and if $\alpha \in \Delta_{s\varepsilon}^1$ is the image of $\alpha' \in \Delta_0^1$ then $\varpi_0(s\varepsilon(\alpha)) = \varpi_0(s\varepsilon(\alpha'))$. Consequently $\varpi_0(s\varepsilon(\alpha)) \geq 0$. We conclude that

$$\varpi_0(T-s\varepsilon T) + \sum c_\varpi \varpi_0(\varpi - s\varepsilon\varpi) \leq C \quad .$$

Let

$$X = \sum_{\varpi_0 \in \hat{\Delta}_1^Q} c_{\varpi_0} \varpi_0 \quad .$$

Multiplying by c_{ϖ_0} and summing we conclude that

$$(X, T-s\varepsilon T) + (X, X) - (X, s\varepsilon X) \leq C\|X\| \quad .$$

Now $X \in \mathfrak{a}_1^{Q_+}$. Thus if we assume (and we shall) that T is ε-invariant then

$$(X, T-s\varepsilon T) = (X, T-sT) \geq 0 \quad .$$

Moreover there is a constant $\delta > 0$ such that

$$(X, X) - (X, s\varepsilon X) \geq \delta \|X\|^2$$

on $\mathfrak{a}_1^{Q_+}$. To see this we have only to verify that

$$\min_{\{X \in \mathfrak{a}_1^{Q_+} \mid \|X\|=1\}} ((X, X) - (X, s\varepsilon X)) > 0 \quad .$$

The minimum is certainly not negative. If it is 0 then for some $Y \in \mathfrak{a}_1^{Q_+}$

$$Y = s\varepsilon Y \quad .$$

The set of roots α in Δ_0^Q such that $(\alpha, Y) \geq 0$ define a standard parabolic subgroup between P_1 and Q which is properly smaller than Q. Since it is invariant under $s\varepsilon$ it is invariant under both ε and s. This contradicts the definition of Q.

We conclude that

$$\|X\|^2 \leq c^1 \|X\|$$

and thus that $\|X\|$ is bounded. We obtain finally inequalities

$$\varpi_0(T-sT) \le C'' , \qquad \varpi_0 \in \hat{\Delta}_1^Q .$$

These inequalities can be violated for T sufficiently regular unless $s \in \Omega^{M_1}$. This leads to the desired conclusion.

So we are to consider

$$(-1)^{a_Q^\varepsilon} \int_{P_1 \cap \varepsilon^{-1}(P_1)\backslash \mathbf{G}_\varepsilon^1} \varepsilon\sigma_1^2(H(g)-T)\Lambda^{T,P_1} K_{P_1,\chi}(g, \varepsilon(g))dg .$$

It was shown in Lectures 10 and 11 that this could be expanded as

$$(-1)^{a_Q^\varepsilon}\sum_P \frac{1}{(2\pi)^{a_P} n_1(P)} \int_{P_1 \cap \varepsilon^{-1}(P_1)\backslash \mathbf{G}_\varepsilon^1} \varepsilon\sigma_1^2(H(g)-T)\int_{\Sigma(P)} \Lambda^{T,P_1} K_{P_1\chi,P,\sigma}(g,\varepsilon(g))d\sigma dg ,$$

$\Sigma(P)$ being the set of possible $\{\sigma\}$. Recall that

$$K_{P_1,\chi,P,\sigma}(g, \varepsilon(g)) = \sum_j E_{P_1}(g, \rho_\sigma(\phi)\Phi_j)\overline{E_{P_1}(\varepsilon(g), \Phi_j)} .$$

Thus we have an integral over the space parametrized by P, σ, j, and g. Some care is necessary because there is an element of conditional convergence, which we recall explicitly. The group $i\mathfrak{a}_1$ acts on Σ and we can clearly decompose $\Sigma(P)$ as a product $\Sigma_1(P) \times i\mathfrak{a}_1$, the connected components of Σ_1 being affine spaces over $i\mathfrak{a}_P^1$ (The attempt to distinguish between spaces and their duals becomes too much of a burden on the notation and I abandon it).

If $\sigma_1 \in \Sigma_1$, $\lambda_1 \in i\mathfrak{a}_1$ denote $\rho_{\sigma_1 \otimes \chi_{\lambda_1}}$ by $\rho_{\sigma_1,\lambda_1}$. The integral is obtained by iterating two other integrals, each of which is absolutely convergent although their iteration may not be. The first is

$$(9)\qquad \frac{1}{(2\pi)^{a_P} n_1(P)} \int_{i\mathfrak{a}_1} \sum_j \Lambda^{T,P_1} E_{P_1}(g, \rho_{\sigma_1}, \lambda_2(\phi)\Phi_j)\overline{E_{P_1}(\varepsilon(g), \Phi_j)}\,|d\lambda| \quad .$$

The second is over

$$\{(P, \sigma)\,|\,P \supseteq P_0,\ \sigma_1 \in \Sigma_1(P)\} \times P_1 \cap \varepsilon^{-1}(P_1)\backslash \mathbf{P}^1 \times K \times \mathfrak{a}^1_{1,\varepsilon}$$

where ${}_\varepsilon\mathfrak{a}^1_1$ is the set of $H \in \mathfrak{a}_1$ such that $d\chi(H) = 0$ for every ε-invariant character χ of G defined over $\mathbf{Q}$. The integrand for the second is the product of (9) with ${}_\varepsilon\sigma^2_1(H-T)e^{-2\rho_{P_1}(H)}$, g being $p(\exp H)k$. For this we do not need to assume that ϕ is K-finite. However for the first part of the proof of the fine χ-expansion we do, the assumption and the theory of Eisenstein series assuring us that the set of (P, σ, j), σ distinguished, which yield a non-zero contribution for a given χ is finite, so that the sum over these parameters presents no analytical problems. Thus until we explicitly return to the general case ϕ will be K-finite.

The integrand of (9) is clearly an entire function of λ_1. It is shown in Lectures 10 and 11 that the contour can be deformed to $\mathrm{Re}\,\lambda_1 = -\Lambda$, $\Lambda \in \mathfrak{a}_1$ arbitrary, without changing the value of the integral, which remains absolutely convergent, the parameter implicit in $E_{P_1}(\varepsilon(g), \Phi_j)$ being $-\overline{\lambda}_1$. We choose Λ such that $(\Lambda, \alpha) \gg 0$ for all $\alpha \in \Delta_1$.

Then, and this will be shown in Lectures 10 and 11, the double integral

$$(10)\int_{P_1\cap\varepsilon^{-1}(P_1)\backslash P_1^1\times K}\int_{\operatorname{Re}\lambda_1=-\Lambda}\Lambda^{T,P_1}E_{P_1}(pak,\rho_{\sigma_1,\lambda_1}(\phi)\Phi_j)\overline{E_{P_1}(\varepsilon(pak),\Phi_j)}\,|d\lambda_2|\,dp\,dk,\quad a=\exp H\ ,$$

is absolutely convergent. Given the properties of the truncation operator this follows from an estimate

$$(11)\qquad \sum_{\gamma\in P_1\cap\varepsilon^{-1}(P_1)\backslash P_1}|E_{P_1}(\varepsilon(\gamma pak),\ \Phi_j)|\le C(a)\,|m|^N\ .$$

Here $p = nm$, N is some fixed real number depending on Λ but not on λ_1 with $\operatorname{Re}\lambda_1 = -\Lambda$, and m lies in a Siegel domain of $\mathbf{M}_1$.

The parameter in the Eisenstein series is λ_1. All we need do is estimate

$$(12)\qquad \sum_{\gamma\in P_1\backslash G}|E_{P_1}(\gamma g,\ \Phi_j)|$$

on a Siegel domain of $\mathbf{G}$, for taking $g = \varepsilon(pak)$ we majorize (10). That (12) is bounded by $C|g|^N$ follows from the elements of the theory of Eisenstein series (see the remarks following Lemma 4.1 of my notes on the subject).

Apart from a finite sum over P, distinguished σ, and j and a constant

$$(-1)^{a_Q^{\varepsilon}}\frac{1}{(2\pi)^{a_P}n_1(P)}$$

we have to consider

$$(13)\quad \int_{\varepsilon\mathfrak{a}_1^1} \varepsilon\sigma_1^2(H-T)e^{-2\rho_{P_1}(H)} \int_{\mathrm{Re}\,\lambda=-\Lambda} \int_{P_1\cap\varepsilon^{-1}(P_1)\backslash \mathbf{P}_1^1\times K} a(pak,\lambda) = A^T(\phi)$$

where

$$a(pak,\lambda) = \Lambda^{T,P_1} E_{P_1}(pak, \rho_{\sigma,\lambda}(\phi)\Phi_j \overline{(E_{P_1}(\varepsilon(pak), \Phi_j)} ,$$

$a = \exp H$, and $_{\varepsilon}\mathfrak{a}_1^1 \subseteq \mathfrak{a}_1$ is the intersection of the kernels of the ε-invariant rational characters of G.

In contrast to the integrals appearing in the ordinary trace formula, (13) does not seem to admit a useful explicit expression even when Φ_j is a cusp form. So we derive an approximate formula for it, anticipating the needs of the arguments in the two Amer. Jour. papers. Recall that they involve substituting $\phi_\gamma = \phi_H$ for ϕ where $\gamma = \gamma_H$ is the distribution

$$\gamma = \frac{1}{\Omega} \sum_{s\in\Omega} \gamma_{s^{-1}H}$$

and ϕ_γ is obtained from ϕ by applying the multiplier associated to γ. Then $A^T(\phi_H)$ is a function of T and H, $H \in \mathfrak{h}$ (Arthur works with a subspace $\mathfrak{h}^1 \subseteq \mathfrak{h}$, but with our formulations $\mathfrak{h}$ is better).

In order to simplify the formulation at various places, we formalize the inequality (5.1) of Amer. Jour. I into a definition. Fix an integer $d_0 \geq 0$. If $\psi^T(H)$ is a function of T and H we write

$$\psi^T(H) \sim 0$$

if there are positive constants ε and C and for every invariant differential operator D on $\mathfrak{h}$ a constant c_D such that

$$|D\psi^T(H)| \leq c_D e^{-\varepsilon\|T\|}(1+\|T\|)^{d_0}$$

whenever $d(T) > C(1+\|H\|)$. Recall that

$$d(T) = \min_{\{\alpha, Q \mid \alpha \in \Delta_Q, Q \supseteq P_0\}} \alpha(T) \quad .$$

Set $\mu = -\varepsilon^{-1}(\lambda)$ and define Ψ_j by

$$\Phi_j(\varepsilon(g)) = \Psi_j(g) \quad .$$

It is a function in $\mathfrak{A}_{\varepsilon^{-1}(\chi), \varepsilon^{-1}(\sigma)}(\varepsilon^{-1}(P))$. Thus if $P' \subseteq \varepsilon^{-1}(P_1) \cap P_1$ and $s \in \Omega^{\varepsilon^{-1}(P_1)}(\mathfrak{a}_{\varepsilon^{-1}(P)}, \mathfrak{a}_{P'})$ we may build the associated Eisenstein series $E_{P_1}(g, M_{P'|\varepsilon^{-1}(P)}(s, \bar{\mu})\Psi_j)$. Set

$$(14)\quad b(pak,\lambda) = \sum_{P'}\sum_{s} \Lambda^{T,P_1} E_{P_1}(pak, \rho_{\sigma,\lambda}(\phi)\Phi_j)\bar{E}_{P_1}(pak, M_{P'|\varepsilon^{-1}(P)}(s, \bar{\mu})\Psi_j) \quad .$$

The sum over P' is a sum over associate classes within $\varepsilon^{-1}(P_1) \cap P_1$. Thus we take only one representative from each class, several classes appearing only because they become associate in $\varepsilon^{-1}(P_1)$. The variable s runs over

$$\Omega^{\varepsilon^{-1}(P_1)\cap P_1}(\mathfrak{a}_{P'}, \mathfrak{a}_{P'}) \setminus \Omega^{\varepsilon^{-1}(P_1)}(\mathfrak{a}_{\varepsilon^{-1}(P)}, \mathfrak{a}_{P'}) \quad .$$

Finally set

$$B^T(\phi) = \int_{\varepsilon \mathfrak{a}_1^1} \varepsilon \sigma_1^2(H-T) e^{-2\rho_{P_1}(H)} \int_{\operatorname{Re}\lambda=0} \int_{M_1 \backslash \mathbf{M}_1^1 \times K} b(mak, \lambda) \quad .$$

We shall show that the iterated integral converges and that

$$A^T(\phi_H) - B^T(\phi_H) \sim 0 \quad .$$

We begin by studying

$$\int_{P_1 \cap \varepsilon^{-1}(P_1) \backslash \mathbf{P}_1^1 \times K} a(pak, \lambda)$$

when Φ_j is a cusp form. This is best regarded as a triple integral, over

$$(M_1 \backslash \mathbf{M}_1^1 \times K) \times (M_1 \cap \varepsilon^{-1}(P_1) \backslash M_1) \times N_1 \backslash \mathbf{N}_1 \quad ,$$

and we begin with the integral over $N_1 \backslash \mathbf{N}_1$.

Since

$$\Lambda^{T,P_1} E_{P_1}(ng, \rho_{\sigma,\lambda}(\phi)\Phi_j) = \Lambda^{T,P_1} E_{P_1}(g, \rho_{\sigma,\lambda}(\phi)\Phi_j)$$

we are led to consider

$$\int_{N_1 \backslash \mathbf{N}_1} E_{P_1}(\varepsilon(ng), \Phi_j) dn = \int_{N_1 \backslash \mathbf{N}_1} E_{\varepsilon^{-1}(P_1)}(ng, \Psi_j) dn \quad .$$

I claim that it is equal to

$$(15) \qquad \sum_{P'} \sum_s E_{P_1 \cap \varepsilon^{-1}(P_1)}(g, M_{P'|\varepsilon^{-1}(P)}(s, \overline{\mu})\Psi_j) \quad ,$$

the range of summation being the same as in (14).

It is enough to verify this equality in the domain of absolute convergence of the Eisenstein series. The proof will be easier to follow if for a few brief moments we change the notation, letting $\varepsilon^{-1}(M_1)$ be G, $P_1 \cap \varepsilon^{-1}(M_1)$ be Q, and $\varepsilon^{-1}(P)$ be P. Our integral is then

$$(16) \qquad \int_{N_Q \backslash \mathbf{N}_Q} E_G(ng, \Psi_j)\, dn \quad .$$

Let

$$F(g, \Psi_j) = \Psi_j(g) e^{(\mu+\rho_P)(H_P(g))} \quad ,$$

so that (16) is equal to

$$(17) \qquad \sum_{\gamma \in P\backslash G/Q} \int_{N_Q \backslash \mathbf{N}_Q} \sum_{\delta \in Q \cap \gamma^{-1} P \gamma \backslash Q} F(\gamma\delta ng, \Psi_j)\, dn \quad .$$

Each γ may be chosen to lie in the normalizer of $\mathfrak{a}_0$ and thus to represent an element s^{-1} of the Weyl group $\Omega(\mathfrak{a}_0, \mathfrak{a}_0)$. We have sufficient freedom to suppose that $s\alpha > 0$ for $\alpha \in \Delta_0^{\varepsilon^{-1}(P)}$. The group

$$\gamma\delta Q \delta^{-1} \gamma^{-1} \cap M_P = \gamma Q \gamma^{-1} \cap M_P$$

is then a standard parabolic subgroup of M_P with unipotent radical $\gamma N_Q \gamma^{-1} \cap M_P$. Since Ψ_j is a cusp form the term of (17) associated to γ is 0 unless $\gamma N_Q \gamma^{-1} \cap M_P = 1$ and thus unless $\gamma M_Q \gamma^{-1} \supset M_P$.

We now assume this and in addition that $s^{-1}\alpha > 0$ for $\alpha \in \Delta_0^Q$

which implies that $s\alpha > 0$ for $\alpha \in \Delta_0^P$. The group

$$P' = N_Q(\gamma^{-1}P\gamma \cap M_Q) = N_Q(\gamma^{-1}N_P\gamma \cap M_Q)\cdot\gamma^{-1}M_P\gamma$$

is a parabolic subgroup associate to P and $s \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$.

The term associated to γ is equal to

$$\sum_{\delta \in Q \cap \gamma^{-1}P\gamma \backslash Q/N_Q} \int_{N_Q \cap \delta^{-1}\gamma^{-1}P\gamma\delta \backslash \mathbf{N}_Q} F(\gamma\delta n g, \Psi_j)\, dn$$

or

$$\sum_{\delta \in Q \cap \gamma^{-1}P\gamma \backslash Q/N_Q} \int_{N_Q \cap \gamma^{-1}P\gamma \backslash \mathbf{N}_Q} F(\gamma n\delta g, \Psi_j)\, dn \quad .$$

The domain of integration is $N_Q \cap \gamma^{-1}N_P\gamma \backslash \mathbf{N}_Q$ and may be replaced by $\mathbf{N}_Q \cap \gamma^{-1}\mathbf{N}_P\gamma \backslash \mathbf{N}_Q$. Since $N_{P'} = N_Q(\gamma^{-1}N_P\gamma \cap M_Q)$ and $\gamma^{-1}N_P\gamma \cap M_Q \subseteq \gamma^{-1}N_P\gamma$ the domain of integration may in fact be taken to be $\mathbf{N}_{P'} \cap \gamma^{-1}\mathbf{N}_P\gamma \backslash \mathbf{N}_{P'}$. Hence the integration yields

$$F(\delta g, M_{P'|P}(s, \overline{\mu})\Psi_j)$$

The range of summation is

$$(M_Q \cap \gamma^{-1}P\gamma)N_Q \backslash Q = P' \backslash Q \quad .$$

So we obtain

$$\sum_{P' \backslash Q} F(\delta g, M_{P'|P}(s, \overline{\mu})\Psi_j) \quad .$$

Summing over γ and reverting to our original notation we obtain (15).

The next step is to replace g by γg in (15) and to sum over $\gamma \in M_1 \cap \varepsilon^{-1}(P_1) \backslash M_1$. Interchanging the order of summation we obtain

$$(17)\quad \sum_{P'} \sum_{s} \sum_{\gamma \in M_1 \cap \varepsilon^{-1}(P_1) \backslash M_1} E_{P_1 \cap \varepsilon^{-1}(P_1)}(\gamma g, M_{P'|\varepsilon^{-1}(P)}(s, \overline{\mu})\Psi_j) \quad .$$

To justify the interchange we must show that the inner sum converges absolutely. If so, it yields

$$E_{P_1}(g, M_{P'|\varepsilon^{-1}(P)}(s, \overline{\mu})\Psi_j) \quad .$$

For absolute convergence we need

$$\mathrm{Re}(\alpha, s\mu_1) = (\alpha, s\varepsilon^{-1}(\Lambda)) \gg 0$$

for all $\alpha \in \Delta^{P_1}_{P_1 \cap \varepsilon^{-1}(P_1)}$ or for $\alpha \in \Delta_0^{P_1} - \Delta_0^{P_1 \cap \varepsilon^{-1}(P_1)}$, which is of course $\Delta_0^{P_1} - \Delta_0^{\varepsilon^{-1}(P_1)}$, a subset of $\Delta_0 - \Delta_0^{\varepsilon^{-1}(P_1)}$. Here $\mu_1 = -\varepsilon^{-1}(\lambda_1)$ where λ_1 is the projection of λ on $\mathfrak{a}_1$. Since $s\varepsilon^{-1}(\Lambda) = \varepsilon^{-1}(\Lambda)$,

$$(\alpha, s\varepsilon^{-1}(\Lambda)) = (\varepsilon(\alpha), \Lambda)$$

and $\varepsilon(\alpha) \in \Delta_0 - \Delta_0^1$. Hence $(\varepsilon(\alpha), \Lambda) \gg 0$ by assumption.

We are left with the evaluation of

$$\sum_{P'} \sum_{t} \int_{M_1 \backslash \mathbf{M}_1^1 \times K} \Lambda^{T,P_1} E_{P_1}(mak, \rho_{\sigma,\lambda}(\phi)\Phi_j) E_{P_1}(mak, M_{P'|\varepsilon^{-1}(P)}(t, \overline{\mu})\Psi_j)\, dmdk \quad ,$$

in which, for convenience later, the variable of summation s has been replaced by t. These integrals are evaluated by the inner product formula of the Compositio paper, which has (in effect) been proved in Lecture 12. The integral corresponding to P' and t is equal to $e^{2\rho_{P_1}(H)}$ times

$$(18)\quad \sum_{P''}\sum_{s_1,s_2} \frac{e^{(s_1\lambda+s_2t\mu)(T^1+H)}}{\theta^1_{P''}(s_1\lambda+s_2t\mu)}(M_{P''|P}(s_1,\lambda)\rho_{\sigma,\lambda}(\phi)\Phi_j,\ M_{P''|\varepsilon^{-1}(P)}(s_2t,\overline{\mu})\Psi_j) \quad .$$

Here $P_0 \subseteq P'' \subseteq P_1$, $s_1 \in \Omega^{P_1}(\mathfrak{a}_P, \mathfrak{a}_{P''})$, $s_2 \in \Omega^{P_1}(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$. The projection of T on $\mathfrak{a}^1_0$ is denoted T^1.

In general the terms of (18) are not individually defined on the domain of integration, $\text{Re}\,\lambda = \text{Re}\,\lambda_1 = -\Lambda$, because of the zeros of the denominator, which we now examine more closely. Apart from a constant $\theta^1_{P''}(s_1\lambda+s_2t\mu)$ is equal to

$$\prod_{\alpha\in\Delta^1_{P''}} \alpha(s_1\lambda+s_2t\mu) = \prod_{\alpha\in\Delta^1_{P''}} \alpha(s_1\lambda-s_2t\varepsilon^{-1}(\lambda)) \quad .$$

Moreover

$$\text{Re}\ \alpha(s_1\lambda-s_2t\varepsilon^{-1}(\lambda)) = \alpha(s_2t\varepsilon^{-1}(\Lambda)) = \beta(\Lambda) \quad ,$$

with $\beta = \varepsilon(t^{-1}s_2^{-1}(\alpha))$, and β either is identically 0 on $\mathfrak{a}_1$ or vanishes nowhere on $\mathfrak{a}^+_1$. If it vanishes identically and s_α, s_β are the reflections corresponding to α and β then

$$s_\alpha s_2 t\varepsilon^{-1} = s_2 t\varepsilon^{-1}(s_\beta)\varepsilon^{-1} \quad .$$

If we sum (18) over P' and t we obtain sums over P', P'', and s_2, whose ranges of summation are to be specified, of

$$(19)\ \sum_{s_1,t} \frac{e^{s_1\lambda+s_2t\mu(T^1+H)}}{\mathcal{O}^1_{P''}(s_1\lambda+s_2t\mu)} (M_{P''|P}(s_1,\lambda)\rho_{\sigma,\lambda}(\phi)\Phi_j,\ M_{P''|\varepsilon^{-1}(P)}(s_2t,\overline{\mu})\Psi_j)) \quad ,$$

s_1 running over $\Omega^{P_1}(\mathfrak{a}_P, \mathfrak{a}_{P''})$ and t over $\Omega^{\varepsilon^{-1}(P_1)}(\mathfrak{a}_{\varepsilon^{-1}(P)}, \mathfrak{a}_{P'})$. The remarks above allow us to apply the usual arguments and to conclude that the zeros of the denominators do not contribute to the singularities of (19). The sum is over P'' which we associate to P in P_1, and for each P'' over a set of representatives P' for the associate classes in $P_1 \cap \varepsilon^{-1}(P_1)$ which lie in the associate class of P'' in P_1. Once P'' and P' are fixed,

$$s_2 \in \Omega^{P_1}(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})/\Omega^{\varepsilon^{-1}(P_1)\cap P_1}(\mathfrak{a}_{P'}, \mathfrak{a}_{P'}) \quad .$$

We are taking $(\Lambda, \alpha) \gg 0$, $\alpha \in \Delta_1$. Thus the numerators of (19) are well-behaved functions and we can consider

$$(20) \int_{\mathrm{Re}\,\lambda_1=-\Lambda} \sum_{s_1,t} \frac{e^{(s_1\lambda+s_2t\mu)(T^1+H)}}{\theta^1_{P''}(s_1\lambda+s_2t\mu)} (M_{P''|P}(s_1,\lambda)\rho_{\sigma,\lambda}(\phi)\Phi_j, M_{P''|\varepsilon^{-1}(P)}(s_2t,\mu)\Psi_j))\,|d\lambda_1|$$

I claim that this is zero for T sufficiently regular and ${}_\varepsilon\sigma_1^2(H-T) \neq 0$ unless $s_2 \in \Omega^{\varepsilon^{-1}(P_1)\cap P_1}$. The reader will note that sufficiently regular means $d(T) \geq C(1 + \|H_0\|)$ when $\phi = \phi_{H_0}$.

Recall that Q is the smallest ε-invariant parabolic containing P_1. Let $H = T_1+X+Y$ with $Y \in \mathfrak{a}_Q$ and with $X \in \mathfrak{a}_1^Q$. Then $T = T_1+T^1$ and $\alpha(X) > 0$ for all $\alpha \in \Delta_1^Q$. We deform the contour to $\mathrm{Re}\,\lambda_1 = -\Lambda-tX$. Then

$$e^{s_1\lambda+s_2t\mu(T^1+H)}$$

is multiplied by

$$e^{-t(X-s_2t\varepsilon^{-1}(X),T)-t(X-s_2t\varepsilon^{-1}(X),X)} \quad .$$

Now, as we saw above,

$$(X-s_2t\varepsilon^{-1}(X),X) \geq \delta\|X\|^2$$

and

$$(X-s_2t\varepsilon^{-1}(X),T) = (X-\varepsilon s_2t\varepsilon^{-1}(X),T) \geq 0 \quad .$$

Since we can estimate

$$(M_{P''|P}(s_1, \lambda)\rho_{\sigma,\lambda}(\phi_{H_0})\Phi_j, M_{P''|\varepsilon^{-1}(P)}(s_2t, \mu)\Psi_j)$$

when $\mathrm{Re}\,\lambda = -\Lambda-tX$ by

$$e^{c\|X\|\|H_0\|}f(\mathrm{Im}\,\lambda) \quad ,$$

with f integrable, we see that the integral vanishes unless $\|X\| \leq c\|H_0\|$.

For $\|X\| \leq c\,\|H_0\|$ we take $\alpha \in \Delta_1^Q$ and deform the contour to $\mathrm{Re}\,\lambda_1 = -\Lambda-t\varpi_\alpha$. If $\varepsilon s_2t^{-1}\varepsilon^{-1}(\varpi_\alpha) \neq \varpi_\alpha$ then

$$(\varpi_\alpha - \varepsilon s_2 t^{-1}(\varpi_\alpha),\ T) \geq c\|T\| \quad .$$

If $s_2 \notin \Omega^{\varepsilon^{-1}(P_1)}$ we can choose α such that $\varepsilon s_2 t^{-1}\varepsilon^{-1}(\varpi_\alpha) \neq \varpi_\alpha$ and then we obtain vanishing for $\|T\| \geq c(1+\|H_0\|)$. Note that the value of the constant c changes from line to line.

If $s_2 \in \Omega^{\varepsilon^{-1}(P) \cap P_1}$ then it may be taken to be 1. Then the integral (20) has a very useful property. Neither $M_{P''|P}(s_1, \lambda)$ nor $M_{P''|\varepsilon^{-1}(P)}(t, \mu)$ depends on λ_1 but only on the projection of λ onto $\mathfrak{a}_P^1$. Thus they have no singularities to obstruct the deformation of the contour, which may therefore be taken to be defined by any Λ with $(\alpha, \Lambda) < 0$ for all $\alpha \in \Delta_1$, or even Δ_1^Q. We may not however allow the (α, Λ) to become zero, for the zeros of the denominators could then cause trouble.

To obviate this we choose δ such that none of the functions

$$M_{P''|\varepsilon^{-1}(P)}(s_2 t, \mu)\Psi_j$$

which appear in (19) have singularities in the region $\|\mathrm{Re}\,\lambda\| < \delta$, $(\mathrm{Re}\,\lambda, \alpha) \leq 0$, $\alpha \in \Delta_1^Q$, even if $s_2 \notin \Omega^{\varepsilon^{-1}(P_1) \cap P_1}$. Then we choose a Λ with $\|\Lambda\| < \delta$. Having deformed the contour we take once again the sum over all P', P'' and s_2, thereby introducing an error which must be estimated. This done the zeros of the denominator no longer cause any trouble; so we can deform to $\Lambda = 0$. Putting back the factor $e^{2\rho_{P_1}(H)}$, then integrating over $\mathfrak{a}_P^1$, and finally multiplying by $\varepsilon\sigma_1^2(H-T)e^{-2\rho_{P_1}(H)}$ (as we have in effect already done) and integrating over ${}_{\varepsilon}\mathfrak{a}_1^1$ we obtain $B^T(\phi)$.

The error is a sum of integrals

$$\int_{{}_{\varepsilon}\mathfrak{a}_1^1} {}_{\varepsilon}\sigma_1^2(H-T) \int_{i\mathfrak{a}_P^1} C(\lambda, H)\,|d\lambda|\,dH \quad ,$$

where $C(\lambda, H)$ is given by (20) with $s_2 \notin \Omega^{\varepsilon^{-1}(P_1)\cap P_1}$ but with $\|\Lambda\| < \delta$. The estimations, which establish incidentally that the integrals defining $B^T(\phi)$ converge, mimic the earlier proof of vanishing.

Let $H = T_1+X+Y$ as before then the same arguments establish that for $\|X\| \geq c\|H_0\|$ we have

$$\|C(\lambda, \ H)\| \leq c_1 e^{-c\|X\|} \quad .$$

Notice that the ε-form of Lemma 7.3 implies that $\|Y\| \leq c\|X\|$ if ${}_{\varepsilon}\sigma_1^2(X+Y) \neq 0$ (c is a highly variable constant). On the other hand if $d(T) > C(1+\|H_0\|)$ and $\|X\| \leq c\|H_0\|$ with $C \gg c$ then

$$C(\lambda, \ H) \leq c_2 e^{-c\|T\|} \ .$$

Since

$$\int_{\{H=X+Y \mid \|X\| \geq c\|H_0\|\}} {}_{\varepsilon}\sigma_1^2(X+Y)\, e^{-c\|X\|} \leq c_3 e^{-c\|H_0\|}$$

the asserted estimates follow easily. (The reader will have observed that the arguments are often sketchy. This is partly because they will ultimately be included in the notes of the earlier lectures.)

To show that

$$A^T(\phi_H) - B^T(\phi_H) \sim 0$$

even when Φ_j is not a cusp form we have to use techniques from the second Duke Journal paper. There will not be time to discuss this paper; so we merely sketch the argument envisaged, referring for a careful exposition to the revised lecture notes.

An Eisenstein series on the group P_1 associated to P may be built up with residues of Eisenstein series associated to cusp forms on groups Q contained in P. Recall that taking a residue involves nothing more than a contour integration over a small cycle surrounding the point at which the residue is wanted. These cycles lie in $\mathfrak{a}_Q^P \otimes \mathbf{C}$ and the parameter which is important for the transition from $A^T(\phi)$ to $B^T(\phi)$ was $\lambda_1 \in \mathfrak{a}_1$. So they do not interfere with each other.

Hence we are able, in imitation of Lemma 3.1 of the Duke Journal paper, to show that all operators commute with the formation of residues, thereby deducing the general statements from those for cusp forms. For example, this is certainly so of the integrations over $N_1 \backslash \mathbf{N}_1$ and $M_1 \backslash \mathbf{M}_1^1 \times K$ and of the summation over $M_1 \cap \varepsilon^{-1}(P_1) \backslash M_1$ that appeared in the treatment of $A^T(\phi)$. So we will obtain formulas like (18) but by no means so simple. Nonetheless these terms whose apparent singularities prevent us from deforming the contour back to $\operatorname{Re} \lambda_1 = -\Lambda$, $(\Lambda, \alpha) > 0$, $\|\Lambda\| < \delta$ can still be shown by the previous arguments to be zero. So we can deform the contour and then restore these terms and estimate the error introduced as before.

<u>More loose strands</u>. To obtain the fine χ-expansion for the twisted case we imitate the arguments in the Amer. Jour. papers but, once again for lack of time, we can only sketch the modifications.

We know that $J_\chi^T(\phi)$ is a polynomial (presumably for $d(T) \geq c(1+\|H\|)$ if ϕ is replaced by ϕ_H). It is given by

$$(21)\quad \sum_P \frac{1}{(2\pi)^{a_P}} \sum_\sigma \sum{}'_{P\subset P_1 \subset P_2} \frac{(-1)^{a_Q^\varepsilon}}{n_1(P)} \int_{\varepsilon\mathfrak{a}_1^1} \varepsilon\sigma_1^2(X-T) e^{-2\rho_{P_1}(X)} \int_{\mathrm{Re}\,\lambda=0} \Psi_\sigma^T(X,\lambda,\phi)\, d\lambda\, dX$$

plus an error term, $E^T(\phi)$. The error term satisfies

$$E^T(\phi_H) \sim 0 \quad .$$

The sum over P, σ, which is effectively finite provided ϕ is K-finite runs over $P \supseteq P_0$ and distinguished σ for which $\mathfrak{a}_{\chi,\sigma}(P) \neq 0$. The sum over $P_1 \subset P_2$ runs over pairs of standard parabolics which are separated by a unique ε-invariant standard parabolic Q. The expression $\Psi_\sigma^T(X, \lambda, \phi)$ which implicitly depends on P, is equal to

$$\sum_{P'} \sum_s \sum_j \int_{M_1\backslash \mathbf{M}^1 \times K} \Lambda^{T,P_1} E_{P_1}(mak, \rho_{\sigma,\lambda}(\phi)\Phi_j) \overline{E}_{P_1}(mak, M_{P'|\varepsilon^{-1}(P)}(s,\overline{\mu})\Psi_j)\, dm\, dk \quad .$$

If we let $P^T(H)$ be the polynomial in T which equals $J_\chi^T(\phi_H)$ for $d(T) > c(1+\|H\|)$ and if we let $\psi^T(H)$ be the value of (21) when $\phi = \phi_H$ we still have

$$P^T(H) - \psi^T(H) \sim 0$$

and

$$\psi^T(H) = \sum_\Gamma \psi_\Gamma^T(H) e^{X_\Gamma(H)} .$$

Thus Prop. 5.1 of the first Amer. Jour. paper allows us to write

$$P^T(H) - \sum_\Gamma P_\Gamma^T(H) e^{X_\Gamma(H)} ,$$

the $P_\Gamma^T(H)$ being polynomials in T, and various estimates obtaining.

At this point we can take over the argument of Amer. Jour. I almost literally. It allows us first of all to consider not (21) itself, but (21) with $\Psi_\sigma^T(X, \lambda, \phi)$ replaced by

$$\Psi_\sigma^T(X, \lambda, \phi) B_\sigma(\lambda) ,$$

where B is a Weyl group invariant function on the Schwartz space of $i\mathfrak{h}$. Then §7 of the paper allows us to replace $\Psi_\sigma^T(X, \lambda, \phi)$ by a much more convenient expression, namely $e^{2\rho_{P_1}(X)}$ times

$$(22) \quad \sum_{P'} \sum_{s,t} \frac{e^{(s\lambda+t\mu)(T^1+X)}}{\theta_{P'}^1(s\lambda+t\mu)} \operatorname{tr}(\varepsilon M_{P'|\varepsilon^{-1}(P)}(t, \varepsilon^{-1}(\lambda))^{-1} M_{P'|P}(s,\lambda)\rho_{\sigma,\lambda}(\phi)) .$$

Here P' runs over standard parabolic subgroups of P_1 associate to P and $s \in \Omega^{P_1}(\mathfrak{a}_P, \mathfrak{a}_{P'})$. On the other hand t runs over all elements of $\Omega^Q(\mathfrak{a}_{\varepsilon^{-1}(P),P'})$ which can be expressed as a product $t_1 t_2$ with

$t_1 \in \Omega^{P_1}(\mathfrak{a}_{P''}, \mathfrak{a}_{P'})$, $t_2 \in \Omega^{\varepsilon^{-1}(P_1)}(\mathfrak{a}_{\varepsilon^{-1}(P)}, \mathfrak{a}_{P''})$.

After making these two substitutions we obtain a function of T which depends on B. All we need do is to find a polynomial $P^T(B)$ to which it is approximately equal for $d(T) > \delta\|T\| \gg 0$ and to see what happens to $P^T(B)$ as $B \longrightarrow 1$ in an appropriate sense. The primary purpose of this lecture was to deal with the first step, mimicking the second Amer. Jour. paper. We need only consider compactly supported B.

Symmetrization. In (21), with the substitutions indicated, we may sum over all pairs $P_1 \subset P_2$ provided we remove $(-1)^{a_Q^\varepsilon}$ from before the integral and insert

$$\sum_{P_1 \subset Q \subset P_2} (-1)^{a_Q^\varepsilon}$$

after it, the sum on Q now being taken over all ε-invariant parabolics between P_1 and P_2.

There are also a number of simple modifications of (22) to be made. First of all

$$\varepsilon M_{P'|\varepsilon^{-1}(P)}(t, \varepsilon^{-1}(\lambda))^{-1} M_{P'|P}(s,\lambda) = M_{\varepsilon(P')|P}(\varepsilon(t),\lambda)^{-1} M_{\varepsilon(P')|\varepsilon(P)}(\varepsilon(s),\varepsilon(\lambda))\varepsilon \quad ,$$

and by the functional equation the right side is equal to

$$M_{\varepsilon(t^{-1}(P'))|P}(1,\lambda)^{-1}\varepsilon(t)^{-1}\varepsilon(t) M_{\varepsilon(t^{-1}(P'))|\varepsilon(P)}(\varepsilon(t^{-1}s),\varepsilon(\lambda))$$

times

$$e^{\left\langle -\lambda+\rho_{\varepsilon(t^{-1}(P'))}+\varepsilon t^{-1}s\lambda-\rho_{\varepsilon(t^{-1}(P'))},T_0-\varepsilon(t^{-1})T_0\right\rangle}\varepsilon \ .$$

So we change the notation, letting $\varepsilon(t^{-1}(P'))$ become P', $\varepsilon(t)$ become t, and $\varepsilon(t^{-1}s)$ become s, thereby simplifying the product to

$$M_{P'|P}(1,\ \lambda)^{-1}M_{P'|\varepsilon(P)}(s,\ \varepsilon(\lambda))e^{\left\langle s\varepsilon\lambda-\lambda,T_0-t^{-1}T_0\right\rangle} \ .$$

The new s lies in $\Omega(\mathfrak{a}_{P'},\ \mathfrak{a}_{\varepsilon(P)})$ and the sole condition on it is that it be expressible as a product

$$s = t^{-1}s_1$$

with $s_1 \in \Omega^{\varepsilon(P_1)}(\mathfrak{a}_{\varepsilon(P)},\ \mathfrak{a}_{t(P')})$. Observe also that t is determined by the condition that t applied to the new P' be standard.

With the new notation the denominator is replaced by

$$\theta_{P'}^{t^{-1}(\varepsilon(P_1))}(s\varepsilon\lambda-\lambda) \ .$$

Moreover we can combine the two exponential factors that appear in the numerator to obtain

$$e^{\left\langle s\varepsilon\lambda-\lambda,t^{-1}\varepsilon(X-T_1)\right\rangle+\left\langle s\varepsilon\lambda-\lambda,Y_{P'}(T)\right\rangle}$$

where

$$Y_{P'}(T) = t^{-1}(\varepsilon(T)) + T_0 - t^{-1}T_0 \ .$$

We are thus concerned with

$$(23)\quad \sum_{P} \frac{1}{(2\pi)^{a_P}} \sum_{\sigma} \sum_{P_1} \int_{\varepsilon \mathfrak{a}_1^1} \sum_{P_1 \subset Q \subset P_2} (-1)^{a_Q^{\varepsilon}} {}_{\varepsilon}\sigma_1^2(X-T) \frac{1}{n_1(P)} \int_{\mathrm{Re}\,\lambda=0} \Omega_{\sigma}^{T}(X,\lambda)\, d\lambda\, dX$$

where $\Omega_{\sigma}^{T}(X, \lambda)$ is the sum over the indicated P' and s of

$$(24)\quad \frac{e^{\langle s\varepsilon\lambda-\lambda,\, Y_{P'}(T)+t^{-1}\varepsilon(X-T_1)\rangle}}{\theta_{P'}^{\varepsilon(P_1)}(s\varepsilon\lambda-\lambda)} \operatorname{tr}(M_{P'|P}(1,\lambda)^{-1} M_{P'|\varepsilon(P)}(s,\varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi))B_{\sigma}(\lambda) \quad .$$

Recall that $n_1(P)$ is the number of parabolic subgroups of P_1 with a given Levi factor in common with P. The next step is to replace $n_1(P)$ by $n(P)$, defined in the same way but with G replacing P_1.

Suppose $r \in \Omega(\mathfrak{a}_{P''}, P)$. We replace in (23) and (24) the variable λ by $r\lambda$ and σ by $r\sigma$. The expression (24) becomes the product of

$$(25)\qquad \frac{e^{\langle r^{-1}s\varepsilon(r)\varepsilon\lambda-\lambda,\ Y_{P'''}(T) - T_0 + r^{-1}(T_0) + r^{-1}t^{-1}\varepsilon(X-T_1)\rangle}}{\theta_{P'''}^{r^{-1}t^{-1}\varepsilon(P_1)}(r^{-1}s\varepsilon(r)\varepsilon\lambda-\lambda)} \quad ,$$

with $P''' = r^{-1}(P')$, and

$$(26)\qquad \operatorname{tr} M_{P'|P}(1,\ r\lambda)^{-1} M_{P'|\varepsilon(P)}(s,\ \varepsilon(r\lambda))\varepsilon\rho_{r\sigma,r\lambda}(\phi))B_{r\sigma}(r\lambda) \quad .$$

Since B is invariant under the Weyl group,

$$B_{r\sigma}(r\lambda) = B_{\sigma}(\lambda) \quad .$$

Moreover the functional equations allow us to rewrite

$$M_{P'|P}(1,\ r\lambda)^{-1}M_{P'|\varepsilon(P)}(s,\ \varepsilon(r\lambda))$$

as the product of

$$M_{P|P''}(r,\lambda)M_{P'|P''}(r,\lambda)^{-1}M_{P'|\varepsilon(P'')}(s\varepsilon(r),\varepsilon(\lambda))M_{\varepsilon(P)|\varepsilon(P''')}(\varepsilon(r),\varepsilon(\lambda))^{-1} \quad .$$

We also have

$$M_{\varepsilon(P)|\varepsilon(P'')}(\varepsilon(r),\ \varepsilon(\lambda))^{-1}\varepsilon = \varepsilon M_{P|P''}(r,\lambda)^{-1}$$

and

$$M_{P|P''}(r,\ \lambda)^{-1}\rho_{r\sigma,r\lambda}(\phi) = \rho_{\sigma,\lambda}(\phi)M_{P|P''}(r,\ \lambda)^{-1} \quad .$$

Thus (26) is equal to

$$\mathrm{tr}(M_{P'|P''}(r,\ \lambda)^{-1}M_{P'|\varepsilon(P'')}(s\varepsilon(r),\ \varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi))B_{\sigma}(\lambda) \quad .$$

Making use of the functional equations as before we see that this is equal to the product of

$$\mathrm{tr}(M_{r^{-1}(P')|P''}(1,\ \lambda)^{-1}M_{r^{-1}(P')|\varepsilon(P'')}(r^{-1}s\varepsilon(r),\ \varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi))B_{\sigma}(\lambda)$$

and

$$e^{\langle r^{-1}s\varepsilon(r)\varepsilon(\lambda)-\lambda,T_0-r^{-1}(T_0)\rangle} \quad .$$

Notice that this exponential cancels part of the numerator of (25).

Now we have to try to simplify the results. The sum in (23) is over all standard parabolics P, P_1 containing P. The invariance with respect to r just established allows us, for a given P_1, to replace P by a fixed element in its associate class provided we replace $n_1(P)$ by $|\Omega^{P_1}(\mathfrak{a}_P, \mathfrak{a}_P)|$. However we can then use the invariance once again to sum over all P'' and all r provided we replace $n_1(P)$ by $n(P)$. So we change the notation, denoting P'' by P, P''' by P', tr by t, and $r^{-1}s\varepsilon(r)$ by s.

We obtain

$$\sum_P \sum_s \frac{1}{(2\pi)^{a_P} n(P)} \sum_\sigma \sum_{P_1} \int_{\varepsilon\mathfrak{a}_1^1} \sum_{P_1 \subset Q \subset P_2} (-1)^{a_Q^\varepsilon} {}_\varepsilon\sigma_1^2(X - Y_1(T)) \int_{i\mathfrak{a}_P} \sum_{P'} \omega_{P'}^T(X, \lambda)$$

where $\omega_{P'}^T(X, \lambda)$ is equal to

$$\frac{e^{\langle s\varepsilon\lambda - \lambda, Y_{P'}(T) + (X - Y_1(T)) \rangle}}{\theta_{P'}^{P_1}(s\varepsilon\lambda - \lambda)} \operatorname{tr}(M_{P'|P}(1, \lambda)^{-1} M_{P'|\varepsilon(P)}(s, \varepsilon(\lambda)) \varepsilon\rho_{\sigma,\lambda}(\phi)) B_\sigma(\lambda) \quad .$$

There are further changes in the notation to explain. The group formerly labelled $r^{-1}t^{-1}\varepsilon(P_1)$ is now labelled P_1, and the condition on P_1 is that it contain P'. Moreover P_2 is the former $r^{-1}t^{-1}\varepsilon(P_2)$. The new t is determined by the condition that $t(P')$, and thus $t(P_1)$ and $t(P_2)$ be standard. The function ${}_\varepsilon\sigma_1^2$ is then defined by transport of structure and $Y_1(T)$ is the projection of $t^{-1}T$ on $\mathfrak{a}_1$. So are the groups Q, but they can clearly be defined intrinsically. If $(L, \mathfrak{a})$ is the ε-special pair attached to s then the groups Q

are the elements of $\mathcal{F}_\varepsilon(L, \mathfrak{a})$ which contain P_1.

We can unburden ourselves of P_2 if we recall from Lecture 9 that

$$\sum_{\{Q,P_2|P_1 \subset Q \subset P_2\}} (-1)^{a_Q^\varepsilon} {}_\varepsilon\sigma_1^2 = \sum_{P_1 \subset Q} (-1)^{a_Q^\varepsilon} \tau_1^Q {}_\varepsilon\hat{\tau}_Q \quad .$$

The sum over P, s, and σ can now be forgotten as can $\frac{1}{(2\pi)^{a_P}} \cdot \frac{1}{n(P)}$ for they appear in the statement of the fine χ-expansion, the transition from P to M being effected as in the second Amer. Jour. paper.

Thus we are reduced to considering

$$(27) \quad \sum_{P_1} \int_{\varepsilon\mathfrak{a}_1^1} \sum_{Q \supset P_1} (-1)^{a_Q^\varepsilon} \tau_1^Q(X-Y_1(T))\, {}_\varepsilon\hat{\tau}_Q(X-Y_1(T)) \int_{i\mathfrak{a}_P} \sum_{P'} \omega_{P'}^T(X, \lambda) \quad ,$$

in which we have still to be precise about which P' and P_1 occur.

Choose the unique standard P'' and the unique $t \in \Omega(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$ such that $t(P') = P''$. If we review the calculations that led to this point, we see that

$$ts\varepsilon(t^{-1}) = s_1 s_2 \varepsilon(s_1^{-1})$$

with $s_1 \in \Omega^{t(P_1)}(\mathfrak{a}_{P''}, {}_{t(P')})$, $s_2 \in \Omega^{t(P_1)}(\mathfrak{a}_{P''}, \mathfrak{a}_{\varepsilon(P'')})$, where $P'' \in \mathcal{F}(M_0)$. We see indeed that the necessary and sufficient condition that P' and P_1 occur is that $ts\varepsilon(t^{-1})$ have this form. In particular if one P' occurs then, as we should expect, all parabolics associate to it in P_1 occur. Thus for a given P_1 either no $P' \subseteq P_1$ occur or

P' runs over $\mathcal{P}^{M_1}(M_P)$.

Combinatorics. We take P_1 with $\mathfrak{a}_{P_1} \subseteq \mathfrak{a}_P$ and consider

(28) $\int_{\varepsilon\mathfrak{a}_1^1} \sum_{\{Q \in \mathcal{F}_\varepsilon(L,\mathfrak{a}) | Q \supseteq P_1\}} (-1)^{a_Q^\varepsilon} \tau_1^Q(X-Y_1(T))_\varepsilon \hat{\tau}_Q(X-Y_1(T)) \int_{i\mathfrak{a}_P} \sum_{P' \in \mathcal{P}^{P_1}(M_P)} \omega_{P'}^T(X,\lambda$

without asking whether it actually occurs in (27). We shall see that if it does not occur then it approaches 0 as T approaches ∞, and therefore may be added to the sum (27), in whose behavior we are interested only for large T.

Let $(L, \mathfrak{a})$ be the ε-special pair determined by s. We introduce new coordinates on $i\mathfrak{a}_P$, replacing λ by the pair (ν, Λ), with ν the projection of λ on $i\mathfrak{a}$ and $\Lambda = s\varepsilon\lambda - \lambda$. It is this change of coordinates which introduces the factor $\frac{1}{\Delta}$ into the statement of the fine χ-expansion.

We may write $Y_{P'}(T) + X - Y_1(T)$ as $X + Y_{P'}^1(T)$, where $\{Y_{P'}^1(T) | P' \in \mathcal{P}^1(M_P)\}$ is an $A_{M_P}^1$-orthogonal family. $Y_{P'}^1(T)$ is given by

$$(r^{-1}T)^1 + T_0 - r^{-1}T_0 \quad ,$$

where $(r^{-1}T)^1$ is the projection of $r^{-1}T$ on $\mathfrak{a}_0^1$, r being any element of the Weyl group such that $r(P^1)$ is standard.

The inner integral in (28) is equal to

$$(29) \qquad \int e^{\langle \Lambda, X \rangle} \operatorname{tr}(M_M(\lambda) M_{P|\varepsilon(P)}(s, \varepsilon(\lambda)) \varepsilon\rho_{\sigma,\lambda}(\phi)) B_\sigma(\lambda) \, |d\Lambda d\lambda|$$

with

$$\mathcal{M}_M(\lambda) = \sum_{P'} \frac{M_{P'|P}(1,\lambda)^{-1} M_{P'|P}(1,\lambda+\Lambda)}{\theta^1_{P'}(\Lambda)} e^{\langle \Lambda, Y^1_{P'}(T) \rangle}$$

and $M = M_P$. We have used the identity

$$M_{P'|P}(1,\lambda)^{-1} M_{P'|\varepsilon(P)}(s,\varepsilon(\lambda)) = M_{P'|P}(1,\lambda)^{-1} M_{P'|P}(1,s\varepsilon(\lambda)) M_{P|\varepsilon(P)}(s,\varepsilon(\lambda)) \quad .$$

We denote the value of (28) by $f(X)$, the function f being defined on $\mathfrak{a}_1$ or on $\mathfrak{a}_1/\mathfrak{a} \cap \mathfrak{a}_1$ because Λ is orthogonal to $\mathfrak{a}$. We remark that if D is any differential operator with constant coefficients on $i\mathfrak{a}_1$ then

$$(30) \qquad |D(\mathrm{tr}(\mathcal{M}_M(\lambda) M_{P|\varepsilon(P)}(s,\varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi)) B_\sigma(\lambda))| \leq c_D(1+\|T\|)^{d_0}$$

where d_0 is independent of D. I omit the verification, which is easy with the help of the expansion $\sum c^S_M d^1_{\mathfrak{s}}$ to be developed below. The essential observation is that $\langle \Lambda, Y^1_{P'}(T) \rangle = 0$ for $\Lambda \in i\mathfrak{a}_1$.

We deduce from (30) that if $X = U+V, V \in i\mathfrak{a}$, U orthogonal to $i\mathfrak{a}$ then for any n we have

$$(31) \qquad |f(X)| \leq c_n(1+\|U\|)^{-n}(1+\|T\|)^{d_0} \quad .$$

LEMMA 6. <u>The expression</u> (28) <u>goes to</u> 0 <u>as</u> T <u>approaches</u> ∞ <u>unless</u> $\mathfrak{a}_1$ <u>is invariant under</u> $s\varepsilon$ <u>and</u> M_1 <u>is the centralizer of</u> $\mathfrak{a}_1^{s\varepsilon}$.

To prove this lemma it is useful to write

$$\sum_{\substack{Q \supset P_1 \\ Q \in \mathfrak{F}_\varepsilon(L, \mathfrak{a})}} (-1)^{a_Q^\varepsilon} \tau_1^Q(X-Y_1(T))\, {}_\varepsilon\hat{\tau}_Q(X-Y_1(T))$$

once again as

$$\sum_{P_2 \supset Q \supset P_1}' (-1)^{a_Q^\varepsilon} {}_\varepsilon\sigma_1^2(X-Y_1(T))$$

the sum running over those P_2 such that there is exactly one element of $\mathfrak{F}_\varepsilon(L, \mathfrak{a})$ between P_1 and P_2.

It is clear that we may reduce ourselves to the case that P_1 is standard. We shall show that if P_1 is standard and (28) does not approach 0 as T approaches ∞ then P_1 is ε-invariant and $s \in \Omega^{P_1}$. A consequence will be that if (28) does not approach 0 then it actually occurs in (27).

It will approach 0 if there is a positive constant c such that

$$\|U\| \geq c\|V\| \ , \ \|U\| \geq c\|T\| \tag{32}$$

whenever ${}_\varepsilon\sigma_1^2(X-Y_1(T)) \neq 0$ for some P_2 such that there is a unique ε-invariant parabolic between P_1 and P_2.

Recall that $d(T) \geq c\|T\|$. Consequently $\|T_1\| \geq c\|T\|$, $T_1 = Y_1(T)$ being the projection of T on $\mathfrak{a}_1$ (unless $P_1 = G$, the trivial case).

If (32) doesn't hold then we can find a sequence $\{(X_n, T_n)\}$ such that

$$\frac{U_n}{\|U_n\|+\|V_n\|+\|T_n\|} \longrightarrow 0 \ .$$

Taking the limit we find a non-zero pair (X, T) with $X \in \mathfrak{a} \cap \mathfrak{a}_1$ and with

$$\alpha(X-T_1) \geq 0, \ \alpha \in \Delta_1^2 , \quad \alpha(X-T_1) \leq 0 \quad \alpha \in \Delta_1 - \Delta_1^2 ,$$

$$_\varepsilon\varpi_\alpha(X-T_1) \geq 0 , \quad \alpha \in {_\varepsilon\Delta_0} - {_\varepsilon\Delta_0^Q} .$$

Let $X = X_1^Q + X_Q$, $X_Q \in \mathfrak{a}_Q$, $X_1^Q \in \mathfrak{a}_1^Q$, $T_1 = T_1^Q + T_Q$. Since $X_1^Q \in \mathfrak{a}_1^Q \cap \mathfrak{a}$ and since Q is the smallest ε-invariant parabolic containing P_1 we conclude that $X_1^Q = 0$. Then $X_1^Q \in \mathfrak{a}^{Q+}$

$$\alpha(-T_1^Q) \geq 0 , \qquad \alpha \in \Delta_1^Q .$$

If $P_1 \neq Q$, so that Δ_1^Q is not empty, and if $\alpha \in \Delta_1^Q$ then

$$\alpha(T_1^Q) = \alpha(T_1) \geq \alpha(T) \geq c\|T\| .$$

We deduce that $T = 0$ if $P_1 \neq Q$. On the other hand the proof of Lemma 7.3 of Lecture 7 shows that

$$\|X_1^Q - T_1^Q\| \geq c\|X_Q - T_Q\| , \qquad c > 0 .$$

Thus $X_1^Q = T = 0$ implies that $X_Q = X = 0$. We conclude that $P_1 = Q$.

The upshort is that in (27) we are free to sum over all P_1 with $\mathfrak{a}_{P_1} \subset \mathfrak{a}_P$ or only over those P_1 such $M_{P_1} = M_Q$ for some $Q \in \mathcal{F}_\varepsilon(L, \mathfrak{a})$. We take the former, larger set.

We set

Replace $d_{p_1}(\lambda, \Lambda)$, λ, Λ varying freely

by $d_{p_1}(\lambda, \Lambda) A(\Lambda)$, $A(\Lambda)$ compact support

Since $\Lambda = s_\varepsilon\lambda - \lambda$, $B_\sigma(\lambda) \neq 0$, Λ is in effect is

compactly supported

$$c^1_{P'}(\Lambda) = e^{\langle \Lambda, Y^1_{P'}(T) \rangle}, \qquad \Lambda \in \mathfrak{a}_{P'}$$

But can multiply with fn of compact support $A(\Lambda)$

and we set

$\Lambda = s\varepsilon\lambda - \lambda \Rightarrow$ compact support. No!

$$d_{P'}(\lambda,\Lambda) = \mathrm{tr}\{M_{P'|P}(1,\lambda)^{-1} M_{P'|P}(1,\lambda+\Lambda) M_{P|\varepsilon(P)}(s,\varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}'(\phi)\}B_\sigma(\lambda) \quad \hat{A}(\Lambda)\,.$$

Moreover for convenience in the following discussion we denote the variable X appearing in (29) by H_1. Then by Lemma 6.3 of the Annals paper the expression (29) is equal to

$$\int e^{\langle \Lambda, H_1 \rangle} \sum_{S \in \mathcal{F}^1(M)} c^S_M(\Lambda)\, d_S^{P_1}(\lambda, \Lambda)\, d\Lambda\, d\nu \quad .$$

Recall that if $S \in \mathcal{F}^1(M)$ then we attach to S the point $Y^1_S(T)$, obtained by projecting any $Y^1_{P'}(T)$ on $\mathfrak{a}_S$, $P' \subset S$, and the collection $\mathcal{Y}^S_M$

$$\mathcal{Y}^S_M = \{Y^1_{S(R)}(T) - Y^1_S(T) \,|\, R \subset S,\ R \in \mathcal{P}^1(M)\} \quad .$$

~~Each point of $\mathcal{Y}^S_M$ projects to $Y^1_S(T)$.~~ If $\Gamma^S_M(\cdot, \mathcal{Y}^S_M)$ is the characteristic function of the convex hull of the points in $\mathcal{Y}^S_M$ then

$$c^S_M(\Lambda) = e^{\langle \Lambda, Y^1_S(T) \rangle} \int_{\mathfrak{a}^S_M} e^{\Lambda(H^S_M)}\, \Gamma^S_M(H^S_M, \mathcal{Y}^S_M)\, dH^S_M \quad .$$

The notation for the function $d_S^{P_1}$ is not good. For example d_S^G is the function formerly denoted $d_S^!$. In any case for each fixed λ we express $d_{P'}(\lambda, \Lambda)$ as a Fourier transform

$$d_{P'}(\lambda, \Lambda) = \int_{i\mathfrak{a}_M} \hat{d}_{P'}(\lambda, U) e^{\langle \Lambda, U \rangle} dU \quad .$$

Then, as was observed in Lecture 13,

$$d_S^{P_1}(\lambda, \Lambda) = \int_{i\mathfrak{a}_M} dU \int_{\mathfrak{a}_S^1} dH_S^1 \, \hat{d}_{P'}(\lambda, U) e^{\langle \Lambda, H_S^1 \rangle + \langle \Lambda, U_1 \rangle} \Gamma_S^1(H_S^1, U_S^1)$$

if $P' \subseteq S$. Since we are dealing with a (G, M) family this may also be written

$$\int_{\mathfrak{a}_S} dU_S \int_{\mathfrak{a}_S^1} dH_S^1 \hat{d}_S(\lambda, U_S) e^{\langle \Lambda, H_S^1 \rangle + \langle \Lambda, U_1 \rangle} \Gamma_S^1(H_S^1, U_S^1)$$

with

$$\hat{d}_S(\lambda, U_S) = \int_{\mathfrak{a}_M^S} \hat{d}_{P'}(\lambda, U_S + V) dV \, , \qquad P' \subseteq S \quad .$$

Putting this all together we see that (28) is equal to

$$(33) \int_{\varepsilon\mathfrak{a}_M^1} \sum_S \int_{\mathfrak{a}_S} \sum_{Q \supset P_1 \supset S} (-1)^{a_Q^\varepsilon} {}_\varepsilon\hat{\tau}_Q(H_1 - Y_1(T) - U_1) \tau_{P_1}^Q(H_1 - Y_1(T) - U_1) \Gamma_S^1(H_S^1 - Y_S^1(T), U_S^1) \Gamma_M^S(H_M^S, Y_M^S) \phi_S(H, U_S) \, dH \quad ,$$

where $H = H_1 + H_S^1 + H_M^S$ and the inner sum is over Q. The function $\phi_S(H, U_S)$ is given by

$$\phi_S(H, U_S) = \int d\Lambda d\nu \, e^{\langle \Lambda, H \rangle} \hat{d}_S(\lambda, U_S) \quad .$$

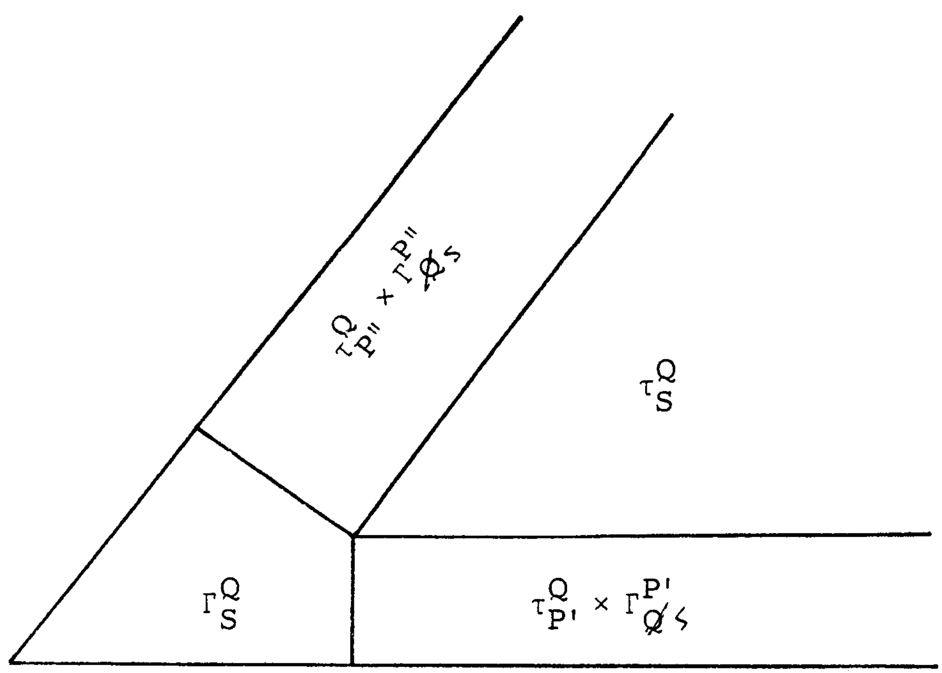
Recall that $\lambda = \lambda(\Lambda, \nu)$, where $\Lambda = s\varepsilon\lambda - \lambda$ and ν is invariant under $s\varepsilon$.

In (33) the space ${}_{\varepsilon}\mathfrak{a}_M^1$ is the set of all X in $\mathfrak{a}_M$ such that $d\chi(X) = 0$ for all ε-invariant rational characters of G (The notation is not ambiguous, but almost so, for there is a danger of confounding ${}_{\varepsilon}\mathfrak{a}_M^1$ with $\mathfrak{a}_M^1$). The point is that the domain of integration is independent of P_1, so that we can take the sum of (27) under the integral sign obtaining (33) again but with the summation extending not only over Q and S but also over P_1.

So we can simplify it, because

$$\sum_{P_1} \tau_{P_1}^Q(H_1 - Y_1(T) - U_1)\Gamma_S^1(H_S^1 - Y_S^1(T), U_S^1) = \tau_S^Q(H_S - Y_S(T)) \quad .$$

This formula is implicit in §2 of the Annals paper, and corresponds to the following diagram:



Thus the sum over P_1 of the integrals (33) becomes the sum over S of

$$(34)\quad \int_{\mathfrak{a}_S} \int_{{}_\varepsilon\mathfrak{n}_M^1} \sum_{Q\supset S} (-1)^{a_Q^\varepsilon}\, {}_\varepsilon\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q)\tau_S^Q(H_S^Q - Y_S^Q(T))\Gamma_M^S(H_M^S, Y_M^S)\phi_S(H, U_S)\, dH \ .$$

LEMMA 7. <u>The integral</u> (34) <u>converges</u>. <u>It approaches</u> 0 <u>as</u> T <u>approaches infinity unless</u> $\mathfrak{a}_S \subseteq \mathfrak{a}_L$.

For the purposes of this lemma it is best not to work with (34) but to return to (33), removing the sum over S and replacing

$$\sum_Q (-1)^{a_Q^\varepsilon}\, {}_\varepsilon\hat{\tau}_Q(H_1 - Y_1(T) - U_1)(\tau_{P_1}^Q(H_1 - Y_1(T) - U_1)$$

by

$$\sum_{\mathcal{P}_\nu}{}' \; {}_\varepsilon\sigma_1^2(H_1 - Y_1(T) - U_1) \ .$$

The function $\hat{d}_S$ is a Schwartz function of Λ, ν, U. Thus ϕ_S is a Schwartz function on $\mathfrak{a}_M/\mathfrak{n} \times \mathfrak{a}_S$. For convergence we need to show that if

$$H_S = X + V$$

with $V \in \mathfrak{n}$, X orthogonal to $\mathfrak{a}$ then there is a positive constant c such that

$$\|X\| + \|U_S\| \geq c\, \|V\|$$

on the support of the integrand, T being for the moment fixed. If this were not so then the usual argument shows the existence of a non-zero $V \in \mathfrak{n} \cap \mathfrak{n}_1$ which takes in the closure of the support of ${}_{\varepsilon}\sigma_1^2$. This contradicts the proof of Lemma 7.3 of Lecture 7.

We use a similar argument to show that the integral approaches 0 if $\mathfrak{n}_S$ is not contained in $\mathfrak{n}_L$. For this we write

$$H = H_M = X+V \quad ,$$

$V \in \mathfrak{n}$, X orthogonal to $\mathfrak{n}$ and show that for some $c > 0$

$$\|X\| + \|U_S\| \geq c\|V\| \ , \quad \|X\| + \|U_S\| \geq c\|T\|$$

on the support of the integrand. Because of Lemma 6, or rather because of its proof, we can simplify the situation somewhat. We can suppose that $P_1 = Q$ is ε-invariant and standard and that s acts trivially on $\mathfrak{n}_Q$. Thus $\mathfrak{n}_L \supset \mathfrak{n}_Q$, $\mathfrak{n} \supseteq \mathfrak{n}_Q^{\varepsilon}$. We may also suppose that S is standard.

If we cannot find the constant c then we can find a non-zero pair (V, T) such that V-T is in the closure of the support of ${}_{\varepsilon}\sigma_1^2$ and such that

$$(35) \qquad \begin{aligned} &\Gamma_S^1(V_S^1-T_S^1), 0) \neq 0 \\ &\Gamma_M^S(V_M^S, \bar{y}_M^S) \neq 0 \quad , \end{aligned}$$

where $\bar{y}_M^S$ is defined like y_M^S but with $T_0 = 0$. Moreover $V \in \mathfrak{n}$. Thus $V^1 \in \mathfrak{n}$, for $P^1 = Q$ is invariant under s and ε. However the first of the inequalities (35) implies that

$$Y_S^1 = T_S^1 \quad . \tag{36}$$

Since $d(T) \geq c\|T\|$ we have $T_S^1 = 0$ only if $T = 0$. If $T = 0$ the second inequality implies that $V_M^S = 0$. However the proof of Lemma 7.3 shows that $V_1 - T_1 = 0$. Thus T cannot be 0 without V being 0. We conclude that $T_S^1 \neq 0$. The inequalities (35) actually imply that V is in the convex hull of the collection

$$\{r^{-1}(T_M) \mid r \in \Omega^S(M)\} \quad .$$

If α is a root of S which does not vanish on $\mathfrak{a}_S$ then there is a positive constant c such that

$$\alpha(r^{-1}(T_M)) \geq c\|T\|$$

for all $r \in \Omega^S(M)$. We conclude that $\alpha(V) \geq c\|T\| \neq 0$. However if $\mathfrak{a}_S$ is not contained in $\mathfrak{a}_L$ we can find a root α which vanishes on $\mathfrak{a}_L$ and thus on V but not on $\mathfrak{a}_S$: Changing its sign if necessary we obtain a root in S and then a contradiction.

We continue to work with the modified (33) assuming that $\mathfrak{a}_S \subseteq \mathfrak{a}_L$. We show that we may substitute $\Gamma_L^S(H_L^S, y_L^S)$ for $\Gamma_M^S(H_M^S, y_M^S)$. For lack of time we simply quote two lemmas from the second Amer. Jour. paer. The first states that

$$\Gamma_L^S(H_L^S, Y_L^S) - \Gamma_M^S(H_M^S, Y_M^S) \geq 0 \quad .$$

The second states that where the difference is not 0,

$$\|H_M^S - H_L^S\| \geq c\|T\| \quad ,$$

c being as usual a positive constant. Since $H_M^S - H_L^S$ is the projection of X on $\mathfrak{a}_M^L$ the conclusion follows readily. See diagram at end of lecture.

Final combinatorics. We are reduced to considering

$$(37) \quad \int_{\varepsilon \mathfrak{a}_M^1} \sum_{\mathfrak{a}_S \subseteq \mathfrak{a}_L} \int_{\mathfrak{a}_S} \sum_{Q \supset S} (-1)^{a_Q^\varepsilon} \, {}_\varepsilon\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q)\tau_S^Q(H_S^Q - Y_S^Q(T))\Gamma_L^S(H_L^S, Y_L^S)\phi_S(H, U_S) \quad .$$

We first treat

$$(38) \quad \sum_{\mathfrak{a}_S \subseteq \mathfrak{a}_L} \int_{\mathfrak{a}_S} \sum_{Q \supset S} (-1)^{a_Q^\varepsilon} \, {}_\varepsilon\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q)\tau_S^Q(H_S^Q - Y_S^Q(T))\Gamma_L^S(H_L^S, Y_L^S)\phi_S(H, U_S) \quad .$$

We may interchange the order of summation and integration. We write $U_S = U_Q^{s\varepsilon} + V$ with $U_Q^{s\varepsilon}$ in $\mathfrak{a}_Q^{s\varepsilon}$ and V orthogonal to $\mathfrak{a}_Q^{s\varepsilon}$ and integrate first with respect to V.

$$(39) \quad \int_{s\varepsilon \mathfrak{a}_S^Q} \phi_S(H, U_Q^\varepsilon + V)\, dV = \int d\Lambda d\nu \, e^{\langle \Lambda, H \rangle} \int_{s\varepsilon \mathfrak{a}_S^Q} \hat{d}_S(\lambda, U_Q^\varepsilon + V)\, dV$$

where ${}_{s\varepsilon}\mathfrak{a}_S^Q$ is the orthogonal complement of $\mathfrak{a}_Q^{s\varepsilon}$ in $\mathfrak{a}_S$. If $P' \in P^S(M)$ the right side is equal to

$$\int d\Lambda d\nu \; e^{\langle \Lambda, H\rangle} \int_{s\varepsilon \mathfrak{a}_M^Q} \hat{d}_{P'}(\lambda,\; U_Q^{\varepsilon}+V)\,dV \;.$$

Impossible notation! $s\varepsilon\mathfrak{a}_M^Q = s\varepsilon\mathfrak{a}_S^Q + \mathfrak{a}_M^S$

However

$$\int_{s\varepsilon \mathfrak{a}_M^Q} \hat{d}_{P'}(\lambda, U_Q^{\varepsilon}+V) = \int_{s\varepsilon \mathfrak{a}_Q^Q} dV_1 \int_{\mathfrak{a}_M^Q} dV_2 \;\; \hat{d}_{P'}(\lambda, U_Q^{\varepsilon}+V_1+V_2)$$

and

$$\int_{\mathfrak{a}_M^Q} \hat{d}_{P'}(\lambda, U_Q+V_2)\,dV_2$$

is the Fourier transform of the restriction of $d_{P'}$ to $\mathfrak{a}_Q$, and therefore the same for all $P' \subseteq Q$. Thus (39) is equal to

$$\int d\Lambda d\nu \; e^{\langle \Lambda, H\rangle} \int_{s\varepsilon \mathfrak{a}_Q^Q} \hat{d}_Q(\lambda, U_Q^{\varepsilon}+V)\,dV \quad ,$$

and, in particular, is independent of S.

Set

$$\psi_Q(H, U_Q) = \int d\Lambda d\nu \; e^{\langle \Lambda, H\rangle} \hat{d}_Q(\lambda, U_Q) \quad .$$

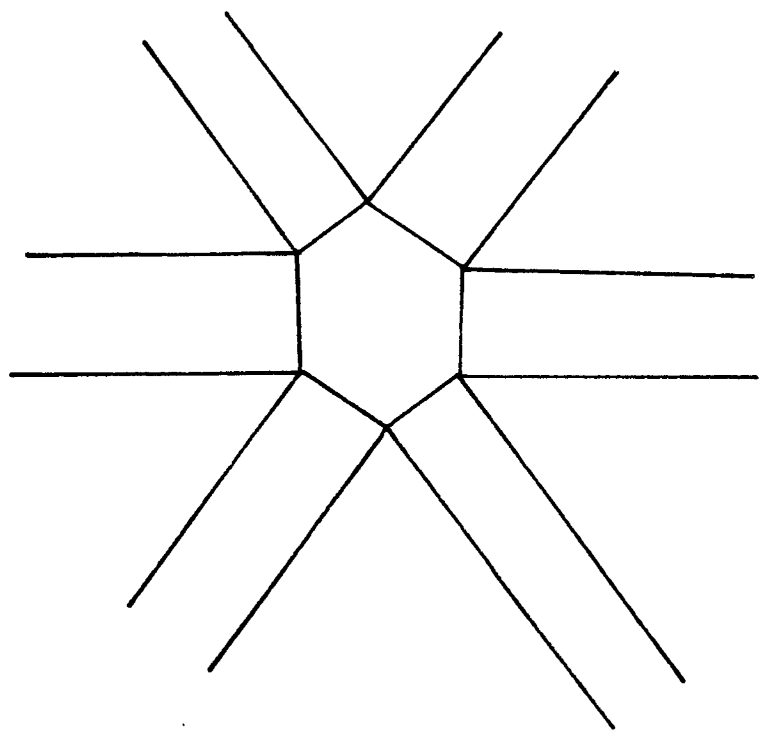
The sum (38) is equal to

$$(40) \qquad \sum_Q (-1)^{a_Q^{\varepsilon}} \int_{\mathfrak{a}_Q} {}_{\varepsilon}\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q)\psi_Q(H, U_Q)$$

because

$$\sum_{S \in \mathcal{P}^Q(L)} \tau_S^Q(H_S^Q - Y_S^Q(T))\Gamma_L^S(H_L^S, y_L^S) \equiv 1 \quad .$$

This identity, which I do not prove formally, is another form of Lemma 6.3 of the Annals paper and corresponds to the following diagram:



The outer integral in (37) can be taken first over ${}_{s\varepsilon}\mathfrak{a}_M^M$ and then over $\mathfrak{a}_M^{s\varepsilon} = \mathfrak{a}$. To integrate (40) over ${}_{s\varepsilon}\mathfrak{a}_M^M$ we have to take

$$\int_{{}_{s\varepsilon}\mathfrak{a}_M^M} \psi_Q(H, U_Q) \quad .$$

Since ${}_{s\varepsilon}\mathfrak{a}_M^M$ is the domain in which $-i\Lambda$ varies this integration yields

$$\int_{i\mathfrak{a}} \hat{d}_Q(\lambda, U_Q)\, d\lambda \quad .$$

We next integrate this over ${}_{s\varepsilon}\mathfrak{a}_Q^Q$ taking

$$\int_{{}_{s\varepsilon}\mathfrak{a}_Q^Q} dV \int_{i\mathfrak{a}} \hat{d}_Q(\lambda, U_Q^{s\varepsilon}+V)\, d\lambda$$

and observing that

$$\int_{{}_{s\varepsilon}\mathfrak{a}_Q^Q} \hat{d}_Q(\lambda, U_Q^{s\varepsilon}+V)\, dV = \hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon}) \quad ,$$

where we now regard Q as an element of $\mathcal{F}_\varepsilon(L, \mathfrak{a})$ and where $\bar{d}_Q$ is defined by the $(G, L, \mathfrak{a})$-family attached to $\{d_{P'}\}$.

Summing up the results so far, we see that (37) is equal to

$$\int_{\mathfrak{a}} \left\{ \sum_Q \int_{\mathfrak{a}_Q^{s\varepsilon}} (-1)^{a_Q^\varepsilon}\, {}_\varepsilon\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q^{s\varepsilon}) \left\{ \int_{i\mathfrak{a}} \hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon})\, d\lambda \right\} dU_Q^{s\varepsilon} \right\} dH \quad .$$

(The notational difficulties and inconsistencies are growing more and more severe.) Let L' run over the Levi factors of groups in $\mathcal{F}_\varepsilon(L, \mathfrak{a})$.

Using the results of §2 of the Annals paper we expand

$$(-1)^{a_Q^\varepsilon}\, {}_\varepsilon\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q^{s\varepsilon})$$

as

$$\sum_{R \supset Q} (-1)^{a_Q^\varepsilon - a_R^\varepsilon}\, {}_\varepsilon\hat{\tau}_Q^R(H_Q - Y_Q(T))\, {}_\varepsilon\Gamma_R^G(H_Q - Y_Q(T), U_Q^{s\varepsilon}) \quad ,$$

it being understood that the sum is over $R \in \mathcal{F}_\varepsilon(L, \mathfrak{a})$.

We postpone consideration of

$$\int_{\mathfrak{a}_Q^{s\varepsilon}} {}_{\varepsilon}\Gamma_R^G(H_Q - Y_Q(T), U_Q^{s\varepsilon})\,\hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon})\,dU_Q^{s\varepsilon}$$

observing for the moment only that it depends on R alone and not on Q because

$$\int \hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon} + V)\,dV$$

does, the integration being taken over the orthogonal complement of $\mathfrak{a}_R^{s\varepsilon}$ on $\mathfrak{a}_Q^{s\varepsilon}$.

This observation allows us to sum

$$\sum_{Q \subseteq R} (-1)^{a_Q^{\varepsilon} - a_R^{\varepsilon}}\, {}_{\varepsilon}\hat{\tau}_Q^R(H_Q - Y_Q(T))$$

obtaining (cf. Lemma 5.3.5 of Lecture 5)

$${}_{\varepsilon}\Gamma_L^R(H_L^R, {}_{\varepsilon}y_L^R) \quad .$$

Since this is a function with compact support we see that (37) is equal to the integral over $i\mathfrak{a}$ of the sum over R of the product of

$$\int_{\mathfrak{a}_L^R} {}_{\varepsilon}\Gamma_L^R(H_L^R, {}_{\varepsilon}y_L^R) = \bar{c}_L^R(T)$$

Poor notation

and

$$\int_{\mathfrak{a}_R^G} \int_{\mathfrak{a}_Q^{s\varepsilon}} {}_{\varepsilon}\Gamma_R^G(H_Q - Y_Q(T), U_Q^{s\varepsilon})\,\hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon})\,dU_Q^{s\varepsilon}\,dH_Q$$

which equals

$$\bar{d}_R^G(\lambda) \quad .$$

Since Λ is not equal to 0 the multiplication by A(Λ) has no effect.

We obtain finally

$$\int_{i\mathfrak{a}} \sum_R \bar{c}_L^R(T)\bar{d}_R^G(\lambda)\,d\lambda \quad ,$$

a polynomial in T. Using Lemma 6.3 of the Annals paper to collapse the sum and examining the definitions we see that this is equal to

$$\int_{i\mathfrak{a}} \operatorname{tr}({}_{\varepsilon}\mathcal{M}_L(P,\lambda)M_{P|\varepsilon(P)}(s,\varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi))B_{\sigma}(\lambda)\,|d\lambda| \quad .$$

All that is left is to rid ourselves of the $B_{\sigma}(\lambda)$ and for this we must consider the normalization of intertwining operators.

Normalization of intertwining operators. In the second Amer. Jour. paper Arthur assumes a normalization of the local intertwining operators with the properties to be stated below. I now want to point out, with minimal explanation, that such a normalization can be easily deduced from results already in the literature.

References 1. J. Arthur, On the invariant distributions attached to weighted orbital integrals (preprint).

2. K. F. Lai, Tamagawa number of reductive algebraic groups, Comp. Math.

3. A. Silberger (i) Introduction to harmonic analysis on reductive p-adic groups P.U.P. (ii) Special representations of reductive p-adic groups are not integrable, Annals. (iii) On Harish-Chandra μ-functions for p-adic groups, Transactions.

I observe that we do not need the formula of Th. 1.6 of paper [3.ii], which, as Shahidi has pointed out to me, is not correct. The source of error is perhaps the assertion preceding Lemma 1.2.

If G is a group over a local field with standard parabolic P_0 then for any $P \supseteq M_0$ and any unitary representation σ of $M = M_P$ we can introduce as usual the induced representations $\rho_{\sigma,\lambda}$ on the space $\mathfrak{A}_\sigma(P)$. Here λ lies in the complex dual of $\mathfrak{a}_M$. We also introduce the intertwining operators $M_{Q|P}(1,\lambda)$, $M_{Q|P}(\sigma,\lambda) = M_{Q|P}(1,\sigma,\lambda)$ which send $\phi \in \mathfrak{A}_\sigma(P)$ to $\phi' \in \mathfrak{A}_\sigma(Q)$ with

$$\phi'(g) = \int_{N_Q \cap N_P \backslash N_P} \phi(ng) e^{(\lambda+\rho_P)(H_P(ng))-(\lambda+\rho_Q)(H_Q(g))} dn \quad .$$

The global operators are tensor products of these local operators.

We need decompositions

$$M_{Q|P}(\sigma,\lambda) = n_{Q|P}(\sigma,\lambda) N_{Q|P}(\sigma,\lambda) \quad ,$$

where $n_{Q|P}(\sigma,\lambda)$ is a scalar and both functions on the right are meromorphic in λ, $\lambda \in \mathfrak{a}_M^* \otimes \mathbf{C}$. The following conditions are to be satisfied.

(i) $$N_{R|P}(\sigma,\lambda) = N_{R|Q}(\sigma,\lambda) N_{Q|P}(\sigma,\lambda)$$

(ii) $$N_{Q|P}(\sigma,\lambda)^* = N_{Q|P}(\sigma,-\bar{\lambda})$$

(iii) $$(N_{S(R')|S(R)}(\sigma,\lambda)\phi)_k = N_{R'|R}\phi_k \quad .$$

(iv) If σ and G are unramified and ϕ is fixed by a hyperspecial K, then

$$N_{Q|P}(\sigma,\lambda)\,\phi$$

is independent of λ.

(v) If σ is tempered then $n_{Q|P}(\sigma,\lambda)$ has neither zeros nor poles in the positive chamber attached to P.

(vi) If the local field is non-archimedean then $N_{Q|P}(\sigma,\lambda)$ is a rational function of $\{q^{-\alpha(\lambda)} \mid \alpha \in \Delta_P\}$.

In the paper [1] Arthur has established the existence of such a normalization for real groups. Much of his argument is also applicable to p-adic groups and shows that it is enough to verify the existence of $n_{Q|P}$ and $N_{Q|P}$ when σ is tempered, P is maximal, and $Q = \overline{P}$ is opposite to P.

In this case, by [3]

$$M_{\overline{P}|P}(\sigma,-\overline{\lambda})^{*} M_{\overline{P}|P}(\sigma,\lambda) = c\,\mu(\sigma;\lambda)$$

where c is a positive constant and μ is the function appearing in Harish-Chandra's Plancherel formula. Again by [3], this function is a rational function of $z = q^{-\alpha(\lambda)}$ (α is now the unique simple root)

$$\mu(\sigma,\lambda) = U(\sigma,z) \quad .$$

All we need do is to decompose $U(\sigma,z)$ as

$$U(\sigma, z) = V_P(\sigma, z)\overline{V}_P(\sigma, \overline{z}^{-1})$$

where $V_P(\sigma, z)$ is a rational function with neither zero pole in $|z| < 1$ and with $V_P(\sigma, z) = \overline{V}_{\overline{P}}(\sigma, \overline{z})$, for we then set

$$n_{\overline{P}|P}(\sigma, \lambda) = \sqrt{c}\, V_P(\sigma, q^{-\alpha(\lambda)})$$

and

$$N_{\overline{P}|P}(\sigma, \lambda) = \frac{M_{\overline{P}|P}(\sigma, \lambda)}{n_{\overline{P}|P}(\sigma, \lambda)} \quad .$$

Since

$$M^*_{\overline{P}|P}(\sigma, \lambda) = M_{P|\overline{P}}(\sigma, -\overline{\lambda})$$

and

$$\overline{n}_{\overline{P}|P}(\sigma, \lambda) = n_{P|\overline{P}}(\sigma, -\overline{\lambda})$$

the condition (ii) is fulfilled. Observe that $\overline{\alpha} = -\alpha$, that is, replacing by $\overline{P}$ entails replacing α by $-\alpha$.

To verify (i) we need only check that

$$N_{P|\overline{P}}(\sigma, \lambda) N_{\overline{P}|P}(\sigma, \lambda) = 1 \quad .$$

By (ii) the left side is

$$N_{\overline{P}|P}(\sigma, -\overline{\lambda})^* N_{\overline{P}|P}(\sigma, \lambda)$$

which equals

$$\frac{c\,\mu(\sigma,\lambda)}{\overline{V}_p(\sigma,\overline{z}^{-1})V_p(\sigma,z)} = 1 \quad .$$

That $V_p(\sigma,z)$ can be so chosen that (iv) is satisfied follows from the calculations in [2].

To prove the existence of $V_p(\sigma,z)$ I use an argument of Shahidi. It exploits the following two properties of $U(\sigma,z)$, both consequences of the fact that $U(\sigma,z)$ is real and positive for $|z| = 1$:

(i) $U(\sigma,z) = \overline{U}(\sigma,\overline{z}^{-1})$.

(ii) Any zero of $U(\sigma,z)$ on $|z| = 1$ is of even multiplicity.

It follows from (i) that if α is a root of $U(\sigma,z) = 0$ then $\overline{\alpha}^{-1}$ is also. The same assertion is valid for poles. Thus we may write

$$U(\sigma,z) = a\,\frac{\prod_{i=1}^{r}(1-\alpha_i z)(1-\overline{\alpha}_i^{-1}z)}{\prod_{i=1}^{r}(1-\beta_i z)(1-\overline{\beta}_i^{-1}z)}$$

where $|\alpha_i| \leq 1$, $|\beta_i| \leq 1$, $1 \leq i \leq r$, and

$$a\,\frac{\prod \alpha_i}{\prod \beta_i} > 0 \quad .$$

We let

$$b\overline{b}\,\frac{\prod \overline{\alpha}_i}{\prod \overline{\beta}_i} = a$$

and set

$$V_P(\sigma, z) = b \frac{\prod_{i=1}^{r} (1-\alpha_i z)}{\prod_{i=1}^{r} (1-\beta_i z)} \quad .$$

Then

$$\bar{V}_P(\sigma, \bar{z}^{-1}) = \frac{\prod \bar{\alpha}_i}{\prod \bar{\beta}_i} \frac{\prod_{i=1}^{r} (1-\bar{\alpha}_i^{-1} z)}{\prod_{i=1}^{r} (1-\bar{\beta}_i^{-1} z)}$$

and

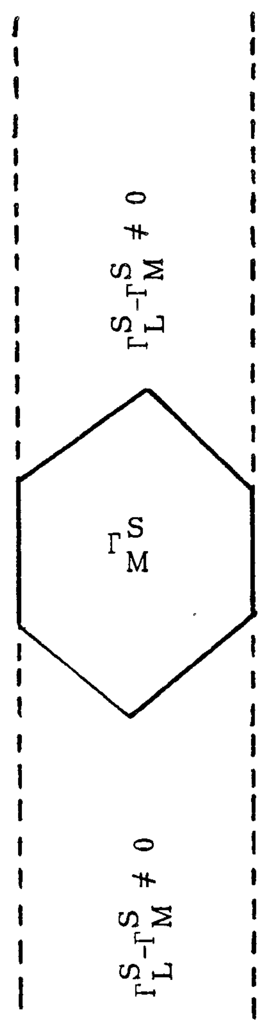
$$U(\sigma, z) = V_P(\sigma, z)\bar{V}_P(\sigma, \bar{z}^{-1}) \quad .$$

If we replace P by $\bar{P}$ then $\mu(\sigma, \lambda)$ is not changed but z is replaced by z^{-1} and $U(\sigma, z)$ by $U(\sigma, z^{-1})$, which equals

$$\bar{a} \frac{\prod_{i=1}^{r} (1-\bar{\alpha}_i z)(1-\alpha_i^{-1} z)}{\prod_{i=1}^{r} (1-\bar{\beta}_i z)(1-\beta_i^{-1} z)} \quad .$$

So we may take

$$V_{\bar{P}}(\sigma, z) = \bar{b} \frac{\prod_{i=1}^{r} (1-\bar{\alpha}_i z)}{\prod_{i=1}^{r} (1-\bar{\beta}_i z)} \quad .$$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