On the global correspondence between GL(n) and division algebras

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1a. Let D be a division algebra of degree n^2 over a global field F of characteristic zero. We suppose that for each place v of F, D_v is $M(n,F_v)$ or a division algebra. We will use the comparison between the trace formula on $GL(n)$ and D^x and local results to get the global correspondence between automorphic representations of $GL(n)$ and D^x.

The case $n = 2$ is already known (JL or GJ). We suppose $n > 2$. Then at infinity D_∞ is $M(n,F_\infty)$. At a finite place v, Zelevinski (Z) introduced a duality in the Grothendieck group $K(GL(n,F_v))$ of the equivalence classes of the representations of finite length of $GL(n,F_v)$. This duality generalizes the duality introduced by Alvis and Curtis for finite groups, and exchanges the class of the Steinberg representation with the trivial one.

We denote by A the adele ring of F. Recall that an irreducible subrepresentation of $L^2(G_F\backslash G_A,\omega)$ for some central character ω is called a discrete automorphic representation of G_A. We denote by S the set of finite places v of F where D_v is a field. Let π_A be an equivalence class of irreducible representations of $GL(n)_A$, such that for every $v \in S$, π_v is square integrable or the dual of a square integrable representation. By the local correspondence (BDKV), we associate to π_A an equivalence class π'_A of irreducible representation of D^x_A:

$$\pi'_v = \pi_v \qquad \text{if } v \notin S$$

- π_v' is such that the characters of π_v' and π_v on the regular elliptic conjugacy classes satisfy

$$\chi_{\pi_v'} = \varepsilon(\pi_v)\chi_{\pi_v} \qquad \text{where } \varepsilon(\pi_v) \in \{\pm 1\} \quad .$$

We will prove the following theorem:

1b. THEOREM: The map $\pi_A \longrightarrow \pi_A'$ induces a bijection from the set of automorphic discrete representations of $GL(n)_A$ such that for every $v \in S$, π_v is square integrable or the dual of a square integrable representation onto the set of automorphic representations of D_A^x.

With the natural definition of duality at infinity for $n = 2$, this theorem includes the theorem of Jacquet-Langlands. We restrict ourselves to the case where D_v is $M(n,F_v)$ or a division algebra because of our ignorance of the residual spectrum for $GL(n)$. In §2, we collect some results on local representations of $GL(n)$. We determine the irreducible representations of $GL(n,F_v)$ whose characters do not vanish on the set G_v^{ell} of regular elliptic conjugacy classes, and we prove that the square-integrable representations and their dual are the only ones which are unitarizable. This last result is a sharpening of a theorem of Casselman (BW). We will use these local results to prove the theorem in §3.

2a. We suppose F local, non-archimedean, of characteristic zero. We let $G = GL(n,F)$ and $E(G)$ be the set of equivalence classes of irreducible representations of G. We denote by $E^2(G)$, $E^2(G)^t$, $E^0(G)$ the subsets given by the quasi-square-integrable, dual of quasi-square-integrable, quasi-cuspidal representations respectively. Recall that a quasi-square-integrable representation is the product of a square-integrable one by a power of ν, where $\nu(g) = |\det g|$, $g \in G$.

Let us recall the classification of $E^2(G)$ given in (Z). Let $X^2(G)$ be the set of (m,ρ) where $m|n$ and $\rho \in E^0(GL(d,F))$ if $md = n$. The unitarily induced representation

$$\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho = i_{P_d}^G(\rho \otimes \nu\rho \otimes \cdots \otimes \nu^{m-1}\rho)$$

where P_d is the standard parabolic whose Levi factor is isomorphic to $GL(d,F)^m$, has a unique irreducible quotient. This quotient denoted by $St_m(\rho)$ is quasi-square-integrable. Every quasi-square-integrable irreducible representation of G is equivalent to a unique $St_m(\rho)$. The representation $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$ has a unique submodule. It is the dual $St_m(\rho)^t$ of $St_m(\rho)$. In this classification the Steinberg representation is

$$St_n\left(\nu - \frac{n-1}{2}\right) .$$

2b. <u>THEOREM</u>:

(1) The representations $St_m(\rho)$ and $St_m(\rho)^t$ are unitarizable if and only if their central character is unitary.

(2) No other subquotient of $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$ is unitarizable.

The part (1) is known: it is clear for $St_m(\rho)$ and is proved in (B) for $St_m(\rho)^t$ which is a "segment" in the classification of (Z). The part (2) generalizes a theorem of Casselman, which corresponds to $m = n$. Our proof given in 2d follows closely the proof of this theorem given in (BW,X1,§4, p. 340-343).

2c. Let us recall the description of the Jordan-Hölder composition series J of $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$ given in (Z,§2,p. 176-180), generalizing (BW,X,4.6 and 4.2). We know that it is combinatorial, and depends only on m. We set:

$$\delta = \nu^{\frac{m-1}{2}} \otimes \cdots \otimes \nu^{-\frac{m-1}{2}} .$$

The functor $i = i_{P_d}^G$ of unitary induction is related to the ordinary induction functor $I = I_{P_d}^G$ by the relation

$$i = I\delta .$$

Their left-adjoints r, R verify

$$r = \delta^{-1} R .$$

Let Σ be the standard set of roots of $GL(m)$, Δ the subset of simple positive roots, and W the Weyl group. Given a subset I of Δ, we set

$$W(I) = \{w \in W \; , \; w(\alpha) > 0 \; , \; \forall\alpha \in I \; , \; w(\alpha) < 0 \; , \; \forall\alpha \in \Delta - I\}$$

W acts naturally by permutation on $GL(d)^m$ and by "transport de structure" on the representations of $GL(d)^m$. It is easy to deduce from (Z):

<u>PROPOSITION</u>: J has a composition series whose successive quotients are the irreducible representations π_I such that:

$$R(\pi_I) = \bigoplus_{w \in W(I)} w(\rho \otimes \nu\rho \otimes \cdots \otimes \nu^{m-1}\rho) \cdot \delta$$

each occurring with multiplicity one.

If $I = \phi$, $\pi_\phi = St_m(\rho)$ and if $I = \Delta$, $\pi_\Delta = St_m(\rho)^t$. When $m = n$ and $\rho = \nu^{-(n-1)/2}$, then

$$R(\pi_I) = \bigoplus_{w \in W(I)} w(\delta^{-1}) \cdot \delta .$$

2d. Proof of (2). Suppose $I \neq \phi, \Delta$ and that the central character of π_I is unitary. Then the central character of $w(\rho \otimes \nu\rho \otimes \cdots \otimes \nu^{m-1}\rho) \cdot \delta$ verifies

$$|\chi_w| = w(\delta^{-1}) \cdot \delta .$$

Let W^1 be the longest element of the Weyl group of $\Delta - I$. Then $w^1 \in W(I)$. There is a canonical isomorphism of the center S of $GL(d,F)^m$ to the diagonal group of $GL(m,F)$, then a natural action of Σ on S. The character $|\chi_{w^1}|$ acts trivially on the set of elements

$$C = \{c \in S \ , \ |c^\alpha| \leq 1 \ , \text{ if } \alpha \in I \ , \ |c^\alpha| = 1 \text{ if } \alpha \in \Delta - I\} .$$

This set is unbounded modulo the center Z of G.

Recall a theorem of Casselman: if $v \in \pi_I$ and $\tilde{v} \in \tilde{\pi}_I$ the contragredient of π_I, and $a \in A^-(\varepsilon)$ where

$$A^-(\varepsilon) = \{a \in S \ , \ |a^\alpha| \leq \varepsilon \ , \ \forall\alpha \in \Delta\} \qquad \varepsilon > 0 \text{ small enough}$$

we have

$$\langle \pi_I(a)v \, , \tilde{v} \rangle = \langle R(\pi_I)(a)u \, , \tilde{u} \rangle$$

if $u, \tilde{u}$ are the canonical images of $v, \tilde{v}$ respectively in $R(\pi_I), R(\tilde{\pi}_I)$.

Let us choose $u, \tilde{u}$ such that $\langle u, \tilde{u}\rangle \neq 0$ in $W^1(\rho \otimes \cdots \otimes \nu^{m-1}\rho)\cdot\delta$ and its contragredient. Let $v, \tilde{v}$ which map onto $u, \tilde{u}$ under the canonical projections. For $a \in A^-(\varepsilon)$, $Ca \subset A^-(\varepsilon)$ and

$$(*) \qquad |\langle \pi_I(ca)v, \tilde{v}\rangle| = |\chi_{w^1}(a)|\, \langle u, \tilde{u}\rangle \quad .$$

There exist a unitary character ν^{ix}, $x \in \mathbb{R}$, of $G = GL(n,F)$ such that $\pi_I \nu^{ix}$ is trivial on a subgroup $Z' \subset Z$, with G/Z', with compact center Z/Z'. We can apply to π_I the Howe theorem: if π_I is unitarizable then the coefficients of π_I vanish at infinity. It follows from (*) since C is unbounded modulo the center that π_I is not unitarizable. Then π_I is not unitarizable.

2e. We determine now the irreducible representations π of G whose characters χ_π do not vanish on the set G^{ell} of elliptic regular conjugacy classes.

We know (Z) that the products (unitary induction) of quasi-square-integrable representations form a **Z**-basis of $K(G)$. Denote by $[\pi]$ the image of π in $K(G)$. For every $\pi \in E(G)$, we have:

$$[\pi] = \Sigma\, n(\pi, \pi_1 \times \cdots \times \pi_r)\, [\pi \times \cdots \times \pi_r]$$

where $\pi_i \in E^2(GL(n_i,F))$, $\Sigma\, n_i = n$. The sum is finite, contains at most one $St_m(\rho)$, and

$$n(St_m(\rho)^t,\ St_m(\rho)) = (-1)^{m-1} \quad .$$

We know (BDKV) that the restriction to G^{ell} of the characters of $E^2(G)$ form a complete orthonormal system. Moreover, for every $\pi \in H^2(G)$ there exists $\phi_\pi \in H(G)$ in the Hecke algebra $H(G)$, called a pseudo-coefficient of π such that

$$\langle \pi , \phi_\pi \rangle = 1$$

$$\langle \pi \ , \phi_\pi \rangle = 0$$

if $\pi = St_m(\rho)$ and π is not a subquotient of $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$, $\pi \ \in E(G)$.

We deduce from this the following:

PROPOSITION:

(1) $\chi_\pi = 0$ on G^{ell} if π is not a subquotient of some $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$ and $\chi_\pi = n(\pi , St_m(\rho))\chi_{St_m(\rho)}$ otherwise

(2) The square-integrable-irreducible representations and their duals are the only irreducible unitary representations whose character do not vanish on G^{ell}.

3a. We suppose now F global of characteristic zero. Denote by S a finite set of non-archimedean places of F. We set

$$G_S = \prod_{v \in S} G_v \quad , \quad G_A = G_S G^S \quad .$$

By convention $X_S = (X_v)_{v \in S}$ satisfies (P) if and only if each component X_v satisfies (P). We deduce from 2e the following corollary

COROLLARY: Let $\pi_A = \pi_S \times \pi^S$ be an automorphic representation of $GL(n)_A$. The character of π_S does not vanish on G_S^{ell} if and only if

$$- \pi_S \in E^2(G_S) \qquad \text{if } \pi_A \text{ is cuspidal}$$

$$- \pi_S \in E^2(G_S)^t \qquad \text{if } \pi_A \text{ is not cuspidal} \quad .$$

Proof: From 2e (2) we know that each component π_v of π_S belongs to $E^2(G_v)$ or $E^2(G_v)^t$. If one of them is square integrable, then π_A is cuspidal (this seems to be well known and was indicated to me by Jacquet, it results from the characterization of square integrable representation by the exponents from Jacquet functors, and the computation of the constant terms by Harish-Chandra). It follows that π_v is not degenerated (Sh) at all non-achimedean places v of F. Therefore, for $v \in S$, π_v is square-integrable, because the elements of $E^2(G_v)^t$ are degenerate.

3b. PROPOSITION: A cuspidal automoprhic representation of $GL(n)_A$ and a non-cuspidal one do not have the same G^S-component.

Proof: If they had, their L-function would be equal. This is incompatible with the existence of a pole for an L-function $L(s, \pi_A \times \sigma_A)$ for σ_A cuspidal of $GL(m)_A$, $m < n$ when π_A is automorphic for $GL(n)_A$ is not cuspidal (J.Sh).

3c. We now proceed to the proof of the global correspondence. Let D be as in §1, and S be the set of places v of F where D_v is a division algebra. The comparison of the trace formulas on GL(n) and D^x made by Langlands (L) gives:

(1) trace $\rho(f) = \text{trace } \rho_d(f)$

for all $f = \pi f_v$, $f = \pi f_v$ associated to f via orbital integrals:

- $f^s = f^s \in H(GLn)^s)$

- The orbital integrals of f_s on regular elements are zero outside of G_s^{ell}, and equal to the orbital integrals of f_s on $D_s^{\times ell}$ naturally isomorphic to G_s^{ell}.

We use the notations of (BDKV) that we quickly recall: a central characger ω is fixed, ρ is the regular representation of $GL(n)_A$ in $L^2(GL(n,F)\backslash GL(n,A),\omega)$, ρ_d its discrete part, ρ the one for $D_A^{\times}$.

Using the standard simplification argument (JL) we write (1) in the equivalent form: for all $\pi^s \in E(GL(n)^s)$.

(2) $\Sigma\, n(\pi_s \otimes \pi^s)$ trace $\pi_s(f_s) = \Sigma\, n(\pi_s \otimes \pi^s)$ trace $\pi_s(f_s)$ where $n(\pi_A)$ is the multiplicity of π_A in ρ_d, and $n(\pi_A)$ the one for ρ.

The following properties are equivalent:

- (2) does not vanish for all f_s
- π^s is the G^s-component of some $\pi_A \subset \rho$
- π^s is the G^s-component of some $\pi_A \subset \rho_d$ such that the character π_s does not vanish on G_s^{ell}.

We suppose that they are satisfied. We deduce from 3a, 3b, the strong multiplicity one theorem for cuspidal representations of $GL(n)_A$, and the local correspondence (1a), that two disjoint possibilities A, B can occur:

A] π^S is the G^S-component of π_A cuspidal. Then π_A is unique, with multiplicity $n(\pi_A) = 1$, π_s is square-integrable. Let $\pi_s^0 \in E(D_s^x)$ associated to π_s by the local correspondence and $\pi_A^0 = \pi_s^0 \otimes \pi^S$. We have for all $f_s \in H(D_s^x)$:

$$\Sigma\, n\,(\pi_s \otimes \pi^S) \text{ trace } \pi_s(f_s) = \text{trace } \pi_s^0(f_s) \quad .$$

By linear independence we deduce that π^S is the G^S-component of a unique automorphic representation of D_A^x, equal to π_A^0, with multiplicity $n(\pi_A^0) = 1$.

B] π^S is the G^S-component of π_A residual. Then π_s is the dual of a square integrable representation. Let $\pi_s^0 \in E(D_s^x)$ associated to π_s by the dual of the local correspondence and $\pi_A^0 = \pi_s^0 \otimes \pi^S$. We have for all $f_s \in H(D_s^x)$:

$$\Sigma\, n\,(\pi_s \otimes \pi^S) \text{ trace } \pi_s(f_s) = \Sigma\, n(\pi_s \otimes \pi^S) \text{ trace } \pi_s^0(f_s)\varepsilon(\pi_s)$$

where $\varepsilon(\pi_s) = \pm 1$.

By linear independence we deduce that the set of automorphic representations of D_A^x with G^S-component π^S is equal to the set of the representations π_A^0, where $\pi_A = \pi_s \otimes \pi^S$ is residual for $GL(n)_A$, with multiplicities $n\,(\pi_A^0) = n(\pi_A)$. Moreover $\varepsilon(\pi_s) = 1$ for all such π_A.

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