Thursday Morning Seminar

DIVISION ALGEBRAS III

R. Langlands

There were three points left unsettled in the previous lecture that I want to deal with here.

The fine σ-expansion. As already suggested it appears best to write the left side of the trace formula as a sum over $\mathcal{L}(M_0)$, the set of Levi subgroups containing M_0. Thus we write

$$\sum_{M\in \mathcal{L}(M_0)} \frac{|\Omega^M|}{|\Omega^G|} J_M^T(\phi) = \sum_M \frac{|\Omega^M|}{|\Omega^G|} J_M^T(\phi) = \sum_M \frac{|\Omega^M|}{|\Omega^G|} J_M(\phi)$$

omitting as convenience suggests both the range of summation and the parameter T.

Arthur expects that $J_M(\phi)$ will be a sum over semi-simple conjugacy classes σ in M,

$$J_M(\phi) = \sum_\sigma J_M(\sigma, \phi) \quad ,$$

the $J_M(\sigma, \phi)$ having a form to be described below. I allow myself to anticipate his results, in spite of the element of uncertainty this introduces. It is intended to present them in the Friday morning seminar.

It will in particular have to be proved that

$$J_M(\gamma, \phi) = |D^G(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(\mathbb{A}_{S'})\backslash G(\mathbb{A}_{S'})} \phi_{S'}(g^{-1}\gamma g) V_M^G(\gamma, g)dg$$

is defined for those γ in $M(\mathbf{A}_{S'})$ such that $1\text{-Ad}(\gamma)$ is invertible on $\mathfrak{g}/\mathfrak{m}$. It will also have to be proved that it extends by continuity to all of $M(\mathbf{A}_S)$.

Fix a semi-simple γ in $\mathfrak{o}$ and let $U_\gamma(M_{S'})$ be the collection of conjugacy classes of unipotent elements in $\mathbf{M}_{S'}$ which commute with γ, or, better, a set of representatives for such classes. Then for S' sufficiently large

$$J_M(\mathfrak{o}, \phi) = c^{S'}(M) \sum_{\delta \in U_\gamma(M_{S'})} c^M_{S'}(\gamma, \delta) J_M(\gamma\delta, \phi) \quad .$$

The constant $c^{S'}(M)$ outside the summation depends only on S', M, and G and is given by

$$c^{S'}(M) = \text{meas}(K^{S'} \cap \mathbf{M}) \backslash K^{S'} \quad .$$

The constant $c^M_{S'}(\gamma, \delta)$ inside the summation depends on S', γ, δ, and M but not in G.

Lemma 1 of the previous lecture remains valid first for γ such that $1\text{-Ad}\ \gamma$ is invertible in $\mathfrak{g}/\mathfrak{m}$, for which it is proved in the same way, and then, by continuity, for all γ.

This makes it natural to modify the notion of a family F^Q_S adapted to f_S

$$(1) \qquad J_M(\gamma, f_S \prod_{v \in S'-S} f_v) = \sum_{\substack{Q \in \mathcal{F}^L(M) \\ Q \neq L}} J_M(\gamma, F^Q_S \prod_{v \in S'-S} f_{Q,v})$$

for all γ. Lemma 2 remains valid, the proof for finite v being that given in the first lecture. The proof for $v = \infty$ will be given below.

This done we can proceed as in the previous lecture to prove the existence of functions ψ^Q such that

$$J_M(\sigma, \phi) = \sum_{\substack{M \subseteq Q \\ Q \neq G}} J_M(\sigma, \psi^Q)$$

for every σ. Then

$$J_M(\phi) = \sum_{\substack{M \subseteq Q \\ Q \neq G}} J_M(\psi^Q)$$

and the measure-theoretic argument can proceed.

Revision of argument. Although I see no reason to doubt its validity I am unable to prove Lemma 2 of the first lecture for $v = \infty$ in the form there stated. I can only prove it when S consists of ∞ alone. This entails some changes in the proof of the existence of the functions ψ^Q.

Notice first that we had fused all the infinite places together into one so that the statement with which the proof began was not strictly correct. All the places (in the usual sense) at which $\Phi(\gamma, \phi_v) \equiv 0$ may be infinite. Then $n = 2m$, m is odd, and $n_1 = n_2 = m$. In particular $\rho(L) = 1$.

Turning to the construction of the f used to define ψ^Q for $Q \in P(M)$, $\rho(M) \geq 2$ we distinguish two cases: (a) $\Phi(\gamma, \phi_\infty)$ is not identically zero for all γ in M_∞ regular in G_∞. (b) It is.

In the first case we may continue to use the argument of the first lecture, noticing that if $M \subset Q$ then the two places at which $\Phi(\gamma, \phi_v) \equiv 0$ for all $\gamma \in M_Q(F_v)$ regular in G_v are both finite. In the second case we use Lemma 6 only to show that the function

$$J_M(\gamma, \phi) - \sum_{\substack{M \subseteq Q \\ Q \neq G \\ Q \notin P(M)}} J_M(\gamma, \psi^Q)$$

is satisfactory away from ∞. (Note that in the definition of satisfactory and in Lemmas 5 and 6 the function $\Phi(\gamma, f)$ should be $|D(\gamma)|^{\frac{1}{2}}\Phi(\gamma, f)$ and $\Phi(\gamma_1, f^{\vee}_{\gamma_2})$ should be $|D(\gamma_1)|^{\frac{1}{2}}\Phi(\gamma_1, f^{\vee}_{\gamma_2})$). To show that it is satisfactory at ∞ we use the argument on pp. 14-18 but with $g_1 \in G(\mathbf{R} \times \mathbf{A}^{S'})$, $g_2 \in G(\mathbf{A}_{S_1})$, $S_1 = S - \{\infty\}$.

<u>A lemma of Arthur</u>. To complete the argument of the first lecture we must therefore prove Lemma 2 when $S = \{\infty\}$. The first step is to reduce ourselves to the case that $S' = S$. For this we need a variant of a lemma of Arthur, but I first give the lemma itself, of which we shall in any case have need.

Recall that the Harish-Chandra homomorphism $z \longrightarrow \Gamma_T(z)$ can be factored through Z_M. $z \longrightarrow \Gamma_M(z) \longrightarrow \Gamma_T(\Gamma_M(z))$, where the Γ_T should really be written Γ_T^M for it refers to the pair T, M. For the next lemma G, M_1 and M' can be any connected reductive groups over $\mathbf{R}$.

LEMMA 1. <u>It is possible to attach to any pair</u> $M \subset M'$, <u>any</u> $T \subseteq M$, <u>any</u> γ <u>in</u> $M(\mathbf{R})$ <u>regular in</u> M', <u>and any</u> $z \in Z_M$ <u>an invariant</u>

<u>differential operator</u> $D_M^{M'}(\gamma, z)$ <u>on</u> T <u>such that</u>

$$J_M^G(\gamma, zf) = \sum_{M \subset M' \subset G} D_M^{M'}(\gamma, \Gamma_{M'}(z)) J_{M'}^G(\gamma, f)$$

<u>for all</u> $z \in Z_G$.

If $M = G$ then $D_M^G(\gamma, z) = \Gamma_T(z)$ is given by the Harish-Chandra homomorphism and the equality is well known. In general we can proceed by induction and it is easy enough to see that we need only show that

$$f \longrightarrow J_M^G(\gamma, zf) - \sum_{M \subset M' \subsetneq G} D_M^{M'}(\gamma, \Gamma_{M'}) J_{M'}^G(\gamma, f)$$

is an invariant distribution, for it is then an elementary consequence of the theory of distributions that it is given by $f \longrightarrow DJ_G^G(\gamma, f)$, because it is obviously concentrated on the orbit of γ. We define $D_M^G(\gamma, z)$ to be this D.

The proof of invariance relies on two simple, and more or less obvious, identities. The first,

$$(zf)_{Q,h} = \Gamma_{M_Q}(z) f_{Q,h}$$

follows easily from the definition on p. 20 of the Annals paper and the definition of the Harish-Chandra homomorphism. The second,

$$\Gamma_M(\Gamma_{M'}(z)) = \Gamma_M(z) \quad ,$$

is almost formal.

This said, we must verify that

$$J_M(\gamma,\ zf^h) - J_M(\gamma,\ zf)$$

is equal to

$$\sum_{M \subset M' \subsetneq G} D_M^{M'}(\gamma,\ \Gamma_{M'}(z))(J_{M'}^G(\gamma,\ f^h) - J_{M'}^G(\gamma,\ f)) \quad .$$

The first difference is equal to

$$\sum_{\substack{Q \in \mathcal{F}(M) \\ Q \neq G}} J_M(\gamma,\ \Gamma_{M_Q}(z) f_{Q,h}) \quad ,$$

which by induction is equal to

$$\sum_{M \subset M' \subset Q \neq G} D_M^{M'}(\gamma,\ \Gamma_{M'}(z)) J_{M'}(\gamma,\ f_{Q,h}) \quad .$$

On the other hand

$$D_M^{M'}(\gamma,\ \Gamma_{M'}(z))(J_{M'}^G(\gamma,\ f^h) - J_{M'}^G(\gamma,\ f))$$

is equal to

$$\sum_{M' \subset Q \neq G} D_M^{M'}(\gamma,\ \Gamma_{M'}(z)) J_{M'}(\gamma,\ f_{Q,h}) \quad .$$

The required equality follows.

Observe that in the Friday afternoon seminar we used a special case of the lemma, that for which $\rho(M) - \rho(G) = 1$, obtaining it by a direct argument.

For the second lemma we consider a function $f = f_S f^S$, f_S being $\prod_{v \in S} f_v$ and f^S being $\prod_{v \in S'-S} f_v$ with f_v spherical for $v \in S'-S$. We observe, for it is the key to the proof of the next lemma, that for a spherical $f = f_v$ the function f_Q depends only on M_Q and thus may be written as $\Gamma_M(f_v)$ if $M = M_Q$. Moreover

$$\Gamma_M(\Gamma_{M'}(f_v)) = \Gamma_M(f_v) \quad .$$

If $h \in G(\mathbf{A}_S)$ then

$$f_{Q,h} = f_{S,Q,h} \Gamma_{M_Q}(f^S) \quad .$$

Thus the next lemma can be proved exactly like Lemma 1.

LEMMA 2. <u>For any</u> $\gamma = (\gamma_1, \gamma_2)$ <u>regular in</u> G <u>there are linear forms</u> $\phi \longrightarrow D_M^{M'}(\gamma, \phi)$ <u>on the Hecke algebra of</u> $M'(\mathbf{A}_S^{S'})$ <u>such that</u>

$$J_M(\gamma, f_S f^S) = \sum_{M \subset M' \subset G} D_M^{M'}(\gamma, \Gamma_{M'}(f^S)) J_{M'}(\gamma_1, f) \quad .$$

Suppose then

$$J_M(\gamma_1, f_S) = \sum_{\substack{Q \in \mathcal{F}^L(M) \\ Q \neq L}} J_M(\gamma_1, F_S^Q)$$

for all $M \subseteq L$. Then by the lemma (with G replaced by L)

$$J_M(\gamma, f_S f^S) = \sum_{M \subset M' \subset L} D_M^{M'}(\gamma, \Gamma_{M'}(f^S)) J_{M'}(\gamma_1, f_S)$$

$$= \sum_{M' \subset Q \neq L} D_M^{M'}(\gamma, \Gamma_{M'}(f^S)) J_{M'}(\gamma_1, F_S^Q)$$

$$= \sum_{M' \subset Q \neq L} D_M^{M'}(\gamma, \Gamma_{M'}(f_Q^S)) J_{M'}(\gamma_1, F_S^Q)$$

$$= \sum_Q J_M(\gamma, F_S^Q f_Q^S) \quad .$$

<u>Proof of Lemma</u> 2 <u>for</u> $S' = S = \{\infty\}$. (Notice that in the statement of that lemma γ is to lie in $L(F_v)$ and be regular in it.) The function f_S is now denoted f and we establish the existence of F^Q by induction on $\rho(Q)$. To warm up we begin with $\rho(Q) - \rho(L) = 1$, the weighting factor being then linear. So Arthur's lemma yields differential equations,

$$(2) \qquad J_M(\gamma, zf) = \Gamma_T(z) J_M(\gamma, f) \quad ,$$

the inhomogeneous term falling away because the orbital integrals of f are zero. Notice that at this point we are working entirely within the group L. From the equations (2) and the estimates of the Inventiones paper, to which we shall return, we deduce that $J_M(\gamma, f)$ defines a piecewise smooth function on $T(\mathbb{R})$.

We need to show that there exists a function F on $M_\infty(\mathbb{R})$, smooth and of compact support, such that

$$J_M(\gamma, f) = J_M(\gamma, F) \quad ,$$

at first for γ regular and semi-simple, and then for all γ. Then we set

$$F^Q = \frac{1}{|P(M)|} F \quad .$$

To do this we first use Shelstad's characterization of orbital integrals to obtain an F in the Schwartz space (Shelstad, Characters and inner forms of a quasi-split group over $\mathbb{R}$, Comp. Math. (1979)), observing that for the groups under consideration orbital integrals are necessarily stable, and then we follow a technique of Clozel (App. to Clozel-Delorme, Le théorème de Paley-Wiener invariant pour les groupes de Lie réductifs) which with the help of this Paley-Wiener theorem replaces F by a compactly supported function.

According to Theorem 11 of Shelstad's paper there are several conditions to verify. The first is formal and trivial to verify. The second, invariance under the Weyl group, is clear from the definition. The others refer to the behavior of $J_M(\gamma, f_\infty)$ near semi-regular elements of T. These are defined by a condition $\alpha(\gamma_0) = 1$ with α real or imaginary.

The conditions for α imaginary are the most difficult to state, but the easiest to verify. The point is that near the orbit of such a γ_0 we can set

$$F(m) = \rho_Q(m) \int_{K_\infty} \int_{N_Q(\mathbb{R})} f_\infty(k^{-1}muk) W_M^G(m, u)\, du\, dk$$

where

$$W_M^G(m, u) = V_M^G(m, n)$$

if $m^{-1}nm = mu$. There is a neighborhood X of the orbit of γ_0 such that $(m, n) \longrightarrow (m, u)$ is a homeomorphism of $X \times N_Q(\mathbf{R})$ with itself. Moreover we can so choose X that $V_M^G(m, n)$ is smooth on $X \times N_Q(\mathbf{R})$. Finally

$$J_M(\gamma, f_\infty) = J_M(\gamma, F)$$

and is thus equal to the orbital integral of a smooth compactly supported function near γ_0. So the conditions at imaginary roots are obvious, and need not even be stated. Indeed the conditions at all roots in M are, for the same reason, clearly satisfied.

The condition at a semi-regular element defined by a real root is smoothness. Continuity is already being taken for granted. Just as in the lecture Cancellation of singularities at the real places it is sufficient to verify that $H_\alpha J_M(\gamma, f_\infty)$ does not jump at γ_0, H_α being defined by

$$\left\langle H_\alpha, H' \right\rangle = \frac{2\alpha(H')}{\langle \alpha, \alpha \rangle} \quad ,$$

for all $H' \in \mathfrak{h}$, and α now being a real root not in M.

This is a consequence of the results in Arthur's paper The characters of discrete series as orbital integrals, Inv. (1976). We need only sort out the notation. First of all, writing $f_\infty = f$ and $L = L_\infty$

$$\left\langle R(\zeta, H : Y : 1), f \right\rangle = \varepsilon |D^L(\gamma)|^{\frac{1}{2}} \int_{\gamma(\mathbf{R}) \backslash (\mathbf{R})} f(g^{-1}\gamma g) v_M^L(g)\, dg \quad ,$$

where $\gamma = \zeta \exp H$, the factor ε is locally constant and equal to

$$\prod_{\beta \in R_I^+} \frac{e^{\frac{\beta(H)}{2}} - e^{-\frac{\beta(H)}{2}}}{\left|e^{\frac{\beta(H)}{2}} - e^{-\frac{\beta(H)}{2}}\right|} ,$$

R_I^+ being the set of imaginary roots positive with respect to some order, and Y is the A-orthogonal set obtained by projecting $\{w^{-1}T\}$ on $\mathfrak{a}$, the split component of M.

The weight $v_M^L(g)$ is defined by projecting $w^{-1}(T-H(wg))$ on $\mathfrak{a}$, obtaining thereby for each chamber W in $\mathfrak{a}$ a point x_W. In essence $v_M^L(g)$ is the volume of the convex hull of the x_W. To define $V_M^L(\gamma, g)$ we have to replace x_W by

$$X_W = x_W - \sum_\beta \ln \frac{|1-\beta(\gamma)|}{|\beta(\gamma)|^{\frac{1}{2}}} \bar{H}_\beta ,$$

where $\bar{H}_\beta$ is the projection of H_β on $\mathfrak{a}$ and $\langle H, H_\beta\rangle = \frac{2\beta(H)}{\langle \beta, \beta\rangle}$. The sum is over all positive roots β whose restriction to $\mathfrak{a}$ is not zero and which separate W from W_+.

<u>Digression</u>. It will be observed that we have modified our formulation of Flicker's trick, replacing $\ell n|1-\beta^{-1}(\gamma)|$ by $\ell n \frac{|\beta(\gamma)-1|}{|\beta(\gamma)|^{\frac{1}{2}}}$. This is of no importance if our only concern is to create a continuous function of γ, one modification serving as well as the other. The new modification has however a symmetry which the old lacks and which we have implicitly used. Namely, replacing γ by $w\gamma w^{-1}$ in

$$|D^L(\gamma)|^{\frac{1}{2}} \int f(g^{-1}\gamma g) v_M^L(g)$$

or in

$$|D^L(\gamma)|^{\frac{1}{2}} \int f(g^{-1}\gamma g) V_M^L(\gamma, g)$$

yields the same result as keeping γ but replacing $v_M^L(g)$ by $v_M^L(wg)$ and $V_M^L(\gamma, g)$ by $V_M^L(w\gamma w^{-1}, wg)$. Now $v_M^L(wg)$ is defined by

$$\{s^{-1}T - s^{-1}H(swg)\} = \{w(w^{-1}s^{-1} - w^{-1}s^{-1}H(swg))\} \quad .$$

Thus $v_M^L(wg) = v_M^L(g)$. So the replacement has no effect on the first integral.

Before considering the second we notice that

$$\sum_\beta \ln \frac{|1-\beta(\gamma)|}{|\beta(\gamma)|^{\frac{1}{2}}} H_\beta = \sum_\beta c_\gamma(\beta) H_\beta$$

where $c_\gamma(\beta)$ is defined for all β and $c_\gamma(-\beta) = c_\gamma(\beta)$. The sum is over all roots separating W from W_+, more precisely over those which are positive on W_+ and negative on W. To replace W_+ by another chamber W' is simply to add a common term, independent of W, to all these sums, and that has no effect on the volume.

The factor $V_M^L(w\gamma w^{-1}, wg)$ is defined by

$$w(w^{-1}s^{-1} - w^{-1}s^{-1}H(swg)) - \sum \ell n \frac{|1-\beta(w\gamma w^{-1})|}{|\beta(w\gamma w^{-1})|^{\frac{1}{2}}} H_\beta \quad .$$

According to the preceding remark we may sum over β which are negative on $s^{-1}W_+$ and positive on wW_+. Thus we write the sum as

$$w(\sum c_\gamma(w^{-1}\beta)H_{w^{-1}\beta}) \quad .$$

If $s\beta$ is negative on W_+ then $sw(w^{-1}\beta)$ is also, for they are equal. Thus the sum is in fact equal to

$$w(\sum_{\substack{\beta>0 \\ s\beta<0}} c_\gamma(\beta)H_\beta)$$

and $V^L_M(w\gamma w^{-1}, wg) = V^L_M(\gamma, g)$. So both integrals are invariant under the substitution $\gamma \longrightarrow w\gamma w^{-1}$, w lying in the Weyl group. This ends the digression.

It is convenient to set $X_W = Y_W + Z_W$, where

$$Z_W = - \sum_{\beta\neq\alpha} \ln \frac{|1-\beta(\gamma)|}{|\beta(\gamma)|^{\frac{1}{2}}} \bar{H}_\beta$$

and Y_W equals x_W if α does not separate W from W_+ and is otherwise equal to

$$x_W - \ln \frac{|1-\alpha(\gamma)|}{|\alpha(\gamma)|^{\frac{1}{2}}} \bar{H}_\alpha \quad .$$

The function Z_W is varying smoothly near γ_0. So it is natural to apply Lemma 6.3 of Arthur's Annals paper to write

$$V_M(\gamma, g) = \sum_{Q\in\mathcal{F}^L(M)} V^Q_M(\gamma, g, \{Y_W\})U_Q(\{Z_W\})$$

the notation being I hope obvious. The singularities of $J_M(\gamma, f)$ at γ_0 are therefore determined by those of $J'_M(\gamma, f_Q)$ the prime indicating that we are using the weight factor determined by the family $\{Y_W\}$ rather than that determined by $\{X_W\}$.

There is a simple relation between $V_M^Q(\gamma, g, \{Y_W\})$ and $v_M^Q(g)$, namely

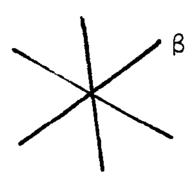
$$V_M^Q(\gamma, g \{Y_W\}) = v_M^Q(g)$$

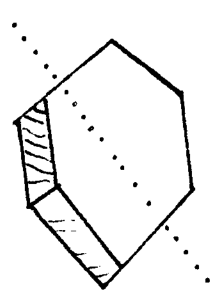
unless α is a root in M_Q. However if α is a root in M_Q and if $M^* \supseteq M$ is the Levi subgroup of M_Q with $\mathfrak{a}^* = \mathfrak{a}_{M^*}$ equal to the null space of α then

$$V_M^Q(\gamma, g, \{Y_W\}) = v_M^Q(g) + \ln \frac{|1-\alpha(\gamma)|}{|\alpha(\gamma)|^{\frac{1}{2}}} v(\bar{H}_\alpha) v_{M^*}^Q(g) \quad ,$$

where $v(\bar{H}_\alpha)$ is the measure of the interval spanned by O and $\bar{H}_\alpha$.

The following diagrams illustrate a typical case of this relation:





The points x_W and Y_W are the same if α is positive on W. The volume of the unshaded region is $v_M^Q(g)$ and that of the shaded region is the difference between $V_M^Q(\gamma, g, \{Y_W\})$ and $v_M^Q(g)$.

To verify the first identity suppose α is not a root M_Q. Then it is either in N_Q or not in Q at all. In the first case $x_W = Y_W$ if $W = W(P)$ and P in $\mathcal{P}(M)$ is contained in Q. In the second case $Y_W = x_W - \ln|1-\alpha^{-1}(\gamma)|\overline{H}_\alpha$. In either case the volumes of their span are equal.

To verify the second we observe that a $P \in \mathcal{P}(M)$ is specified by a $P^* \in \mathcal{P}(M^*)$ and a $P' \in \mathcal{P}^{M^*}(M)$, the second set containing exactly two elements. Moreover if $W = W(P)$ then $x_W = Y_W$ unless α is not a root in P', but then $Y_W = x_W - \ln\frac{|1-\alpha(\gamma)|}{|\alpha(\gamma)|^{\frac{1}{2}}}\overline{H}_\alpha$. At this point one either regards the asserted equality as geometrically obvious or proves it with the algebraic formalism of the Annals paper.

Comparing these relations with the definitions on p. 227 of the Inventiones paper we conclude first of all that

$$J_M'(\gamma, f_Q) = R_{f_Q}(\zeta, H : Y^Q : 1)$$

if α is not a root in M_Q. Observe that in this case an argument used above reduces the study of the weighted orbital integrals near γ_0 to that of ordinary orbital integrals. So we are provided with the required smoothness at no cost. If α is a root in M_Q then

$$J_M'(\gamma, f_Q) = S_{f_Q}^\alpha(\zeta, H : Y^Q : 1) \quad .$$

Incidentally, the family Y^Q is defined in §6 of the Annals paper.

This last formula allows us to apply Theorem 6.1 of the Inventiones paper. It asserts, in particular, that the jump in $H_\alpha J'_M(\gamma, f_Q)$ is equal to

$$n_\alpha(A) J_{M^*}(\gamma_0, f_Q) \quad .$$

We deduce that the jump in $J_M(\gamma, f)$ itself at γ_0 is equal to

$$\sum_{M^* \subseteq Q} |D^G(\gamma)|^{\frac{1}{2}} \int f(g^{-1}\gamma_0 g) v^Q_{M^*}(g)\, dg U_Q(\{Z_W\})$$

and another application of Lemma 6.3 shows that this is equal to

$$J_{M^*}(\gamma_0, f) \quad .$$

Notice that the projection of H_α on $\mathfrak{a}_{M^*}$ is zero. At the moment we are dealing with the case that $\rho(M) = \rho(L) + 1$. Thus $M^* = L$ and $J_{M^*}(\gamma_0, f)$ is the limit of ordinary orbital integrals and consequently zero.

This gives us Shelstad's conditions. To convert the function she provides into a compactly supported function we have to assume that f is K-finite and f_Q therefore $K \cap M$ finite. This assumption was overlooked in the statement of Lemma 2. It is a restriction that does not hinder our real purpose. The use of Clozel's technique was suggested by Arthur.

Let F be a function in the Schwartz class on $M(\mathbf{R})$ with

$$J_M(\gamma, f) = J_M(\gamma, F) \quad .$$

Since the orbital integrals of F are well defined, even though F itself is not, the trace $\mathrm{tr}\,\pi(F)$ is well defined for any tempered representation π of $M(\mathbf{R})$ and does not depend on the choice of F. If z is the Casimir operator, $f = (z-\lambda)f'$, $\lambda \in \mathbf{C}$, and

$$J_M(\gamma, f') = J_M(\gamma, F')$$

for all regular semi-simple γ then we may take $F = (\Gamma_M(z) - \lambda)F'$. We conclude that if π is a tempered representation of $M(\mathbf{R})$ and

$$\pi(\Gamma_M(z)) = \lambda I$$

then

$$\mathrm{tr}\,\pi(F) = \mathrm{tr}\,\pi((\Gamma_M(z) - \lambda)F') = 0 \quad .$$

<u>A difficulty</u>. If f is K-finite the equation $f = (z-\lambda)f' = 0$ is solvable for λ positive and large as a consequence of the Plancherel theorem, but it is solvable only in the Schwartz space (Arthur, <u>Harmonic analysis in the Schwartz space of a reductive Lie group</u> (preprint)). Thus in order to make use of the previous observation we must show that F exists not just for compactly supported functions f but also for functions in the Schwartz space, provided of course that their orbital integrals are zero.

To deal with this larger class of functions we need only establish inequalities

$$|DJ_M(\gamma, f)| \leq C(1 + \|H\|)^{-n} \quad ,$$

where D is an arbitrary invariant differential operator on $T(\mathbb{R})$, $C = C(f, D))$ is a constant, and $\gamma = \zeta \exp H$ with ζ in the maximal compact subgroup of $T(\mathbb{R})$ and with $H = H(\gamma)$ in the Lie algebra of its vector part. It must of course be possible to choose the integer n arbitrary.

Corollary 7.4 of Arthur's paper gives us pretty nearly what we want. He works with $v_M^L(g)$ rather than with $V_M^L(\gamma, g)$. However

$$\left| \ell n \left(\frac{|\alpha(\gamma)-1|}{|\alpha(\gamma)|^{\frac{1}{2}}} \right) \right| \leq C(1 + \|H\|) \quad .$$

So it is easy to convert the corollary to an estimate for $J_M(\gamma, f)$.

Following Arthur we set $L(\gamma)$ equal to the absolute value of the logarithm of the smallest of the numbers $|1-\alpha(\gamma)^{-1}|$, where α runs over the roots of T which do not vanish on $\mathfrak{n}$. The estimate is

$$\text{(3)} \qquad |J_M(\gamma, f)| \leq C(1 + L(\gamma))^{p}(1 + \|H\|)^{-n} \quad ,$$

where $C = C(f, n)$. The question now is whether the technique of the lecture on real groups allows us to rid ourselves of the annoying factor $(1 + L(\gamma))^{p}$ without losing the factor $(1 + \|H\|)^{-n}$. A glance at that lecture and a moment's reflection convinces us that what we need are inequalities

$$\text{(4)} \qquad |DJ_M(\gamma, f)| \leq C\tau(\gamma)^{-p'}(1 + \|H\|)^{-n}$$

valid for an arbitrary invariant differential operator in $T(\mathbb{R})$. The number p' depends on D and $\tau(\gamma)$ is the distance from γ to the set

$$\bigcup_{\alpha} \{t \mid \alpha(t) = 1\} \quad .$$

This is what the second stage of the argument in §8 of the Inventiones paper gives us. As Arthur remarks it is taken from Harish-Chandra. Notice that (3) and the differential equation (2) provide us with an inequality (4) whenever D is a $\Gamma_T(z)$. This is not the place to recapitulate the argument in all its details. It suffices to note that using the fact that the algebra of invariant differential operators is a finite module over $\Gamma_T(Z_L)$ and the existence of a fundamental solution for powers of the Laplace operator in $T(\mathbf{R})$ one finds $D_1, \ldots, D_r$ in $\Gamma_T(Z_L)$ and functions $E_{1,\varepsilon}, \ldots, E_{r,\varepsilon}, B_\varepsilon$ such that

$$D\varphi(\gamma) = \sum_{j=1}^{r} \int D_j \varphi(\tilde{\gamma}) E_{j,\varepsilon}(\tilde{\gamma}^{-1}\gamma)\, d\tilde{\gamma} - \int \varphi(\tilde{\gamma}) B_\varepsilon(\gamma^{-1}\tilde{\gamma})\, d\tilde{\gamma} \quad .$$

Here φ is any function smooth on the set of regular elements in $T(\mathbf{R})$ and $\varepsilon = \frac{\tau(\gamma)}{5}$, $\tau(\gamma)$ being supposed small. I have allowed myself to work in $T(\mathbf{R})$ rather than in its Lie algebra as Arthur and Harish-Chandra do.

We of course take $\varphi(\gamma) = J_M(\gamma, f)$. The point is that the functions $E_{j,\varepsilon}$ and the function B_ε have support in a ball of radius 3ε. Moreover the $E_{j,\varepsilon}$ are bounded and, as an explicit calculation of the fundamental solution shows,

$$|B_\varepsilon(\tilde{\gamma})| \le C\varepsilon^{-q} \quad .$$

(See §29 of Harish-Chandra, Invariant eigendistributions on a semisimple Lie group, Trans. AMS (1965).)

So the difficulty can be surmounted. We return to Clozel's technique, taking f once again to have compact support. To show that F has the same orbital integrals as a function with compact support we have to show that the conditions of Theorem A.1 of the appendix to the Clozel-Delorme paper are satisfied. Take a cuspidal parabolic P of M and consider the representations $\pi_{\delta,\lambda}$ induced from a discrete series representation $\delta \otimes \lambda$ on $M_P(\mathbf{R})$. That

$$\lambda \longrightarrow \operatorname{tr} \pi_{\delta,\lambda}(F)$$

is of Paley-Wiener type follows from the explicit formulas for the character of $\pi_{\delta,\lambda}$ and the fact that the orbital integrals of F are compactly supported.

Once this is granted all that is left is to verify that for all but finitely many δ, taken modulo central characters of M, the function $\operatorname{tr} \pi_{\delta,\lambda}(F)$ vanishes on an open set of λ. This however follows from the observation that if z is the Casimir and $A >> 0$ then for all but finitely many δ there is a λ such that $\pi_{\delta,\lambda}(z) = \mu I$ with $\mu > A$.

We have now treated the case $\rho(Q) = 1 + \rho(L)$ and we pass to the general case, proceeding by induction.

All we need do is verify that, for all regular semi-simple γ,

$$\gamma \longrightarrow J_M(\gamma, f) - \sum_{\substack{M \subset Q \neq L \\ M \neq M_Q}} J_M(\gamma, F^Q) = T(\gamma, f)$$

is equal to $J_M(\gamma, F)$ where F is smooth and compactly supported on M. Most of the argument has now been given, but we still have to verify the differential equations

$$(5) \qquad T(\gamma, zf) = \Gamma_T(z)T(\gamma, f)$$

and the jump conditions.

To verify the differential equations we have to observe that our inductive assumption allows us to take the family associated to zf to be $\{\Gamma_{M_Q}(z)F^Q\}$. Consequently $T(\gamma, zf)$, which is not well defined unless the family attached to zf is specified for $M \subseteq Q$, $M \neq M_Q$, may be taken to be

$$J_M(\gamma, zf) - \sum_{\substack{Q \in \mathcal{F}(M) \\ Q \notin \mathcal{P}(M)}} J_M(\gamma, \Gamma_{M_Q}(z)F^Q) \quad .$$

Applying Arthur's lemma we see that

$$J_M(\gamma, zf) = \sum_{M \subseteq M'} D_M^{M'}(\gamma, \Gamma_{M'}(z))J_{M'}(\gamma, f)$$

and that

$$J_M(\gamma, \Gamma_{M_Q}(z)F^Q) = \sum_{M \subseteq M' \subseteq Q} D_M^{M'}(\gamma, \Gamma_{M'}(z))J_{M'}(\gamma, F^Q) \quad .$$

Summing over Q and then taking a difference and using the relation

$$J_{M'}(\gamma, f) = \sum_{M' \subseteq Q} J_{M'}(\gamma, F^Q)$$

we obtain

$$\Gamma_T(z)(J_M(\gamma, f) - \sum_{\substack{M \subset Q \\ Q \notin \mathcal{P}(M)}} J_M(\gamma, F^Q) \quad .$$

This yields (5).

The jumps only cause difficulty at the semi-regular points defined by roots not in M. Our discussion of the results of the Inventiones paper, especially of Theorem 6.1. show that the jump in the normal derivative of $T(\gamma, f)$ is given by

$$n_\beta(A)\{J_{M^*}(\gamma_0, f) - \sum_{M^* \subseteq Q} J_{M^*}(\gamma_0, F^Q)\} \quad .$$

By the induction assumption this is zero.

Observe that we are implicitly using a result of Harish-Chandra for which no proof has been published. Namely, the induction assumption asserts that

$$J_{M^*}(\gamma, f) - \sum_{M^* \subseteq Q} J_{M^*}(\gamma, F^Q) = 0$$

for γ semi-simple in $M^*(\mathbf{R})$ and regular in $L(\mathbf{R})$. For the inductive argument and, as we have seen, for the fine σ-expansion we need the equality for any γ such that $1 - \mathrm{Ad}\,\gamma$ is invertible on $\mathfrak{l}/\mathfrak{m}$. This follows easily from the fact that for any $\gamma_0 \in M(\mathbf{R})$ there are differential operators $D_1, \ldots, D_r$ on Cartan subgroups $T_1, \ldots, T_r$ of M and sequences γ_i^n of regular elements in $T_i(\mathbb{R})$ such that

$$J_M(\gamma_0, F) = \lim_{n\to\infty} \sum_i D_i J_M(\gamma_i^n, F)$$

for any compactly supported smooth function on $M(\mathbb{R})$.

The measure-theoretic argument sketched. It involves of course the distributions Θ_M, which have never been explicitly defined. Their definition requires an improved form of Theorem 8.2 in Arthur's second Amer. Jour. paper. This involves two things, replacing the hypothetical normalization of §6 of that paper by one deduced from results of Silberger, and proving that the representation on the discrete spectrum is of trace class. These are for the Friday morning seminar.

Observe first of all that we work with functions ϕ on $\mathbf{G}$ and not on $\mathbf{G}^1$. So the integration over $i\mathfrak{a}_L^*/i\mathfrak{a}_G^*$ that appears in Arthur will be an integration over $i\mathfrak{a}_L^*$.

As it is given by Theorem 8.2 the trace formula appears as an absolutely convergent sum over χ of terms $J_\chi(\phi)$. Each $J_\chi(\phi)$ is itself a sum but Arthur does not assert that the double sum is absolutely convergent. Nonetheless it is hoped to present a proof of this in the Friday morning seminar. So we can rearrange at will. The terms appearing in the expression of $J_\chi(\phi)$ as a sum are indexed by two Levi subgroups $M \subseteq L$ and an irreducible unitary representation π of $M(\mathbb{A})$ or, if one prefers, of $M(\mathbb{A})^1$. Only countably many π actually contribute. There is another index s, but it is unimportant. Indeed it is better to use the definition of $M(P, s)$ given on p. 1309 and to express the sum over M, π, and s as a sum over unitary representations of $L(\mathbb{A})$ (induced from $M(\mathbb{A})$. The result will be attributed to $\Theta_L(\phi)$.

Thus $\Theta_L(\phi)$ is a sum

$$\sum_\sigma \int_{i\mathfrak{a}_L^*} \operatorname{tr}(R_\sigma(\lambda)I(\sigma \otimes \lambda, \phi))d\lambda \quad ,$$

the iterated operation being, as is to be shown, absolutely convergent. The only σ which actually occur are those which are unramified outside of S. Thus if we choose a place v_0 not in S and replace ϕ by $\phi * \phi_{v_0}$ where ϕ_{v_0} is a spherical function at v_0 the sum is replaced by

$$(6) \qquad \sum_\sigma \int_{i\mathfrak{a}_L^*} \operatorname{tr}(R_\sigma(\lambda)I(\sigma \otimes \lambda, \phi))\alpha_{\sigma\otimes\lambda}(\phi_{v_0})d\lambda \quad ,$$

$\alpha_{\sigma\otimes\lambda}$ being the homomorphism of the Hecke algebra into $\mathbf{C}$ attached to $\sigma \otimes \lambda$. Now $v = \alpha_{\sigma\otimes\lambda}$ is the homomorphism attached to the unitary representation $\sigma \otimes \lambda$ and thus satisfies $\overline{\alpha(\phi_{v_0})} = \alpha(\phi_{v_0}^*)$ with $\phi_{v_0}^*(g) = \bar{\phi}_{v_0}(g^{-1})$.

It is well known that the set of all such homomorphisms may be identified with the quotient C of a compact subset of $\mathfrak{a}_v^* \otimes \mathbf{C}$ by the Weyl group. The Hecke algebra may be identified with an algebra of continuous functions in C. Just as in the study of base change for GL(2) its closure is the algebra of all continuous functions and (6) defines a linear form on this algebra, thus a measure on C. It is clear that the measures associated to non-conjugate L are orthogonal.

What about the terms $\Theta_M(\psi^Q)$. We observe that when ϕ is replaced by $\phi * \phi_{v_0}$ the set S' may increase but the set S does not. Now we have been very careful - because it is of crucial importance - to insist that we could nonetheless take the family of functions ψ^Q attached to $\phi * \phi_{v_0}$ to be $\psi_S^Q \cdot \Gamma_{M_Q}(\phi_{v_0})$, where ψ_S^Q depends on ϕ alone, or, to be more precise, on ϕ <u>and</u> the choice of S alone but not on ϕ_{v_0} or the choice of S'.

Thus Θ_M is a sum of terms like

$$\int_{i\mathfrak{a}_M^*} \operatorname{trace}(R(\lambda)I(\sigma \otimes \lambda, \psi_S^Q)\alpha_{\sigma\otimes\lambda}(\Gamma_{M_Q}(\phi_{v_0}))d\lambda \quad .$$

So it is clear that $\phi_{v_0} \longrightarrow \Theta_M(\psi^Q)$ may be regarded as a measure on C and that measures associated to non-conjugate M are orthogonal.

Putting together the measures associated to G and taking ϕ_{v_0} to be the identity we obtain

$$\Theta_G(\phi) = \Theta_{G'}(\phi') \quad .$$

We have had to assume that ϕ is K_∞-finite but that is of no consequence.