Afternoon Seminar

# THE HYPERBOLIC TERMS FOR SL(3) AND SU(3)

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1. Formal properties of $J^T$. For the ordinary trace formula these are taken from Arthur's paper The trace formula in invariant form. For the twisted trace formula they will have to be verified in the morning seminar, no serious modification of the proof being anticipated.*

The first point to keep in mind is that $J^T(\phi)$ is a polynomial in $T$. To express this more precisely we introduce for any standard $\varepsilon$-invariant parabolic the function $\phi_Q$ on $\mathbb{M}$ by

$$\phi_Q(m) = \rho_Q(m) \int_K \int_{\mathbb{N}_Q} \phi(k^{-1}mn\varepsilon(k))dndk \quad .$$

It has the same properties as $\phi$ but with respect to $M$ rather than $G$.

In particular the twisted trace formula for $M$ allows us to introduce $J^T(\phi_Q)$. There are polynomials $p_Q$ on $\mathfrak{a}_Q^\varepsilon/\mathfrak{a}_G^\varepsilon$ of degree equal to $\dim(\mathfrak{a}_Q^\varepsilon/\mathfrak{a}_G^\varepsilon)$ such that

$$(1) \qquad J^T(\phi) = \sum_{Q \supseteq P_0} J^{T_1}(\phi_Q)p_Q(T-T_1) \quad .$$

We conclude that $J^T(\phi)$ is a polynomial in $T$.

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*Observe that, contrary to what has been said more than once in these seminars, the correct domain of integration for obtaining the trace formula from the basic identity is $G\backslash \mathbb{G}_\varepsilon^1$ where

$$\mathbb{G}_\varepsilon^1 = \{g \in G \,\big|\, |\chi(g)| = 1 \,\forall \chi \in X_\varepsilon^*(G)\}$$

the set $X_\varepsilon^*(G)$ being the set of $\varepsilon$-invariant rational characters of $G$ defined over $\mathbb{Q}$.

It is inconvenient to be tied to one $P_0$. So we are going to modify the formula in an essentially trivial way. We do however fix $M_0$, an $\varepsilon$-invariant Levi factor of $P_0$ over $\mathbb{Q}$. If $M \supseteq M_0$ is reductive and $\varepsilon$-invariant we let $\mathcal{L}_\varepsilon(M)$ be the set of $\varepsilon$-invariant reductive groups over $\mathbb{Q}$ containing M. We let $\mathcal{F}_\varepsilon(M)$ be the set of $\varepsilon$-invariant parabolics over $\mathbb{Q}$ containing M and $\mathcal{P}_\varepsilon(M) \subseteq \mathcal{F}_\varepsilon(M)$ the set of $\varepsilon$-invariant parabolics with M as Levi factor. When $\varepsilon = 1$ it is not included in the notation. Thus $\mathcal{P}_1(M) = \mathcal{P}(M)$.

Arthur introduces the notion of K admissible relative to $M_0$. There is no need to rehearse the definition here. The only important point is that the Cartan decomposition $\mathbb{G} = \mathbb{P}_0'K$ is valid for any $P_0' \in \mathcal{P}(M_0)$. Thus if $s \in \Omega^\varepsilon(\mathfrak{a}_0, \mathfrak{a}_0) = \Omega(\mathfrak{a}_0^\varepsilon, \mathfrak{a}_0^\varepsilon)$ is represented by $w = w_s$ and $P_0' = w^{-1}P_0w$ then we can define truncation with respect to $P_0'$. If T lies in the positive chamber with respect to $P_0'$ and is sufficiently regular then $T' = s^{-1}T + H(w_s^{-1})$ lies in the chamber positive with respect to $P_0'$ and truncation with respect to $P_0'$ allows us to introduce $J^{T'}(\phi)$. We take T to be $\varepsilon$-invariant and to ensure that $T'$ is also $\varepsilon$-invariant we take $H^\varepsilon(w_s^{-1})$ to be the projection of $H(w_s^{-1})$, calculated with respect to $P_0$, on $\mathfrak{a}_0^\varepsilon$.

We have

$$J^{T'}(\phi) = J^T(\phi) \quad . \tag{2}$$

Notice that $T' = T_s'$ is determined by s alone and is independent of the choice of $w_s$. An analogue of the identity (1) is valid.

$$J^{T'}(\phi) = \sum_{Q' \supseteq P_0} J^{T_1'}(\phi_{Q'}) p_{Q'}(T'-T_1') \quad .$$

Let $T^{Q'}$ be the projection of $T'$ on $\mathfrak{a}_0^{Q'}$. It follows readily from (2) that

$$J^{T'}(\phi_{Q'}) = J^{T^{Q'}}(\phi_{Q'})$$

depends only on $Q'$ and not on $s$, two different choices of $s$ differing by an element in $\Omega^{M_{Q'}}(\mathfrak{a}_0^{\varepsilon}, \mathfrak{a}_0^{\varepsilon})$. Moreover it follows readily from Arthur's definition that

$$p_{Q'}(T'-T_1') = p_Q(T-T_1) \quad .$$

We denote this polynomial in $T-T_1$ by $p_{M_Q}(T-T_1)$.

The identity (1) may be written

$$(3) \qquad J^T(\phi) = \sum_{Q \in \mathcal{F}_\varepsilon(M_0)} J^{T_1^Q}(\phi_Q) \frac{|\Omega^{M_Q}(\mathfrak{a}_0^{\varepsilon}, \mathfrak{a}_0^{\varepsilon})|}{|\Omega^G(\mathfrak{a}_0^{\varepsilon}, \mathfrak{a}_0^{\varepsilon})|} p_{M_Q}(T-T_1) \quad ,$$

$T$ and $T_1$ being sufficiently regular and in the Weyl chamber positive with respect to $P_0$. We abbreviate the quotient appearing here to

$$\frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon^G|}$$

If $\mathfrak{o}$ is a semi-simple conjugacy class in $G$ and $M \in \mathcal{L}_\varepsilon(M_0)$ then $\mathfrak{o} \cap M = \mathfrak{o}_1 \cup \ldots \cup \mathfrak{o}_r$ where the $\mathfrak{o}_i$ are semi-simple $\varepsilon$-conjugacy classes in $M$. If $P \in \mathcal{P}(M)$ set

$$J^{T^P}_{\mathfrak{o}}(\phi_P) = \sum_i J^{J^P}_{\mathfrak{o}_i}(\phi_P) \quad .$$

There is an analogue of (3),

$$(4) \qquad J^T_{\mathfrak{o}}(\phi) = \sum_{Q \in \mathcal{F}_\varepsilon(M_0)} J^{T_1^Q}_{\mathfrak{o}}(\phi_Q) \frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon^G|} p_{M_Q}(T-T_1) \quad .$$

Notice that the value of $J^{T^Q}_{\mathfrak{o}}(\phi_Q)$ is 0 if $\mathfrak{o} \cap M_Q$ is empty. Now associated to $\gamma$ we have the split center $A$ of the $\varepsilon$-centralizer of $\gamma$ and $\delta(\mathfrak{o}) = \dim A - \dim \mathfrak{a}_G^\varepsilon$ is an invariant of $\mathfrak{o}$. Clearly $\mathfrak{o} \cap M_Q$ is empty unless $\dim \mathfrak{a}_Q^\varepsilon / \mathfrak{a}_G^\varepsilon \leq \delta(\mathfrak{o})$. We conclude that $J^T_{\mathfrak{o}}(\phi)$ is a polynomial of degree at most $\delta(\mathfrak{o})$.

Having recalled how $J^T(\phi)$ and the terms $J^T_{\mathfrak{o}}(\phi)$ of the coarse $\mathfrak{o}$-expansion depend upon $T$, we now see how far $J^T$ and $J^T_{\mathfrak{o}}$ depart from $\varepsilon$-invariance. If $h \in \mathbb{G}$ set

$$\phi^h(g) = \phi(hg\varepsilon(h^{-1})) \quad .$$

Observe that we can write $h = ah'$ with $a \in A_G^\varepsilon$ and $h' \in G_\varepsilon^1$. This is important if the argument in §3 of Arthur's paper is to be imitated to yield

$$(5) \qquad J^T(\phi^h) = \sum_{Q \in \mathcal{F}(M_0)} J^{T^Q}(\phi_{Q,h}) \frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon^G|}$$

where

$$\phi_{Q,h}(m) = \rho_Q(m) \int_K \int_{\mathbb{N}_Q} \phi(k^{-1}mn\varepsilon(k)) u'_Q(k, h)\, dn\, dk \quad .$$

Although we use the same notation as Arthur for the weight factor it does depend on $\varepsilon$.

There is a formula analogous to (5) for $J^T_{\mathfrak{o}}(\phi^h)$.

$$J^T_{\mathfrak{o}}(\phi^h) = \sum_{Q \in \mathcal{F}(M_0)} J^{T^Q}_{\mathfrak{o}}(\phi_{Q,h}) \frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon^G|} \qquad (5')$$

We will be guided by these formulas in our definition of $J^T_M(\phi)$. We shall seek to impose at least two conditions:

$$J^T_M(\phi) = \sum_{Q \in \mathcal{F}(H)} \frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon|} J_M^{T_1^Q}(\phi_Q) p_{M_Q}(T-T_1) \qquad \text{(a)}$$

$$J^T_M(\phi^{h'}) = \sum_{Q \in \mathcal{F}(M)} \frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon|} J_M^{T^Q}(\phi_{Q,h}) \qquad \text{(b)}$$

Observe that $\mathcal{F}(G)$ consists of $G$ alone, and that $p_G$ is identically 1. Thus (a) and (b) assert in particular that $J^T_G$ is independent of $T$ and $\varepsilon$-invariant.

2. <u>The hyperbolic terms for a Chevalley group</u>. We are concerned with the ordinary trace formula and we want to define $J^T_{M_0}(\phi)$ as a sum

$$\frac{1}{|\Omega^G|} \sum_{\gamma \in M_0} J^T_{M_0}(\gamma, \phi) .$$

If $\gamma_0$ lies in $M_0$ and $M_0$ is the connected component of the centralizer of $\gamma_0$ then the contribution of the semi-simple class

$\mathfrak{o} = \mathfrak{o}(\gamma_0)$ containing $\gamma_0$ has been described in the morning seminar. If we divide that contribution among the conjugates of $\gamma_0$ in $M_0$ we see that

$$J^T_{\mathfrak{o}}(\phi) = \frac{1}{|\Omega^G|} \operatorname{meas}(M_0 \backslash \mathbb{M}_0^1) \sum_{\gamma} \int_{\mathbb{M}_0 \backslash \mathbb{G}} \phi(g^{-1}\gamma g) v^G_{M_0}(g)\,dg \quad .$$

Recall that $v^G_{M_0}(g)$ is the volume of the compact convex set spanned by the points

$$\{s^{-1}T - s^{-1}H(w_s g) \,|\, s \in \Omega\} \quad ,$$

when $\Omega = \Omega(\mathfrak{a}_0, \mathfrak{a}_0)$. Notice also that

$$\{s^{-1}T - s^{-1}H(w_s w_r g) \,|\, s \in \Omega\} = \{r(s^{-1}T - s^{-1}H(w_s g))\} \quad .$$

Since the action of the Weyl group preserves volume, we conclude that

$$\int_{\mathbb{M}_0 \backslash \mathbb{G}} \phi(g^{-1}\gamma g) v^G_{M_0}(g)\,dg = J^T_{M_0}(\gamma, \phi)$$

is a symmetric function of $\gamma$.

It is for the moment defined only for quasi-regular elements in $M_0$. To define it more generally we write the integral as

(6) $$\int_K \int_{\mathbb{N}_0} \phi(k^{-1}n^{-1}\gamma n k) v^G_{M_0}(nk)\,dn\,dk \quad ,$$

observing that

$$v^G_{M_0}(g) = v^G_{M_0}(nk) - v^G_{M_0}(n)$$

if $g = ank$. For $\gamma$ in $M_0$ and quasi-regular the transformation

$$n \longrightarrow u = \gamma^{-1}n^{-1}\gamma n$$

is a measure-preserving bijection from $\mathbb{N}_0$ to itself. So we can change variables in (6) to obtain

$$(7) \qquad \int_K \int_{\mathbb{N}_0} \phi(k^{-1}\gamma u(k) v_{M_0}^G(\gamma, u)dudk \quad .$$

We have set

$$v_{M_0}^G(\gamma, u) = v_{M_0}^G(n) \quad .$$

If (7) were defined on all of $\mathbb{M}_0$ we would simply define $J_{M_0}^T(\gamma, \phi)$ to be its value at $\gamma$. It is not. If $W = W(s) = s^{-1}(W_+)$ is the Weyl chamber associated to $s$ we set

$$x_W(\gamma, u) = s^{-1}T - s^{-1}H(w_s n) \quad .$$

The difficulty is that some of the $x_W(\gamma, u)$ may go off to infinity as $\gamma$ approaches a singular $\gamma_0$, $u$ remaining fixed, so that the volume $v_{M_0}^G(\gamma, u)$ becomes infinite. We could tolerate this for some $u$ but if it happens for all $u$ there is no chance of defining the integral (7) at $\gamma_0$.

We examine first the group $SL(2)$ for which

$$x_{W_+} = T \qquad\qquad x_{W_-} = -T + H(w_s n) \quad .$$

We have

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad w_s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $H(w_s n)$ is the sum of local contributions $H(w_s n_v)$. Since

$$w_s n = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$$

the standard choice of $K$ yields the following results:

(i) $v$ real,

$$H(w_s n_v) = -\ell n \sqrt{1 + x_v^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(ii) $v$ complex,

$$H(w_s n_v) = -\ell n (1 + |x_v|^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(iii) $v$ finite

$$H(w_s n_v) = -\ell n \max\{1, |x_v|\} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Notice that

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}_0 .$$

If

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and

$$u = \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix}$$

then

$$x' = \left(1 - \frac{b}{a}\right) x \quad .$$

Thus, for a fixed $x'$, both $|x|$ and $H(w_s n_v)$ run off to infinity as $\frac{b}{a} \longrightarrow 1$.

To rectify the situation we notice the following simple identities:

(i) v real,

$$\ln \sqrt{1 + x_v^2} = \ell n \sqrt{\left(1 - \frac{b}{a}\right)^2 + x_v'^2} - \ell n \left|1 - \frac{b}{a}\right|_v$$

(ii) v complex,

$$\ln(1 + |x_v|^2) = \ell n \left|1 - \frac{b}{a}\right|^2 + |x_v'|^2 - \ell n \left|1 - \frac{b}{a}\right|_v$$

(iii) v finite,

$$\ell n \max\{1, |x_v|\} = \ell n \max\{\left|1 - \frac{b}{a}\right|_v, |x_v'|\} - \ell n \left|1 - \frac{b}{a}\right|_v \quad .$$

In all three cases the first term behaves reasonably well as $\left|1 - \frac{b}{a}\right|_v \longrightarrow 0$ provided $|x_v'| \neq 0$. The second term behaves badly, but we begin with regular, retional $\gamma$ and for these

$$\sum_v \ell n \left|1 - \frac{b}{a}\right|_v = 0 \quad .$$

Thus we could have begun by defining $v^G_{M_0}(\gamma, u)$ to be the volume of the convex hull of

$$x'_{W_+} = T$$

and

$$x'_{W_-} = -T + H(w_s n) - \sum_v \ell n \left|1 - \frac{b}{a}\right|_v H \quad .$$

There is no difference for regular, rational $\gamma$.

This leads to difficulties. A little more care is called for. We first observe that for a fixed $\phi$ there are only finitely many $\gamma$ in $M_0$ for which $\phi(k^{-1}\gamma uk)$ does not vanish identically in $u$ and $k$. So there is a finite set of places $S(\phi)$ such that $\left|1 - \frac{b}{a}\right|_v = 1$ if $v \notin S(\phi)$, $\gamma$ is quasi-regular in $M'_0$, and $\phi(k^{-1}\gamma uk)$ does not vanish identically as a function of $u$ and $k$. Thus we begin by defining $V^G_{M_0}(\gamma, u)$ to be the volume of the convex hull of

$$X_{W_+} = T$$

and

$$X_{W_-} = -T + H(w_s n) - \sum_{v \in S(\phi)} \ell n \left|1 - \frac{b}{a}\right|_v H \quad .$$

We so choose $S(\phi)$ that it contains $S$.

With this definition of $V^G_{M_0}(\gamma, u)$ we have, for $\gamma$ regular in $M_0$,

$$(8)\quad J^T_{M_0}(\gamma, \phi) = c\chi(\gamma) \int_K \int_{N_0(\mathbb{A}_{S(\phi)})} \phi_{S(\phi)}(k^{-1}\gamma uk) V^G_{M_0}(\gamma, u)\,du\,dk \quad .$$

Here $\chi(\gamma)$ is the characteristic function of the set of elements in $\mathbb{M}_0$ whose projection on $G(\mathbb{A}^{S(\phi)})$ lies in $K^{S(\phi)}$ and

$$c = \int_{N_0(\mathbb{A}^{S(\phi)}) \cap K} du \quad .$$

We are assuming that $K = \prod_v K_v$.

To see this we have to observe first of all that $\chi(\gamma) \neq 0$ and $\phi_{S(\phi)}(k^{-1}\gamma uk) \neq 0$ imply that $\phi(k^{-1}\gamma uk) \neq 0$. We also have to observe that when this is so then

$$\max\{1, |x_v|\} = 1$$

on $N_0(\mathbb{Q}_v) \cap K_v$, which is the support of $u \longrightarrow \phi_v(k^{-1}\gamma uk)$.

The sequence of steps leading to (8) I refer to as Flicker's trick. Before seeing how it works for other Chevalley groups we examine the right side of (8) more carefully for $SL(2)$. If $d$ is the volume of the interval $[0, H]$ the double integral is equal to the product of

$$d \text{ meas } K^{S(\phi)}$$

with the sum of

$$2T \int_{K_{S(\phi)}} \int_{N_0(\mathbb{A}_{S(\phi)})} \phi_{S(\phi)}(k^{-1}\gamma uk)\,du\,dk$$

and

$$\sum_{v \in S(\phi)} \int_{K_{S(\phi)}} \int_{N_0(\mathbb{A}_{S(\phi)})} \phi_{S(\phi)}(k^{-1}\gamma u k) V^G_{M_0}(\gamma, u_v) du dk$$

where

$$v^G_{M_0}(\gamma, u_v)H = -s^{-1}H(w_s n_v) - \ell n \left|1 - \frac{b}{a}\right|_v H \quad .$$

The previous formulas make it clear that we can dominate $v^G_{M_0}(\gamma, u_v)$ by a locally integrable function of $u$ on any compact set of $\gamma$ and thus that (8) defines $J^T_{M_0}(\gamma, \phi)$ as a continuous function of $\gamma$ on $\mathbb{M}_0$. Thus we can define $J^T_{M_0}(\phi)$ to be

$$\frac{1}{|\Omega^G|} \sum_{\gamma \in M_0} J^T_{M_0}(\gamma, \phi) \quad .$$

One establishes directly for regular $\gamma$ and by continuity for general $\gamma$ that

$$J^T_{M_0}(\gamma, \phi) = \sum_{Q \in F(M_0)} \frac{|\Omega^{M_Q}|}{|\Omega^G|} J^{T_1^Q}_{M_0}(\gamma, \phi_Q) p_{M_Q}(T - T_1) \tag{9}$$

and that

$$J^T_{M_0}(\gamma, \phi^h) = \sum_{Q \in F(M_0)} \frac{|\Omega^{M_Q}|}{|\Omega^G|} J^{T^Q}_{M_0}(\gamma, \phi_{Q,h}) \quad . \tag{10}$$

Observe that if $Q$ is minimal then

$$J^{T^Q}_{M_0}(\gamma, \phi_Q) = \phi_Q(\gamma)$$

and

$$J_{M_0}^{T^Q}(\gamma, \phi_{Q,h}) = \phi_{Q,h}(\gamma) \quad .$$

We now set

$$J_G^T(\mathfrak{o}, \phi) = J_{\mathfrak{o}}^T(\phi) - \frac{1}{|\Omega^G|} \sum_{\gamma \in \mathfrak{o} \cap M_0} J_{M_0}^T(\gamma, \phi) \quad .$$

We readily deduce from (4), (5), (9), (10) and the definition that

(a) $J_G^T(\mathfrak{o}, \phi) = 0$ if $\mathfrak{o}$ is hyperbolic (that is, not elliptic)

(b) $J_G^T(\mathfrak{o}, \phi)$ is independent of $T$

(c) $J_G^T(\mathfrak{o}, \phi)$ is invariant.

We shall set

$$J_G^T(\phi) = \sum_{\mathfrak{o}} J_G^T(\mathfrak{o}, \phi) \quad .$$

For an arbitrary Chevalley group the appropriate modification is to replace $x_W = x_W(\gamma, u)$ by

(11) $$X_W = x_W - \sum_{v \in S(\phi)} \sum_{\alpha} \ell n \left|1 - \alpha^{-1}(\gamma)\right|_v H_\alpha$$

where $H_\alpha$ is defined by

$$\beta(H_\alpha) = (\beta, \alpha)$$

and the inner sum runs over the positive roots separating $W$ from $W_+$. It is up to the morning seminar to justify this assertion. We shall have to examine it more carefully when we discuss the cancellation of singularities, but for the moment we need it only as a guide for the quasi-split case to which we now turn, contenting ourselves with SU(3), which we realize as the unitary group of the form

$$\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$$

taken on a quadratic extension E of the global field F. (Notice that by working only with groups over $\mathbb{Q}$ we have backed ourselves into a corner, for we have no formalism for dealing easily with an arbitrary number field. It is however easy to imagine what that would be, and even easier to construct it. So we feel free to use it, and in particular to apply the above treatment to Chevalley groups over F.)

The hyperbolic terms for SU(3). The weighting factor is at first defined by

$$x_{W_+} = T$$

and

$$x_{W_-} = s^{-1}T - s^{-1}H(w_s n)$$

where s is the one non-trivial element in the Weyl group. It is represented by

$$w_s = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} .$$

As before we regard the resulting weight factor as a function of $\gamma$ and u. Thus

$$v^G_{M_0}(n) = v^G_{M_0}(\gamma, u)$$

where

$$u = \gamma^{-1} n^{-1} \gamma n \quad .$$

We want to modify as before to obtain $V^G_{M_0}(\gamma, u)$. It is clear that the choice of $X_{W_-}(\gamma, u)$ is determined by what we have already done, for the group is split at half the places and we shall need to invoke the product formula.

To make this more precise we observe that the global $\mathfrak{a}_0$, which may be identified in the present case with

$$\left\{ \begin{pmatrix} x_0 & \\ & -x \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$$

is contained in the local $\mathfrak{a}_0$, denoted $\mathfrak{a}^v_0$. If v remains prime in E then $\mathfrak{a}^v_0 = \mathfrak{a}_0$. If it splits then $\mathfrak{a}^v_0$ may be identified with

$$\left\{ \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R},\ x + y + z = 0 \right\} .$$

In addition

$$s^{-1}H(w_s n) = \sum_v s^{-1}H(w_s n_v)$$

and to calculate $s^{-1}H(w_s n_v)$ at a place split in $E$ we first calculate it on $\mathfrak{a}_0^v$ and then project to $\mathfrak{a}_0$, the projection being

$$\begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{x-z}{2} & & \\ & 0 & \\ & & \frac{z-x}{2} \end{pmatrix} .$$

If

$$\gamma = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$$

the modification to be undertaken in the split group is to subtract

$$\ln\left|1 - \frac{b}{a}\right|_v \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} + \ln\left|1 - \frac{c}{b}\right|_v \begin{pmatrix} a & & \\ & 1 & \\ & & -1 \end{pmatrix} + \ln\left|1 - \frac{c}{a}\right|_v \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} .$$

However we are interested only in the projection, which is

$$(12) \quad \ln\left|1 - \frac{b}{a}\right|\left|1 - \frac{c}{b}\right|_v \begin{pmatrix} \frac{1}{2} & & \\ & 0 & \\ & & -\frac{1}{2} \end{pmatrix} + \ln\left|1 - \frac{c}{a}\right|_v \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} .$$

The split center $A_0$ of $M_0$ has two weights in $\mathfrak{a}_0$, say $\alpha_1$ and $\alpha_2$ with $\alpha_2 = 2\alpha_1$. Set

$$H_{\alpha_1} = \begin{pmatrix} \frac{1}{2} & & \\ & 0 & \\ & & -\frac{1}{2} \end{pmatrix} , \quad H_{\alpha_2} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} .$$

Furthermore if $\alpha$ is one of these two weight let $\mathfrak{w}_0(\alpha)$ be the corresponding weight space in $\mathfrak{w}_0$ and consider

$$A_\alpha(\gamma) = \text{ad } \gamma \,|_{\mathfrak{w}_0(\alpha)} .$$

The expression (12) is equal to

$$\ell n \,|\det(1 - A_{\alpha_1}^{-1}(\gamma))\,|_v H_{\alpha_1} + \ell n \,|\det(1 - A_{\alpha_2}^{-1}(\gamma))\,| H_{\alpha_2} .$$

We can therefore expect to define $X_{W_-}(\gamma, u)$ by

$$X_{W_-}(\gamma, u) = x_{W_-}(\gamma, u) - \sum_{v \in S(\phi)} \sum_\alpha \ln|\det(1 - A_\alpha^{-1}(\gamma))\,|_v H_\alpha ,$$

the existence of the set $S(\phi)$ being proved as before. The inner sum runs over $\alpha_1$ and $\alpha_2$, the positive roots separating $W_+$ from $W_-$.

The $X_{W_-}(\gamma, u)$ allow us once again to introduce weight factors $V_{M_0}(\gamma, u)$, but we still must verify that the formula (8) serves to define $J_{M_0}^T(\gamma, \phi)$ as a continuous function on all of $\mathbb{M}_0$. This is a local problem and we carry out the calculations only for the case that $x$ does not split in $E$. The question will have to be taken up afresh in the next lecture anyhow.

Let

$$n_v = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}$$

$$n'_v = \begin{pmatrix} 1 & x' & z' \\ & 1 & y' \\ & & 1 \end{pmatrix}$$

If $v$ does not split in $E$ there are relations to satisfy:

$$b\bar{b} = 1,\ a\bar{c} = 1,\ x = \bar{y},\ z+\bar{z} = x\bar{x},\ x' = \bar{y}',\ z'+\bar{z}' = x'\bar{x}' \quad .$$

The bar denotes conjugation in $E_v/F_v$.

If $v$ is not split in $E$ and is finite we have to choose from amongst those $\lambda$ in $E_v$ with $\mathrm{tr}\lambda = 1$ one for which $|\lambda|$ is minimal. Then an appropriate choice of $K_v$ is apparently the stabilizer in $G(F_v)$ of the lattice

$$\left\{ \begin{pmatrix} u \\ v \\ \lambda w \end{pmatrix} \middle| \ u,\ v,\ w \ \text{integral} \right\} \quad .$$

Thus $K_v$ consists of the matrices $(a_{ij})$ in $G(F_v)$ for which $a_{11}$, $a_{12}$, $a_{21}$, $a_{22}$, $a_{33}$, $\lambda a_{13}$, $\lambda a_{23}$, $a_{31}/\lambda$, $a_{32}/\lambda$ are integral. If $v$ is real then $K_v$ can be taken to be the intersection of $G(F_v)$ with the unitary group attached to the form

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad .$$

Since

$$w_s n_v = \begin{pmatrix} & & 1 \\ & -1 & -y \\ 1 & x & z \end{pmatrix}$$

we see that $-s^{-1}H(w_s n_v)$ is the product of

$$H = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

with the following factor

(i) $-\frac{1}{2}\ln(1 + |x|^2 + |z|^2)$ if $v$ is real.

(ii) $-\ell n(1 + |x|^2 + |z|^2)$ if $v$ is complex.

(iii) $-\ell n \max\{1, |x|, |\lambda z|\} + \ell n|\lambda|$ if $v$ is finite.

The second term in (iii) is harmless, and certainly does not affect the singular behavior at any $\gamma_0$.

It is easy to express $x', y', z'$ in terms of $x, y, z$

$$x' = (1 - \frac{b}{a})x \ , \quad y' = (1 - \frac{c}{b})x \ , \quad z' = (1 - \frac{c}{a})z + (\frac{c}{a} - \frac{b}{a})xy \quad .$$

Thus

$$-s^{-1}H(w_s n_v) - \sum_\alpha \ell n|\det(1 - A_\alpha^{-1}(\gamma))|H_\alpha$$

is equal to the product of $H$ with the factor

(i) $-\frac{1}{2}\ell n(|1 - \frac{b}{a}|^2|1 - \frac{c}{a}|^2 + |x'|^2|1 - \frac{c}{a}|^2 + |z'(1 - \frac{b}{a}) + \frac{b}{a}x'y'|^2$ if $v$ is real,

(ii) $-\ln(|1 - \frac{b}{a}|^2|1 - \frac{c}{a}|^2 + |x'|^2|1 - \frac{c}{a}|^2 + |z'(1 - \frac{b}{a}) + \frac{b}{a}x'y'|^2$ if $v$ is complex,

(iii) $-\ell n(\max\{|1 - \frac{b}{a}||1 - \frac{c}{a}|, |x'||1 - \frac{c}{a}|, |\lambda||z'(1 - \frac{b}{a}) + \frac{b}{a}x'y'|\}) + \ell n|\lambda|$

if $v$ is finite.

This factor we denote $V_{M_0}^{G}(\gamma, u_v)$. We need to verify that

$$\int_{K_{S(\phi)}} \int_{N_0(\mathbb{A}_{S(\phi)})} \phi_{S(\phi)}(k^{-1}\gamma uk) V_{\mathbb{M}_0}^{G}(\gamma, u_v)\, du\, dk$$

defines a continuous function of $\gamma$ in $M_0(\mathbb{A}_{S(\phi)})$.

To do this we choose a small $\varepsilon$ and consider the domains

$$|z'(1 - \frac{b}{a})| \leq \varepsilon |x'y'| , \quad |z'(1 - \frac{b}{a})| > \varepsilon |x'y'|$$

separately. On the first domain we can suppose that

$$V_{M_0}^{G}(\gamma, u_v) \leq c_1 + c_2 \ln |x'y'|$$

when $\phi_{S(\phi)}(k^{-1}\gamma uk)$ is not zero. Since $\ln |x'y'|$ is locally integrable the dominated convergence theorem is applicable. Since $z'$ is bounded on the support of $\phi_{S(\phi)}(k^{-1}\gamma uk)$, we can replace the second domain by

$$|x'| \leq c_3 |\alpha| , \quad |y'| \leq c_3 |\alpha| ,$$

where $\alpha \in E_v$ and $|\alpha^{-2}(1 - \frac{b}{a})|$ is bounded away from infinity and zero. We set $x' = \alpha x''$, $y' = \bar{\alpha} y''$ and change variables to obtain an integral over $|x''| \leq c_3$, $|y''| \leq c_3$, $|z'| < c_4$ of a function dominated by

$$|1 - \frac{b}{a}|(c_5 + c_6 \ln |1 - \frac{b}{a}| + c_7 \ln |\beta z' + x''y''|) ,$$

where $\beta = \beta(\gamma)$ is bounded away from infinity and zero. The dominated convergence theorem is applicable once again.

We have discussed the quasi-split groups SL(2) and SU(3). We could as easily have discussed a group G whose derived group was isogeneous to one of these two groups, introducing the distribution $J_G$. The tactics outlined in the previous lecture demand that we show that

$$J_G(\phi) - \sum_{H \neq G} \iota(G, H) SJ_H(\phi^H)$$

is a stably invariant distribution.

For an anisotropic group G obtained from one of the above by an inner twisting there are no hyperbolic terms and we set

$$J_G(\phi) = J_G^T(\phi) = J^T(\phi) \quad .$$

Now we must show that

$$J_G(\phi) = \sum_H \iota(G, H) SJ_H(\phi^H) \quad .$$

These two problems will be dealt with by Rogawski after Christmas. In the next lecture we will turn to a different matter, cancellation of singularities.