Notes I for the seminar "Analytical Aspects of the Trace Formula II"

Preliminary Facts About Unitary Groups in Three Variables

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§1. Definitions. Let E/F be a separable quadratic extension of fields and let $\mathfrak{G}(E/F) = \{1, \sigma\}$ be the Galois group of E/F. Let D be a simple algebra which is central over E. An involution of the second kind τ is an anti-automorphism of order two of D such that the restriction of τ to the center E of D coincides with σ. If A is a commutative F-algebra, then σ extends to an automorphism of $E \otimes_F A$ and τ extends to an involution of $D \otimes_F A$ whose restriction to $E \otimes_F A$ is σ. Given a pair (D, τ), we obtain an algebraic group U_τ defined over F such that for every F-algebra A, the group of A-rational points is given by:

$$U_\tau(A) = \{g \in (D \otimes_F A)^* : \tau(g)g = 1\} .$$

We call U_τ the unitary group defined by (D, τ). We also obtain the groups

$$SU_\tau(A) = \{g \in (D \otimes_F A)^* : \tau(g)g = 1, \ Nm(g) = 1\}$$

$$GU_\tau(A) = \{g \in (D \otimes_F A)^* : \tau(g)g \in (E \otimes_F A)^*\}$$

where Nm is the reduced norm map.

Let $M_n(R)$ be the algebra of $n \times n$ matrices over R for any ring R. If $D = M_n(E)$, an involution of the second kind τ is of the form $\tau(x) = \Phi^{-1}\sigma({}^t x)\Phi$ for $x \in M_n(E)$ and $\Phi \in GL_n(E)$ is Hermitian, i.e., ${}^t\Phi = \sigma(\Phi)$. In this case, U_τ is the unitary group attached to the Hermitian form $\langle v_1, v_2 \rangle = {}^t\sigma(v_1)\Phi v_2$ where v_1 and v_2 are column vectors in E^n.

Let

$$\Phi_n = \begin{pmatrix} 0 & & & 1 & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & 1 & & & 0 \end{pmatrix} \in GL_n(E)$$

and let U_n denote the unitary group with respect to E/F defined by the Hermitian form Φ_n. Then U_n is quasi-split (a reductive group over a field F is called quasi-split if it contains a Borel subgroup over F) since the subgroup of upper-triangular matrices in U_n is a Borel subgroup over F.

We recall the definition of an inner form of an algebraic group. Let G and G' be algebraic groups over a field F and suppose that there is an isomorphism $\varphi : G \longrightarrow G'$ defined over a Galois extension E/F with Galois group $\mathfrak{G}(E/F)$. For $\sigma \in \mathfrak{G}(E/F)$, $a_\sigma = \varphi^{-1} \circ {}^\sigma\varphi$ is an automorphism of G over E and $a_{\sigma\tau} = a_\sigma \circ \sigma(a_\tau)$ for all $\tau \in \mathfrak{G}(E/F)$. Hence $\{a_\sigma\} \in H^1(\mathfrak{G}(E/F), \mathrm{Aut}(G))$ and if a_σ is an inner automorphism for all $\sigma \in \mathfrak{G}(E/F)$, then G' is called an inner form of G. Every connected reductive group is an inner form of a unique quasi-split reductive group. Hence all unitary groups are inner forms of the groups U_n defined above.

§2. Unitary groups in three variables. In this section, suppose that (D, τ) is a pair as in §1 where $\dim_E D = 9$. The group U_τ is called a unitary group in three variables. We list some facts about unitary groups in three variables.

Fact (i): If $F = \mathbb{R}$ and $E = \mathbb{C}$, all unitary groups in three variables are isomorphic to either the quasi-split form U_3 or the compact form defined by the Hermitian form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Fact (ii): If F is a p-adic field, all unitary groups in three variables are quasi-split, hence isomorphic to U_3.

Now let E/F be a quadratic extension of number fields and let v be a place of F. Let U_τ be a unitary group in three variables with respect to a pair (D, τ) for the extension E/F. From the definition of U_τ and the above facts, we get:

Fact (iii): a) If v is infinite, then $U_\tau(F_v)$ is isomorphic to $U_3(\mathbb{R})$ or its compact form if v is ramified and it is isomorphic to $GL_3(\mathbb{R})$ or $GL_3(\mathbb{C})$ if v is unramified and $F_v = \mathbb{R}$ or $F_v = \mathbb{C}$, respectively.

b) If v is finite and does not split in E, then $U_\tau(F_v)$ is isomorphic to $U_3(F_v)$.

c) If v is finite and splits in E, then $U_\tau(F_v)$ is isomorphic to $(D \otimes_E E_w)^*$ where w is a place of E lying above v (the groups for the two different places are isomorphic).

Fact (iv): The isomorphism class of U_τ is determined by the isomorphism classes of the groups $U_\tau(F_v)$ for all places v of F, i.e., U_{τ_1} is isomorphic to U_{τ_2} if and only if $U_{\tau_1}(F_v)$ is isomorphic to $U_{\tau_2}(F_v)$ for all places v of F, where (D_1, τ_1) and (D_2, τ_2) are pairs as in §1.

§3. L-groups. The L-group of a connected reductive group G over a field F is a complex Lie group of the form ${}^LG = {}^LG^o \rtimes W_{E/F}$ where E/F is a Galois extension over which G splits (we may take E/F as large as desired, or may take E to be an algebraic closure of F) and $W_{E/F}$ is the Weil group of E/F. The L-group depends only on the quasi-split group G^* of which G is an inner form: ${}^LG^o$ is the group of complex points of the connected reductive group over $\mathbb{C}$ whose root data is inverse to that of G^* and the action of $W_{E/F}$ on ${}^LG^o$ is defined through the action of $\mathfrak{G}(E/F)$ on the root data defined by a Borel subgroup of G^* defined over F. We refer to Borel's article in [ ] for precise definitions. We simply state what is needed for this seminar.

(i) Let E/F be a separable quadratic extension and let $G = U_\tau$ be a unitary group in n-variables with respect to E/F. Then ${}^LG^o = GL_n(\mathbb{C})$ and ${}^LG = {}^LG^o \rtimes W_{E/F}$ where $W_{E/F}$ acts on ${}^LG^o$ through its projection onto $\mathfrak{G}(E/F) = \{1, \sigma\}$ and σ acts on ${}^LG^o$ by the automorphism $g \longmapsto J_n^{-1}\, {}^tg^{-1}\, J_n$ with

$$J_n = \begin{pmatrix} 0 & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & \cdot & & & \\ (-1)^{n-1} & & & & 0 \end{pmatrix}$$

(ii) With G, E/F as in (i), let $\tilde{G} = \mathrm{Res}_{E/F}(G)$ where $\mathrm{Res}_{E/F}$ denotes restriction of scalars. Then

$$^L\tilde{G}^o = GL_n(\mathbb{C}) \times GL_n(\mathbb{C}), \quad ^L\tilde{G} = {^L\tilde{G}^o} \rtimes W_{E/F}$$

where $W_{E/F}$ acts on $^L\tilde{G}^o$ through its projection onto $\mathcal{G}(E/F)$ and σ acts on $^L\tilde{G}^o$ by the automorphism:

$$(g_1, g_2) \longmapsto (J_n^{-1}\ {^t g_2^{-1}} J_n,\ J_n^{-1}\ {^t g_1^{-1}} J_n)$$

for $(g_1, g_2) \in {^L\tilde{G}^o}$.

There is a homomorphism $^LG \xrightarrow{\psi_G} {^L\tilde{G}}$ defined by $\psi_G(g\times w) = (g, g) \times w$ for $g \times w \in {^LG^o} \rtimes W_{E/F}$.

§4. Unramified Representations.

Let G be a connected reductive group over a p-adic field F. It is called unramified if G is quasi-split over F and splits over an unramified extension E/F. If G is unramified, it contains a "hyper-special" maximal compact subgroup K (see [ ]) and an irreducible admissible representation π of $G(F)$ is called unramified if the space π^K of vectors in π which are fixed by some such K is non-zero.

Let G be unramified with E/F as above and let $^LG = {^LG^o} \rtimes \mathcal{G}(E/F)$. Let $\Phi \in \mathcal{G}(E/F)$ be the Frobenius element. Then the set $\prod^{un}(G)$ of unramified representations of G is parametrized by the set of $^LG^o$-conjugacy classes in ^LG which contain an element of the form $\gamma \times \Phi$ with $\gamma \in {^LG^o}$ semisimple.

If π is unramified and irreducible, then $\dim \pi^K = 1$ and the Hecke algebra $\mathcal{H}_K$ of bi-K-invariant, compactly supported functions on $G(F)$ acts on π^K through a character $\lambda : \mathcal{H}_K \longrightarrow \mathbb{C}$. The above assertion amounts to an identification of the characters of the commutative algebra $\mathcal{H}_K$ with the conjugacy classes $\{\gamma \times \Phi\}$. See Borel's article in [ ] for details.

§5. Functoriality in the unramified case.

If G_1 and G_2 are connected reductive groups over F (local or global) with L-groups ${}^LG_1 = {}^LG_1^o \rtimes W_{E/F}$ and ${}^LG_2 = {}^LG_2^o \rtimes W_{E/F}$ respectively (we take E large enough to define both L-groups), then a map of L-groups is a homomorphism:

$$\begin{array}{ccc} {}^LG_1 & \xrightarrow{\varphi} & {}^LG_2 \\ & \searrow \quad \swarrow & \\ & W_{E/F} & \end{array}$$

such that the above diagram commutes, where the maps to $W_{E/F}$ are the projections on the second factor.

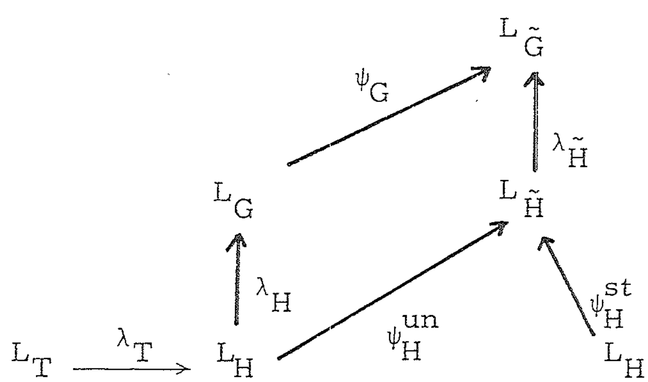
Now assume that F is p-adic and that G_1 and G_2 are unramified. Take E to be an unramified extension of F over which both G_1 and G_2 split. By §4, an L-group map $\varphi : {}^LG_1 \longrightarrow {}^LG_2$ gives a map $\prod^{un}(G_1) \longrightarrow \prod^{un}(G_2)$ by associating the unramified representation corresponding to the ${}^LG_1^o$-conjugacy class of $\gamma \times \Phi$ to the one corresponding to the ${}^LG_2^o$-conjugacy class of $\varphi(\gamma \times \Phi)$.

§6. The basic diagram of L-group homomorphisms.

Let E/F be a quadratic extension of number fields with Galois group $\mathcal{G}(E/F) = \{1, \sigma\}$. We define the following five groups:

$$G = U_3 \quad \text{with respect to} \quad E/F$$
$$H = U_2 \quad \text{with respect to} \quad E/F$$
$$\tilde{G} = \mathrm{Res}_{E/F}(G)$$
$$\tilde{H} = \mathrm{Res}_{E/F}(H)$$
$$T = \mathbb{E}^1 \times \mathbb{E}^1$$

where $\mathbb{E}^1$ is the algebraic group over F defined by the group $E^1 = \{x \in E : N_{E/F}(x) = 1\}$ of norm one elements in E. Thus, for any field extension K/F, $\mathbb{E}^1(K) = \{x \in E \otimes_F K : N_{E \otimes_F K}(x) = 1\}$. We will define a diagram of L-groups:



such that $\psi_G \circ \lambda_H = \lambda_{\tilde{H}} \circ \psi_H^{un}$.

For any number field k, let $\mathbb{A}_k$, $\mathbb{A}_k^*$, and C_k denote the adeles, ideles, and idele classes of k, respectively. Recall that there is an exact sequence $1 \longrightarrow C_E \longrightarrow W_{E/F} \longrightarrow \mathcal{G}(E/F) \longrightarrow 1$. We regard C_F as contained in C_E in a natural way and $N_{E/F}(C_E) \subset C_F$.

All L-groups occurring in the basic diagram have been defined in §3 except ${}^LT$. We have ${}^LT = {}^LT^o \rtimes W_{E/F}$ where

$$ {}^LT^o = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in GL_2(\mathbb{C}) \right\} $$

and $W_{E/F}$ acts on ${}^LT^o$ through its projection onto $\mathfrak{G}(E/F)$, and σ acts on ${}^LT^o$ through the automorphism

$$ \sigma\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} \quad . $$

The maps in the basic diagram are defined as follows.

(i) ψ_G is the map defined in §3.

(ii) ψ_H^{st} is the map ψ_H defined in §3.

Let $\alpha : GL_2(\mathbb{C}) \longrightarrow GL_3(\mathbb{C})$ be the map

$$ \alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b \\ 0 & \delta & 0 \\ c & 0 & d \end{pmatrix} \text{ where } \delta^{-1} = ad-bc \quad . $$

We fix an element $w_\sigma \in W_{E/F}$ which projects to $\sigma \in \mathfrak{G}(E/F)$ and a character μ of C_E whose restriction to C_F is the character of order two associated to E/F by class field theory. We have $w_\sigma^2 \in C_F - N_{E/F}(C_E)$ and hence $\mu(w_\sigma^2) = -1$.

(iii) λ_H maps $h \times 1 \in {}^LH$ to $\alpha(h) \times 1 \in {}^LG$,

$$\lambda_H(1\times z) = \begin{pmatrix} \mu(z) & & \\ & \mu(z)^{-2} & \\ & & \mu(z) \end{pmatrix} \times z \in {}^L G \quad \text{for} \quad z \in C_E$$

$$\lambda_H(1\times w_\sigma) = \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix} \times w_\sigma \in {}^L G \quad .$$

Then λ_H defines a map of L-groups

(iv) $\lambda_{\tilde{H}}$ maps $(h_1, h_2, z) \in {}^L\tilde{H}^o \times C_E$ to $(\alpha(h_1), \alpha(h_2), z) \in {}^L\tilde{G}$ and

$$\lambda_{\tilde{H}}((1, 1) \times w_\sigma) = \left(\begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}\right) \times w_\sigma \in {}^L\tilde{G} \quad .$$

(v) ψ_H^{un} maps $h \times 1 \in {}^L H$ to $(h, h) \times 1 \in {}^L\tilde{H}$ and

$$\psi_H^{un}(1\times z) = \left(\begin{pmatrix} \mu(z) & \\ & \mu(z) \end{pmatrix}, \begin{pmatrix} \mu(z) & \\ & \mu(z) \end{pmatrix}\right) \times z \in {}^L\tilde{H}$$

$$\psi_H^{un}(1\times w_\sigma) = \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right) \times w_\sigma \in {}^L\tilde{H} \quad .$$

(vi) ψ_T maps $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times z \in {}^L T^o \rtimes C_E$ to $\begin{pmatrix} a\mu^{-1}(z) & \\ & b\mu^{-1}(z) \end{pmatrix} \times z \in {}^L H^o$

$$\psi_T(1\times w_\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times w_\sigma \quad .$$

§7. <u>What to expect.</u>

One would like to understand functoriality for the basic diagram as completely as possible. The general problem and techniques for solving it (stable, twisted trace formula and matching orbital integrals) were outlined by Langlands. Contributions (in chronological order) are as follows:

Rogawski: existence of quasi-lifts for λ_H and matching orbital integrals for λ_H (part of thesis, 1980, and unpublished manuscript, 1981)

Kottwitz: matching orbital integrals for ψ_G (unpublished manuscript, 1981)

Flicker: discussion of ψ_H^{un} and ψ_H^{st} (Duke J. Math. 49, no. 3, 1982) and preliminary draft dealing with ψ_G

Further references are:

Labesse-Langlands: L-indistinguishability for SL(2), Can. J. Math. 31 (1979). (This paper treats the analogue of the map λ_T for SL(2) and applies, with minor modifications, to λ_T. It was a motivating example for subsequent work on L-indistinguishability.)

Langlands: Les débuts d'une formule des traces stable, Pub. Math. Paris VII, 1982.

D. Shelstad has an extensive bibliography of papers dealing with L-indistinguishability for real groups. See Langlands Paris VII notes and further notes of this seminar for references.

What follows is a sketch of results which one would like to prove and questions one would like to answer. Some are suggested by the unpublished manuscript of Flicker cited above. In some cases, certain facts are already known and one wants to understand the compatibility of what is known with the formalism of L-groups.

For the definition of an automorphic representation, we refer to the article of Borel and Jacquet in [ ]. An automorphic representation will be called cuspidal, discrete residual, or Eisenstein according as it occurs in the space of cusp forms, residues of Eisenstein series, or in the orthogonal complement of the discrete spectrum, respectively. It occurs discretely if it is cuspidal or residual.

If G is a connected reductive group over a local (resp. global) field F, $\prod(G)$ will denote the set of irreducible, admissible (resp. automorphic) representations of $G(F)$ (resp. $G(\mathbb{A})$). If F is global $\prod_v(G)$ will denote $\prod(G/F_v)$ and $\prod_v^{un}(G)$ will denote the set $\prod^{un}(G/F_v)$.

If G_1 and G_2 are groups over a global field and $\varphi : {}^LG_1 \longrightarrow {}^LG_2$ is a map of L-groups, we obtain maps $\varphi : {}^LG_{1v} \longrightarrow {}^LG_{2v}$ by restriction, where ${}^LG_v$ is the L-group of G/F_v. Hence φ gives rise to maps $\varphi : \prod_v^{un}(G_1) \longrightarrow \prod_v^{un}(G_2)$ for all places v at which G_1 and G_2 are unramified as in §5.

Definition: With φ, G_1, G_2 as above, let $\pi^1 = \otimes\, \pi_v^1 \in \prod(G_1)$ and $\pi^2 = \otimes\, \pi_v^2 \in \prod(G_2)$. Then π^2 is called a quasi-transfer of π^1 (with respect to φ) if $\pi_v^2 = \varphi(\pi_v^1)$ for almost all v at which G_1, G_2 and π_v are unramified.

The following is a list of questions:

Q1. Do quasi-transfers exist for all maps in the basic diagram? This is known so far for all of the maps except ψ_G (for $\lambda_{\tilde{H}}$, it follows from the theory of Eisenstein series and for the other maps from the references cited above).

We want to define transfers locally and globally and there are two points to understand beforehand. First, in general it makes no sense to transfer a representation to an individual representation because, for example, in the local tempered case, transfers should satisfy certain character identities and these identities can only be formulated in terms of stable conjugacy. Postponing details for the time being, it suffices to say that one can only compare certain linear combinations of characters of irreducible π with such linear combinations on other groups and not the characters themselves. The linear combinations are built out of elements in finite sets of representations called L-packets, in the local case. Second, transfers and L-packets for non-tempered representations will generally not obey the same formalism as that which one hopes for in the tempered case. This comes under the rubric of "anomalous" representations discussed below.

To define L-packets for G locally means to partition $\Pi_v(G)$ into finite sets Π_v - it will consist either entirely of tempered or entirely of non-tempered representations. Tempered Π_v should satisfy character identities defined by functoriality and be compatible with global transfer. The definition for non-tempered Π_v is reduced to the tempered case via the Langlands classification.

Suppose local L-packets for a group G have been defined. A global L-packet $\Pi = \otimes \Pi_v$ is then, by definition, obtained by choosing a local L-packet Π_v for all v such that Π_v contains an unramified representation π_v^o for almost all v and setting

$$\Pi = \otimes \Pi_v = \{\otimes \pi_v : \pi_v \in \Pi_v \text{ for all } v,\ \pi_v \text{ unramified for a.a.v.}\} .$$

It should be stressed that apart from the case $F = \mathbb{R}$ or $\mathbb{C}$, there is no general definition of local L-packets. See [Langlands, Paris VII notes] for a description of the formalism one would like L-packets to satisfy. For

$GL(n)$, all L-packets have one element (locally and, hence, globally). For $SL(n)$, local L-packets consist of the sets of representations which are equivalent under conjugation by $GL(n)$, and one may use the same definition globally. For a torus, L-packets consist of one element. These are the only cases in which L-packets have been defined for all places. In particular, this defines L-packets for the groups $\tilde{G}$, $\tilde{H}$, and T in the basic diagram since $\tilde{G}(F) = GL_3(E)$, $\tilde{H} = GL_2(E)$, and T is a torus.

For the group H, L-packets can also be defined because the derived group H^d of H is $SL(2)$. Hence $GL(2)$ acts on H by conjugation and L-packets are defined via conjugation by $GL(2)$ as in the case of $SL(2)$.

To formulate transfer results, it remains to define L-packets for G. We give an ad hoc definition - the trace formula can be expected to show that this definition has the properties wanted. Fix a Borel subgroup $B = AN$ of G and for χ a character of $A(F_v)$, let $i(\chi) = \mathrm{ind}_{B(F_v)}^{G(F_v)} \chi$ (unitary induction).

(i) For χ a character of $A(F_v)$, define an L-packet

$$\prod(i(\chi)) = \begin{cases} i(\chi) & \text{if } i(\chi) \text{ is irreducible} \\ \{\text{constituents of } i(\chi)\} & \text{if } \chi \text{ is unitary} \end{cases} .$$

(ii) If $\pi \in \prod_v(G)$ is non-tempered and is not of the form $i(\chi)$, π is in an L-packet by itself.

By the Langlands classification, the only $\pi \in \prod_v(G)$ not covered by (i) and (ii) are square-integrable. To define the L-packet $\prod(\pi)$ of π

for π square integrable, we shall assume that there exists an element $\pi^o = \otimes \pi^o_w \in \prod(G)$ such that $\pi^o_v = \pi$ and such that there exists a cuspidal quasi-transfer $\tilde{\pi} = \otimes \tilde{\pi}_w$ with respect to ψ_G (see Q1). In this case, we call $\tilde{\pi}_v$ a transfer (via ψ_G) of π_v. The trace formula should imply that $\tilde{\pi}$ exists and that $\tilde{\pi}_v$ is uniquely determined by π_v.

(iii) For square-integrable $\tilde{\pi}$ lifting to $\tilde{\pi}_v$ as above, set

$$\prod(\pi) = \{\pi' \in \prod_v(G) : \pi' \text{ transfers to } \tilde{\pi}_v\} .$$

A global L-packet $\prod = \otimes \prod_v$ is called automorphic if some $\pi \in \prod$ is automorphic.

<u>Q2</u>. Let $\varphi : {}^LG_1 \longrightarrow {}^LG_2$ be one of the maps in the basic diagram and let $\prod$ be a global L-packet for G_1. Set

$$\varphi(\prod) = \{\tilde{\pi} \in \prod(G_2) : \tilde{\pi} \text{ is the quasi-transfer of some } \pi \in \prod \text{ via } \varphi\} .$$

If $\prod$ is tempered, is $\varphi(\prod)$ a global L-packet for G_2? One expects so in all cases and it is known for all maps except λ_H and ψ_G. If it is true, say $\varphi(\prod) = \tilde{\prod} = \otimes \tilde{\prod}_v$, then it is reasonable to set $\varphi(\prod_v) = \tilde{\prod}_v$ and thereby obtain a map for local tempered L-packets. The φ-transfer of non-tempered L-packets is defined in all cases by using the Langlands classification.

For each character μ_v of $E^1(F_v)$, let $St(\mu_v)$ denote the associated Steinberg representation in $\prod_v(H)$. Then $\{St(\mu_v)\}$ is a local L-packet for H. Flicker's paper suggests the following question.

Q3. Is $\lambda_H(\{St(\mu_v)\}) = \{\pi_\mu^+, \pi_\mu^-\}$ where π_μ^+ is square-integrable but not supercuspidal and π_μ^- is supercuspidal?

Q4. Is π_μ^- unipotent in the sense of Lusztig? Does π_μ^+ have Iwahori-fixed vectors and if so, are π_μ^- and π_μ^+ related as in Lusztig's conjectural classification of representations with Iwahori-fixed vectors for Chevalley groups?

Q5. Is the set of local and global L-packets for G with more than one element equal to the image under λ_H of local and global L-packets on H.

Q6. Suppose $\prod \subset \prod(G)$ is a discretely-occurring global L-packet for G such that $\psi_G(\prod)$ does not occur discretely. Is $\prod = \lambda_H(\prod_H)$ for some discretely-occurring global $\prod_H \subset \prod(H)$?

Q7. Suppose $\prod \subset \prod(H)$ is a discretely-occurring global L-packet for H such that $\lambda_H(\prod)$ does not occur discretely. Is $\prod = \lambda_T(\prod_T)$ for some $\prod_T \subset \prod(T)$?

Recall that $\tilde{G}(F) = GL_3(E)$. Let $\pi = \otimes \pi_v$ be a cuspidal automorphic representation of $M(\mathbb{A}_E)$, where M is the Levi factor of a parabolic subgroup P of GL_3. Let ω be the central character of π and suppose that $\mathrm{Re}(|\omega|) \in X_*(A_P) \otimes \mathbb{R}$ lies in the closure of the positive Weyl chamber defined by P, where A_P is the center of M. For all v, the induced representation $I(\pi_v)$ has a unique irreducible quotient π_v' and the global representation $I(\pi)$ has $\pi' = \otimes \pi_v'$ as a quotient. All automorphic representations π' obtained in this way are called <u>isobaric</u>; all other non-cuspidal automorphic representations are called anomalous.

Let $\Pi = \otimes \Pi_v$ be a global automorphic L-packet for G and let $\tilde{\Pi} = \otimes \tilde{\Pi}_v$ where $\tilde{\Pi}_v = \psi_G(\Pi_v)$. If $\tilde{\Pi}$ is anomalous, we call the L-packet Π anomalous (note that $\tilde{\Pi}$ is a single representation).

<u>Q8</u>. Are all discretely-occurring anomalous representations of G obtained as the ψ_H-transfer of a one-dimensional automorphic representation of H?

<u>Q9</u>. Are the tempered components of a discretely-occurring anomalous representation of G of the form $\pi_{\bar{\mu}_v}$?

There are a number of questions regarding multiplicities in the discrete spectrum for G which are suggested by the results for SL(2) of Langlands-Labesse. Let $\Pi_d(G) = \{\pi \in \Pi(G) : \pi \text{ occurs discretely}\}$. If $\Pi \subset \Pi(G)$ is a global L-packet such that $\Pi \cap \Pi_d(G) \neq 0$ we will say that Π occurs discretely. For $\pi \in \Pi_d(G)$, let $m(\pi)$ be the multiplicity of π in the discrete spectrum of G.

<u>Q10</u>. Is $m(\pi) = 1$ for all $\pi \in \Pi_d(G)$?

<u>Q11</u>. If $\Pi \subset \Pi(G)$ occurs discretely and is not in the image of λ_H, is $m(\pi)$ the same for all $\pi \in \Pi$?

<u>Q12</u>. Let $\Pi \subset \Pi(G)$ be a tempered, global L-packet which occurs discretely and suppose that $\Pi = \lambda_H(\Pi_H)$ for some $\Pi_H \subset \Pi(H)$. Is it possible to define a positive rational number $n(\Pi)$ and maps

$$\varepsilon_{iv} : \Pi_v \longrightarrow \mathbf{C}^* \qquad \text{for } i = 1, \ldots, N-1$$

such that for all $\pi = \otimes \pi_v \in \prod_v$,

$$m(\pi) = \frac{1}{N}(n(\prod) + \sum_{i=1}^{N-1} \varepsilon_i(\pi))$$

where $\varepsilon_i(\pi) = \prod_v \varepsilon_{iv}(\pi_v)$ and $\varepsilon_{iv}(\pi_v) = 1$ for a.a.v.?

Q13. If $\prod$ and $\prod'$ are global, isobaric, discretely-occurring L-packets for G such that $\prod_v = \prod'_v$ for a.a.v., is $\prod_v = \prod'_v$ for all v?

One would like to know the image of ψ_G. If π is a local or global admissible representation of $\tilde{G}$, define π^σ by the formula $\pi^\sigma(g) = \pi(\sigma(g))$ and call π σ-invariant if $\pi \approx \pi^\sigma$.

Q14. Does the image of ψ_G consist of σ-invariant representations? If $\tilde{\pi}$ is σ-invariant and not in the image of ψ_G, is it in the image of $\lambda_{\tilde{H}} \circ \psi_H^{st}$?

If $\prod(G)$ can be analyzed, one can attempt to compare G with its inner forms, the other unitary groups in three variables. One question is:

Q15. Do anomalous representations occur discretely for an inner form G' of G if and only if G' is not defined by a pair (D, τ) where D is a division algebra?

Finally, one would like to understand any possible relations between the residual spectrum and the cuspidal spectrum of G. For example, do analogues of the "CAP" representations for $Sp(4)$ constructed by Piatetski-Shapiro occur for G? The residual spectrum, which is spanned by the residues of Eisenstein series, is determined by the behavior of certain L-functions as follows.

Let $B = AN$ be the Borel subgroup of upper-triangular matrices in G, where A is the diagonal subgroup of G and N is the unipotent radical of B. Then

$$A(F) = \left\{ \begin{pmatrix} \alpha\beta & & \\ & \beta & \\ & & \bar{\alpha}\beta^{-1} \end{pmatrix} : \alpha \in E^*,\ \beta \in E' \right.$$

and A is isomorphic to $(\mathrm{Res}_{E/F}(\mathbb{G}_m)) \times \mathbb{E}^1$. Let χ be a character of $A(F) \backslash A(\mathbb{A})$ and let $I(\chi) = \mathrm{ind}_{G(\mathbb{A})}^{G(\mathbb{A})} \chi$. A constituent of $I(\chi)$ is of the form $\otimes\, \pi_v$ with π_v a constituent of $i(\chi_v)$ for all v and π_v unramified for almost all π_v.

<u>Q16</u>. Which constituents of $I(\chi)$ occur in the cuspidal spectrum of G?

Suppose that χ is unitary and consider the Eisenstein series $E(s)$ associated to the character $\chi\alpha_s$ where

$$\alpha_s \left(\begin{pmatrix} \alpha\beta & & \\ & \beta & \\ & & \bar{\alpha}^{-1}_{\beta} \end{pmatrix} \right) = |\alpha|^s \quad \text{for} \quad s \in \mathbb{C},\ \alpha \in \mathbb{A}_E \ .$$

The poles of $E(s)$ are the same as those of the function

$$\xi(s) = \frac{L(s,\ \chi_1)L(2s,\ \chi_2\omega)}{L(s+1,\ \chi_1)L(2s+1,\ \chi_2\omega)}$$

where χ_1 is the character of C_E obtained by restricting χ to the image of the map

$$\alpha \longmapsto \begin{pmatrix} \alpha & & \\ & 1 & \\ & & \bar{\alpha}^{-1} \end{pmatrix} \qquad \alpha \in \mathbb{A}_E^* ,$$

χ_2 is the character of C_F obtained by restricting χ_1 to the image of C_F in C_E, and ω is the character of order two of C_F associated to E/F by class field theory.

The residues of $E(s)$ for $s \in [0.\ 1]$ contribute to the discrete spectrum. If χ_1 is trivial, $\xi(s)$ has a simple pole at $s = 1$. For χ such that χ_1 is non-trivial, $\xi(s)$ has a pole for $s \in [0, 1]$ if and only if $\chi_2\omega$ is trivial and $L(\frac{1}{2}, \chi_1) \neq 0$, in which case the pole occurs at $s = \frac{1}{2}$ and is simple.

Q17. If $\chi_2\omega$ is trivial and $L(\frac{1}{2}, \chi_1) = 0$, so that $\xi(s)$ is finite at $s = \frac{1}{2}$, do constituents of $I(\chi)$ occur in the cuspidal spectrum of G?