Notes II for the seminar "Analytical Aspects of the Trace Formula II"

Stable Conjugacy, Stable Trace Formula   J. Rogawski

§8. Stable conjugacy, twisted conjugacy

[1] Langlands, Les débuts d'une formule des traces stable
Pub. Math., Paris VII

[2] Langlands, Stable conjugacy, definitions and lemmas
Can. J., 31 (1979)

[3] Kottwitz, Rational conjugacy classes in reductive groups
Duke J., 49, No. 4 (1982)

In order to compare the trace formulas of different groups, we need a way of matching conjugacy classes in different groups.

Example: Let G and G^* be connected reductive groups over F and let $\overline{\mathfrak{g}} = \mathrm{Gal}(\bar{F}/F)$. Then $\overline{\mathfrak{g}}$ acts on $\mathrm{Hom}_{\bar{F}}(G,G^*)$. Let $G \xrightarrow{\psi} G^*$ be an isomorphism defined over $\bar{F}$ and suppose that $\sigma\psi = \psi\circ\mathrm{ad}(g_\sigma)$ for some $g_\sigma \in G(\bar{F})$, for all $\sigma \in \overline{\mathfrak{g}}$. Then G is an inner form of G^* and for all $\gamma \in G(F)$, $\psi(g_\sigma \gamma g_\sigma^{-1}) = \sigma\psi(\gamma) = \sigma(\psi(\gamma))$ and hence $\psi(\gamma)$ and $\sigma(\psi(\gamma))$ are conjugate in $G^*(\bar{F})$. In other words, the conjugacy class of $\psi(\gamma)$ is defined over F and we have a map

$$\{\text{Conjugacy classes in } G(F)\} \xrightarrow{\psi} \{\text{Conjugacy classes in } G^*(\bar{F}) \text{ which are defined over } F\} .$$

If $\{\gamma\},\{\gamma'\}$ are conjugacy classes in $G(F)$, then $\psi(\{\gamma\}) = \psi(\{\gamma'\})$ if and only if γ and γ' are conjugate in $G(\bar{F})$.

Kottwitz's Theorem ([3]): If G is connected, reductive and quasi-split over F, and if the derived group of G is simply-connected, then every conjugacy class in $G(\bar{F})$ which is defined over F intersects $G(F)$.

Definition: Let G be connected, reductive, quasi-split over F with simply connected derived group. We say that γ and γ' in $G(F)$ are stably conjugate if they are conjugate in $G(\bar{F})$ (abbreviation: st-conjugate).

In the above example $G \xrightarrow{\psi} G^*$ with $\psi^{-1} \circ \sigma\psi$ an inner automorphism for all $\sigma \in \overline{\mathfrak{G}}$, Kottwitz's theorem shows that if the derived group of G^* is simply-connected and G^* is quasi-split, then we obtain an injection

$$\{\text{stable conj. classes in } G\} \hookrightarrow \{\text{stable conj. classes in } G^*\}$$

$$G(F) \cap \{g^{-1}\gamma g : g \in G(\bar{F})\} \longmapsto G^*(F) \cap \{g^{-1}\psi(\gamma)g : g \in G^*(\bar{F})\}$$

Parametrization of conjugacy classes in a stable class: If $\gamma, \gamma' \in G(\bar{F})$ and $g^{-1}\gamma g = \gamma'$ for some $g \in G(\bar{F})$, then by applying $\sigma \in \overline{\mathfrak{G}}$ to this equality we see that $\sigma(g)g^{-1} \in G_\gamma(\bar{F})$ where G_γ is the centralizer of γ, and $\{a_\sigma = \sigma(g)g^{-1}\}$ is a cocycle in $H^1(\overline{\mathfrak{G}}, G_\gamma)$ whose image in $H^1(\overline{\mathfrak{G}}, G)$ is trivial. The next lemma is easy to check.

Lemma 8.1: Let $\gamma \in G(F)$. The set of $G(F)$-conjugacy classes within the stable conjugacy class of γ is parametrized by the set:

$$\mathcal{D}(\gamma/F) = \operatorname{Ker}\{H^1(\overline{\mathfrak{G}}, G_\gamma) \longrightarrow H^1(\overline{\mathfrak{G}}, G)\}$$

Let G^{sc} denote the simply-connected covering group of the derived group G^{der} of G and for any $\gamma \in G$, let G_γ^{sc} denote the centralizer in G^{sc} of γ under the map $G^{sc} \longrightarrow Ad(G)$. Then $G_\gamma^{sc} \longrightarrow G_\gamma$ by restricting the map $G^{sc} \longrightarrow G^{der}$. Let

$$\mathcal{E}(\gamma/F) = \text{Image of } H^1(\overline{\mathfrak{G}}, G_\gamma^{sc}) \text{ in } H^1(\overline{\mathfrak{G}}, G_\gamma) .$$

Then it is easy to see that $\mathcal{D}(\gamma/F) \subset \mathcal{E}(\gamma/F)$: the images of $G^{sc}(\bar{F})$ and $G(\bar{F})$ in the adjoint group coincide.

From now on we use CSG to abbreviate "Cartan subgroup of G defined over F."

<u>Definition</u>: Let T_1 and T_2 be CSG's of G. We say that T_1 and T_2 are stably conjugate if there is a $g \in G(\bar{F})$ such that $T_2 = g^{-1}T_1 g$ and the map $t \longmapsto g^{-1}tg$ is defined over F (equivalently, T_1 and T_2 are stably conjugate if some regular element of $T_1(F)$ is stably conjugate to an element of $T_2(F)$).

For T a CSG of G, set

$$\mathcal{O}\!\mathcal{L}(T) = \{g \in G(\bar{F}) : g^{-1}Tg \text{ and } t \longrightarrow g^{-1}tg \text{ are defined over } F\} .$$

Set: $\mathcal{D}(T/F) = T(\bar{F})\backslash \mathcal{O}\!\mathcal{L}(T)/G(F)$. It is easy to check that the map

$$\mathcal{D}(T/F) \longrightarrow \text{Ker}\{H^1(\overline{\mathfrak{G}}, T) \longrightarrow H^1(\overline{\mathfrak{G}}, G)\}$$

$$g \longmapsto \{a_\sigma = \sigma(g)g^{-1}\}$$

is a bijection and identifies $\mathcal{J}(T/F)$ with $\mathcal{J}(\gamma/F)$ for any regular $\gamma \in T(F)$. Let

$$\mathcal{E}(T/F) = \text{Image of } H^1(\overline{\mathfrak{g}}, T^{sc}) \text{ in } H^1(\overline{\mathfrak{g}}, T)$$

where T^{sc} is the inverse image of $T^{sc} \cap G^{der}$ in G^{sc}. Then $\mathcal{J}(T/F) \subset \mathcal{E}(T/F)$.

<u>Stabilization of the elliptic regular terms</u>: Let $f = \prod_v f_v$ be a function on $G(\mathbb{A})$ of the usual type to which one applies the trace formula. Assume that $f(zg) = \xi^{-1}(z)f(g)$ for all $z \in Z(\mathbb{A})$, where Z is the center of G and ξ is a character of $Z(F)\backslash Z(\mathbb{A})$. The elliptic regular term of the trace formula is:

$$E(f) = \sum_{\{\gamma\}\ \text{elliptic}}{}' \delta(\gamma)^{-1} \operatorname{meas}(Z(\mathbb{A})G_\gamma(F)\backslash G_\gamma(\mathbb{A}))\Phi(\gamma, f)$$

where Σ' denotes a sum of regular elements, $\{\gamma\}$ ranges over the elliptic regular conjugacy classes in $Z(F)\backslash G(F)$, and $\delta(\gamma)$ is the index of $Z(F)\backslash G_\gamma(F)$ in the centralizer of γ in $Z(F)\backslash G(F)$. Set:

$$\Omega_F^\circ(T,G) = \text{the Weyl group of } T \text{ in } G(F)$$

$$\Omega_{\bar F}(T,G) = \text{the Weyl group of } T \text{ in } G(\bar F)$$

$$\Omega_F(T,G) = \{w \in \Omega_{\bar F}(T,G) : t \longmapsto w^{-1}tw \text{ is defined over } F\}$$

$$\Omega_F^\circ(T,G)_\gamma = \{w \in \Omega_F^\circ(T,G) :\ w^{-1}\gamma w\gamma^{-1} \in Z(F)\}$$

for γ regular in $T(F)$.

For $\gamma \in T'(F)$, $\delta(\gamma) = |\Omega_F^\circ(T,G)_\gamma|$ and the conjugacy class of γ modulo $Z(F)$ intersects $Z(F)\backslash T(F)$ in $|\Omega_F^\circ(T,G)|\delta(\gamma)^{-1}$ points. Hence, if $\mathcal{T}_G$ is a set of representatives for the conjugacy classes of CSG's in G, we have

$$E(f) = \sum_{T \in \mathcal{T}_G} |\Omega_F^\circ(T,G)|^{-1} \operatorname{meas}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A})) \sum_{\gamma \in Z(F)\backslash T(F)}{}' \Phi(\gamma,f) .$$

where Σ' means sum over regular elements. For $\delta \in \mathcal{D}(T/F)$ and $\gamma \in T(F)$, let T^δ and γ^δ denote $h^{-1}Th$ and $h^{-1}\gamma h$ where $h \in \mathcal{O}\!\mathcal{l}(T)$ is any element representing δ. It suffices that T^δ and γ^δ are defined up to $G(F)$-conjugacy because the orbital integral

$$\Phi(\gamma^\delta,f) = \int_{h^{-1}T(\mathbb{A})h\backslash G(\mathbb{A})} f(g^{-1}h^{-1}\gamma hg)dg$$

depends only on δ.

Given a stable conjugacy class $\{T\}_{st}$ of CSG's and $T_0 \in \{T\}_{st}$, the number of T_0-conjugates of the form T^δ for $\delta \in \mathcal{D}(T/F)$ is equal to

$$\frac{|\Omega_F(T,G)|}{|\Omega_F^\circ(T,G)|}$$

and hence we may write:

$$E(f) = \sum_{T \in \mathcal{T}_G^{st}} |\Omega_F(T,G)|^{-1} \operatorname{meas}(Z(\mathbb{A})T(F)\backslash G(\mathbb{A})) \sum_{\gamma \in Z(F)\backslash T(F)}{}' \; \sum_{\delta \in \mathcal{D}(T/F)} \Phi(\gamma^\delta,f)$$

where $\mathcal{T}_G^{st}$ is a set of representatives for the stable conjugacy classes of CSG's of G.

Now fix $T \in \mathcal{T}_G^{st}$ and $\gamma \in Z(F)\backslash T(F)$ and consider the sum

$$\sum_{\delta \in \mathcal{D}(T/F)} \Phi(\gamma^{\delta}, f) \quad .$$

Set:

$$\mathcal{D}(T/\mathbb{A}) = \bigoplus_v \mathcal{D}(T/F_v)$$

$$\mathcal{E}(T/\mathbb{A}) = \bigoplus_v \mathcal{E}(T/F_v) \quad .$$

We have:

$$\begin{array}{ccc} \mathcal{D}(T/F) & \longrightarrow & \mathcal{D}(T/\mathbb{A}) \\ \downarrow & & \downarrow \\ \mathcal{E}(T/F) & \xrightarrow{\psi_T} & \mathcal{E}(T/\mathbb{A}) \end{array}$$

and since $\Phi(\gamma^{\delta}, f) = \prod_v \Phi(\gamma^{\delta}, f_v)$, it is clear that $\Phi(\gamma^{\delta}, f)$ depends only on the image of δ under ψ_T.

Let $\mathfrak{K}(T/F)$ be the set of characters of $\mathcal{E}(T/\mathbb{A})$ which are trivial on $\psi_T(\mathcal{D}(T/F))$. By Tate-Nakayama duality, $|\mathrm{Ker}\ \psi_T| < \infty$ and $[\mathcal{E}(T/\mathbb{A}) : \psi_T(\mathcal{E}(T/F)) < \infty$. The following two lemmas are proved in [1].

<u>Lemma</u> <u>8.2</u>: Let $\delta \in \mathcal{E}(T/F)$ and suppose that $\psi_T(\delta) \in \mathcal{D}(T/\mathbb{A})$. Then $\delta \in \mathcal{D}(T/F)$.

<u>Lemma</u> <u>8.3</u>: The set of places v of F such that $\Phi(\gamma^{\delta_v}, f_v) \neq 0$ for some

$\delta_v \in \mathcal{D}(T/F_v)$ with $\delta_v \neq 1$ is finite.

For $\delta \in \mathcal{E}(T/F) - \mathcal{D}(T/F)$ (resp. $\delta \in \mathcal{E}(T/F_v) - \mathcal{D}(T/F_v)$), set $\Phi(\gamma^\delta, f) = 0$ (resp. $\Phi(\gamma^\delta, f_v) = 0$). The orthogonality relations for finite abelian groups give:

$$\sum_{\delta \in \mathrm{Im}(\psi_T)} \Phi(\gamma^\delta, f) = \sum_{\kappa \in \mathfrak{K}(T/F)} \sum_{\delta \in \mathcal{E}(T/\mathbb{A})} \kappa(\delta)\Phi(\gamma^\delta, f) ,$$

where convergence of the right-hand side is assured by Lemma 8.3. By Lemma 8.2, $\Phi(\gamma^\delta, f)$ depends only on $\psi_T(\delta)$ for $\delta \in \mathcal{E}(T/F)$. Hence

$$\sum_{\delta \in \mathcal{E}(T/F)} \Phi(\gamma^\delta, f) = |\mathrm{Ker}\,\psi_T| \sum_{\kappa \in \mathfrak{K}(T/F)} \sum_{\delta \in \mathcal{E}(T/\mathbb{A})} \kappa(\delta)\Phi(\gamma^\delta, f) .$$

For v a place of F and κ_v a character of $\mathcal{E}(T/F_v)$, set

$$\Phi^{T/\kappa_v}(\gamma, f) = \sum_{\delta \in \mathcal{E}(T/F_v)} \kappa(\delta)\Phi(\gamma^\delta, f) .$$

For $\kappa \in \mathfrak{K}(T/F)$, let κ_v be the restriction of κ to $\mathcal{E}(T/F_v)$. Then Lemma 8.3 gives:

$$\Phi^{T/\kappa}(\gamma, f) = \sum_{\delta \in \mathcal{E}(T/\mathbb{A})} \kappa(\delta)\Phi(\gamma^\delta, f)$$

where $\Phi^{T/\kappa}(\gamma, f) = \prod_v \Phi^{T/\kappa_v}(\gamma, f_v)$. This proves the next proposition.

<u>Proposition</u> <u>8.4</u>:

$$E(f) = \sum_{T \in \mathcal{T}_G^{st}} \frac{|\mathrm{Ker}\,\psi_T|\,\mathrm{meas}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A}))}{|\Omega_F(T,G)| \cdot |\mathfrak{K}(T/F)|} \sum_{\gamma \in Z(F)\backslash T(F)}{}' \sum_{\kappa \in \mathfrak{K}(T/F)} \Phi^{T/\kappa}(\gamma, f) .$$

Twisted conjugacy:

Recall that $\tilde{G} = \mathrm{Res}_{E/F}(G)$. Since G is isomorphic to GL_3 over $\bar{F}$, $\tilde{G}$ is a twisted form of $GL_3 \times GL_3$. The action of $\mathcal{G}$ on $GL_3 \times GL_3$ which defines $\tilde{G}$ is: $((g_1,g_2) \in GL_3(\bar{F}) \times GL_3(\bar{F}))$

$$(g_1,g_2) \longmapsto \begin{cases} (\tau(g_1),\ \tau(g_2)) & \text{if } \tau|_E = 1 \\ (\tilde{\tau}(g_2),\ \tilde{\tau}(g_1)) & \text{if } \tau|_E = \sigma \end{cases}$$

where $\tilde{\tau}(g) = \Phi^{-1}\tau({}^tg^{-1})\Phi$ for $g \in GL_3(\bar{F})$ and $\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. For $g \in GL_3(F)$, $\tilde{\sigma}(g)$ will denote $\Phi^{-1}\sigma({}^tg^{-1})\Phi$. We have

$$\tilde{G}(F) = \{(g,\tilde{\sigma}(g)) : g \in GL_3(E)\} .$$

Let α be the automorphism of $\tilde{G}$ which interchanges the GL_3-factors: $\alpha((g_1,g_2)) = (g_2,g_1)$. Then G embeds in $\tilde{G}$ as the fixed point set of α, i.e.,

$$G = \{(g,g) \in \tilde{G}\} \quad \text{and}$$

$$G(F) = \{(g,g) \in GL_3(E) \times GL_3(E) : \tilde{\sigma}(g) = g\} .$$

We define a norm map

$$N:\ \tilde{G} \longrightarrow \tilde{G}$$

$$g \longmapsto g\alpha(g) .$$

If we identify $\tilde{G}(F)$ with $GL_3(E)$ by projection onto the first factor, then

N on $GL_3(E)$ is the map $g \longmapsto g\tilde{\sigma}(g)$.

Lemma 8.5: Let $g = (g_1, \tilde{\sigma}(g_1)) \in \tilde{G}(F)$. Then the $\tilde{G}(F)$-conjugacy class of $N(g)$ intersects $G(F)$ in a unique stable conjugacy class.

Proof: $N(g) = (g_1\tilde{\sigma}(g_1), \tilde{\sigma}(g_1)g_1)$ and $g_1\tilde{\sigma}(g_1) \in GL_3(E)$. Since $\tilde{\sigma}(g_1\tilde{\sigma}(g_1)) = g_1^{-1}(g_1\tilde{\sigma}(g))g_1$, the conjugacy class of $g_1\tilde{\sigma}(g_1)$ in $G(E) = GL_3(E)$ is defined over F and hence Kottwitz's theorem implies that there is an $x \in GL_3(\bar{F})$ such that $x^{-1}(g_1\tilde{\sigma}(g_1))x \in G(F)$. Let $h = (x, \tilde{\sigma}(g_1)x)$. Then $h^{-1}N(g)h \in G(F) \subset \tilde{G}(F)$ and so the $\tilde{G}(F)$-conjugacy class of $N(g)$ intersects $G(F)$. If (y_1, y_1) and (y_2, y_2) in $G(F)$ are both $\tilde{G}(\bar{F})$-conjugate to $N(g)$, it is clear that y_1 and y_2 are $G(\bar{F}) = GL_3(\bar{F})$-conjugate and the lemma follows.

This lemma gives a map

$$\tilde{G}(F) \longrightarrow \{\text{stable conjugacy classes in } G(F)\}.$$

To describe the fibers of the map, we make the following definitions.

Definition: Let $\gamma_1, \gamma_2 \in \tilde{G}(F)$. We call γ_1 and γ_2 twisted conjugate (t-conjugate for short) if there is a $g \in \tilde{G}(F)$ such that $g^{-1}\gamma_1\alpha(g) = \gamma_2$ and twisted stably conjugate (tst-conjugate for short) if such a $g \in \tilde{G}(\bar{F})$ exists.

Since $N(x^{-1}\gamma\alpha(x)) = x^{-1}N(\gamma)x$, it is clear that the fibers of the above map are tst-conjugacy classes. Let

$$\mathcal{N} : \{\text{tst-conjugate classes in } \tilde{G}(\bar{F})\} \longrightarrow \{\text{st-conjugacy classes in } G(\bar{F})\}$$

be the resulting map. For $\gamma \in \tilde{G}(F)$, let $\{\gamma\}_{tst}$ denote the tst-conjugacy class of γ. We will write $\mathcal{N}(\gamma) = \gamma_0$ to indicate that $\mathcal{N}(\{\gamma\}_{tst}) = \{\gamma_0\}_{st}$.

For $\gamma = (\gamma_1, \gamma_2) \in \tilde{G}(F)$, set

$$\tilde{G}_{\gamma\alpha} = \{g \in \tilde{G} : g^{-1}\gamma\alpha(g) = \gamma\} .$$

The group $\tilde{G}_{\gamma\alpha}$ is defined over F since α is, and if $(g_1, g_2) \in \tilde{G}_{\alpha\gamma}$, then $g_1^{-1}N(\gamma)g_1 = N(\gamma)$. Projection onto the first factor gives an isomorphism $\tilde{G}_{\gamma\alpha} \to G_\gamma$ defined over $\bar{F}$.

If $\gamma_1, \gamma_2 \in \tilde{G}(F)$ and $g \in \tilde{G}(\bar{F})$ is such that $g^{-1}\gamma_1\alpha(g) = \gamma_2$, then

$$\{a_\alpha = \sigma(g)g^{-1}\} \in \mathrm{Ker}\{H^1(\overline{\mathfrak{G}}, \tilde{G}_{\gamma\alpha}) \longrightarrow H^1(\overline{\mathfrak{G}}, \tilde{G})\} .$$

Denote this kernel by $\mathcal{S}_\alpha(\gamma/F)$; it parametrizes the t-conjugacy classes within the tst-conjugacy class of γ.

Stabilization of the twisted elliptic term

For the purposes of the trace formula, we deal only with the F-points $\tilde{G}$ and it is convenient to deal instead with $G(E) = \tilde{G}(F) = GL_3(E)$. The center of $G(E)$ is $Z(E) = E^*$ and the norm map on Z is:

$$N : \quad Z(E) \longrightarrow Z(F)$$
$$z \longmapsto z/\bar{z} \quad .$$

We also have $N : Z(\mathbb{A}_E) \longrightarrow Z(\mathbb{A})$. Let $\tilde{\xi} = \xi \circ N$ where ξ is a character of $Z(F)\backslash Z(\mathbb{A})$ and let $\phi = \prod_v \phi v$ be a function on $G(\mathbb{A}_E)$ of the type to which we can apply the trace formula and assume that $\phi(zg) = \tilde{\xi}(z)^{-1}\phi(g)$.

Let $\mathcal{E}$ be a set of representatives for the t-conjugacy classes of $\gamma \in G(E)$ such that $\mathcal{N}(\gamma)$ is elliptic regular, taken module $Z(E)$.

We are interested first in the contribution from $\mathcal{E}$ to the trace formula applied to the kernel

$$\sum \phi(g^{-1}\gamma\alpha(h)) \quad .$$

As a function along the diagonal $g = h$, it is invariant under $Z(\mathbb{A}_E)$ since $\tilde{\xi}(z^{-1}\tilde{\sigma}(z)) = 1$ for all $z \in Z(\mathbb{A}_E)$. Set

$$\Phi_\alpha(\gamma,\phi) = \int_{Z(\mathbb{A}_E)\tilde{G}_{\alpha\gamma}(\mathbb{A})\backslash G(\mathbb{A}_E)} \phi(g^{-1}\gamma\tilde{\sigma}(g)dg$$

$$\Phi_\alpha(\gamma,\phi_v) = \int_{\tilde{Z}(F_v)\tilde{G}_{\alpha\gamma}(F_v)\backslash\tilde{G}(F_v)} \phi(g^{-1}\gamma\tilde{\sigma}(g)dg \quad .$$

Let $\delta_\alpha(\gamma)$ be the index of $Z(E)\tilde{G}_{\alpha\gamma}(F)$ in the α-centralizer of γ in $Z(E)\backslash G(F)$. Let

$$TE(\phi) = \sum_{\gamma\in\mathcal{E}} \delta_\alpha(\gamma)^{-1} \operatorname{meas}(Z(\mathbb{A})\tilde{G}_{\alpha\gamma}(F)\backslash\tilde{G}_{\alpha\gamma}(\mathbb{A}))\Phi_\alpha(\gamma,f) \quad .$$

This is the contribution of $\mathcal{E}$ to the twisted trace formula. For the next proposition, recall that $\mathcal{J}_G^{st}$ is a set of representatives for the st-conjugacy classes of CSG's of G.

Let $\tilde{F}^* = \{(z,1):\tilde{G}(\bar{F}): z \in F^*\}$. Then we have a map $\tilde{F}^* \longrightarrow \mathcal{D}_\alpha(\gamma/F)$ which sends $(z,1)$ to $\delta(z) = \{\sigma((z,1))(z,1)^{-1}\}$ and $\gamma^{\delta(z)} = z\gamma$. If $\delta \in \mathcal{D}_\alpha(\gamma/F)$ is represented by $\{\sigma(g)g^{-1}\}$, then $\delta(z)\delta$ is represented by $\sigma((z,1)g)$ and $\gamma^{\delta(z)\delta} = z\gamma^\delta$. Hence $\tilde{F}^*$ acts on $\mathcal{D}(\gamma/F)$. Let $\tilde{\mathcal{D}}_\alpha(\gamma/F)$ be the set of orbits of $\tilde{F}^*$ in $\mathcal{D}_\alpha(\gamma/F)$. Since $\tilde{\xi}$ is trivial on $\{z \in Z(E):N(z)=1\} = \{z \in Z(E) : z \in F^*\}$, the twisted orbital integral $\Phi_\alpha(\gamma^\delta,f)$ depends only on the image of δ in $\tilde{\mathcal{D}}_\alpha(\gamma/F)$.

Proposition 8.6:

$$TE(\phi) = \sum_{T \in \mathcal{T}_G^{st}} \sum'_{\gamma_0 \in Z(F)\backslash T(F)} |\Omega_F(T,G)|^{-1} \mathrm{meas}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A})) \sum_{\delta \in \tilde{\mathcal{D}}_\alpha(\gamma/F)} \Phi_\alpha(\gamma^\delta, f)$$

where the last sum is defined by any $\gamma \in \tilde{G}(F)$ such that $\eta(\gamma) = \gamma_0$ (it equals zero if $\{\gamma_0\}_{st}$ is not in the image of η).

Proof: For $\gamma \in \tilde{G}(F)$ and $\gamma_0 \in T(F)$ such that $\eta(\gamma) = \gamma_0$, set

$$\Omega_F(T,G)_{\gamma_0} = \{g \in G(\bar{F}) : g^{-1}\gamma_0 g \gamma_0^{-1} \in Z(F)\}/G_\gamma(\bar{F})$$

$$\Omega(\gamma) = \{g \in \tilde{G}(\bar{F}) : g^{-1}\gamma\alpha(g)\gamma^{-1} \in Z(E)\}/\tilde{F}^*\tilde{G}_{\gamma\alpha}(\bar{F}) .$$

Lemma 8.7: The map $\Omega(\gamma) \longrightarrow \Omega_F(T,G)_{\gamma_0}$ given by projecting $g = (g_1,g_2) \in \Omega(\gamma)$ to the first factor g_1 is an isomorphism.

Proof: If $g = (g_1,g_2)$ represents an element of $\Omega(\gamma)$ is such that $g^{-1}\gamma\alpha(g) = z\gamma$ for some $z \in Z(E)$, then $g_1^{-1}N(\gamma)g_1 = (z/\bar{z})N(\gamma)$. Hence $g_1 \in G_\gamma(\bar{F})$ if and only if $z \in F^* \subset z(E)$. If $g_1^{-1}N(\gamma)g_1 = (z/\bar{z})N(\gamma)$ (every element in $Z(F)$ is of the form $(z/\bar{z})$ for $z \in E^*$), then $g = (g_1,g_2)$ satisfies $g^{-1}\gamma\alpha(g) = z\gamma$, where $g_2 = \gamma_1^{-1} g_1 z\gamma_1$ and $\gamma = (\gamma_1, \tilde{\sigma}(\gamma_1)) \in \tilde{G}(F)$. To see that the map is an isomorphism, we have to show that if $g^{-1}\gamma\alpha(g) = z\gamma$ with $z \in F^*$, then $g \in \tilde{F}^* G_{\gamma\alpha}(\bar{F})$ and this is clear.

Now let

$$\Omega^0(\gamma) = \{g \in \tilde{G}(F) : g^{-1}\gamma\alpha(g)\gamma^{-1} \in Z(E)\}/Z(E)\tilde{G}_{\gamma\alpha}(F) .$$

It is clear that $|\Omega^0(\gamma)| = \delta_\alpha(\gamma)$. If $g \in \tilde{G}(F)$ is such that $g^{-1}\gamma\alpha(g) = z\gamma$ with $z \in F^*$, then $z \in N_{E/F}(E^*)$, as one sees by taking determinants: $z = N_{E/F}(\det(g))^{-1}z^{-2}$. Therefore, the obvious map $\Omega^0(\gamma) \longrightarrow \Omega(\gamma)$ is injective.

To prove the proposition, note that in the sum over $Z(F)\backslash T(F)$, a given stable conjugacy class $\{\gamma_0\}$ occurs $|\Omega_F(T,G)|\cdot|\Omega_F(T,G)_{\gamma_0}|^{-1}$ - times. Let $\delta'_\alpha(\gamma^\delta)$ be the number of $\delta_1 \in \tilde{\mathcal{D}}_\alpha(\gamma/F)$ such that γ^{δ_1} is t-conjugate to $z\gamma^\delta$ for some $z \in Z(E)$. It will suffice to show that $|\Omega_F(T,G)_{\gamma_0}| = \delta_\alpha(\gamma^\delta)\delta'_\alpha(\gamma^\delta)$, or, by the above, that $\delta'_\alpha(\gamma) = [\Omega(\gamma) : \Omega^0(\gamma)]$ (we may take $\delta = 1$). This is clear from the definition of $\tilde{\mathcal{D}}_\alpha(\gamma/F)$.

§9. Conjugacy classes in G

For later use, it will be convenient to have a list of the stable conjugacy classes of CSG's in G. Let A be a fixed CSG of G; for the next Proposition, G can be any connected reductive group. If T is any other CSG, there is a $g \in G(\bar{F})$ such that $g^{-1}Ag = T$ (the map $t \longrightarrow g^{-1}tg$ is not, in general, defined over F). Hence $\{a_\sigma = \sigma(g)g^{-1}\} \in H^1(\bar{\mathfrak{g}},\bar{N})$ where $\bar{N}$ is the normalizer of A in $G(\bar{F})$. It is easy to check that $\{a_\alpha\}$ determines the G(F)-conjugacy class of T. Let $\bar{\Omega}$ be the Weyl group of A in $G(\bar{F})$ and let

$$\psi : H^1(\bar{\mathfrak{g}},\bar{N}) \longrightarrow H^1(\bar{\mathfrak{g}},\bar{\Omega})$$

be the natural map.

Both $\bar{\mathfrak{g}}$ and $\bar{\Omega}$ act on A and the character group $X^*(A) = \mathrm{Hom}(A,GL(1))$. A cocyle $\alpha = \{a_\sigma\} \in H^1(\bar{\mathfrak{g}}, \bar{W})$ defines a twisted action of $\bar{\mathfrak{g}}$ on $X^*(A)$:

$$\tilde{\sigma}(\alpha) = a_\sigma\cdot\sigma(\alpha) \qquad \text{for } \alpha \in X^*(A)$$

and hence $\alpha = \{a_\sigma\}$ defines a form A_α of A.

Proposition 9.1:

(a) Let T_1, T_2 be CSG's of G associated to cocycles $\alpha_1, \alpha_2 \in H^1(\overline{\mathcal{G}}, \bar{N})$. Then T_1 and T_2 are st-conjugate if and only if $\psi(\alpha_1) = \psi(\alpha_2)$.

(b) If G is quasi-split, then every $\bar{\alpha} \in H^1(\overline{\mathcal{G}}, \bar{W})$ is of the form $\psi(\alpha)$ where α arises from a CSG of G, i.e., $\bar{\alpha} = \psi(\alpha)$ for some $\alpha \in \mathrm{Ker}\{H^1(\overline{\mathcal{G}}, \bar{N}) \longrightarrow H^1(\overline{\mathcal{G}}, G)\}$.

Proof: Part (a) follows easily from the definitions. For (b), suppose $\bar{\alpha} \in H^1(\overline{\mathcal{G}}, \bar{W})$ and let $A_{\bar{\alpha}}$ be the twisted form of A that it defines. Let $\gamma \in A_{\bar{\alpha}}(F)$ be regular. There is an isomorphism $\phi: A_{\bar{\alpha}}(\bar{F}) \longrightarrow A_\alpha(\bar{F})$ and $\phi(\gamma) \in A_\alpha(\bar{F})$ is regular. Furthermore, $\sigma(\phi(\gamma)) = a_\sigma \phi(\gamma) a_\sigma^{-1}$ for all $\sigma \in \overline{\mathcal{G}}$, where $\alpha = \{a_\sigma\}$. Hence the conjugacy class of $\phi(\gamma)$ is defined over F. By Steinberg's theorem, $G(F)$ contains an element γ_0 in the $G(\bar{F})$-conjugacy of γ if G is quasi-split. The CSG G_{γ_0} then corresponds to the cocycle α.

We now consider the quasi-split unitary group in three variables $G = U_3$ with respect to a local or global quadratic extension E/F. We may assume that U_3 is the unitary group of the Hermitian form $\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (it is isomorphic to the unitary group of the form $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$). Let A be the diagonal subgroup of G. Then $\bar{W}$ is isomorphic to the symmetric group S_3. Let

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

The elements lie in $\bar{N}$ and we identify $\bar{W}$ with S_3 by mapping w, w_1, w_2 to the transpositions (13), (12), (23) respectively. The Galois group $\bar{\mathfrak{g}}$ acts on G as follows:

$$\tau : g \longmapsto \begin{cases} w^{-1}({}^t\tau(g))^{-1}w & \text{if } \tau|_E = \sigma \\ \tau(g) & \text{if } \tau|_E = 1 \end{cases}$$

where $\mathfrak{g}(E/F) = \{1,\sigma\}$. Under the identification of $\bar{W}$ with S_3, $\tau \in \bar{\mathfrak{g}}$ acts on S_3 trivially if $\tau|_E = 1$ and if $\tau|_E = \sigma$:

$$\tau : \begin{array}{l} (13) \longmapsto (13) \\ (12) \longmapsto (23) \\ (23) \longmapsto (12) \end{array} .$$

Let T be a CSG of G and let L be the centralizer of T in $M_3(E)$. Since L is a maximal, commutative, semi-simple subalgebra of $M_3(E)$, it is isomorphic to a direct sum of field extensions of E and the possibilities are:

(1) $L = E \oplus E \oplus E$

(2) $L = K \oplus E$ with K/E quadratic

(3) L is a cubic extension of E.

To state what we need about stable conjugacy classes of CSG's, we first define some tori. Recall that $\mathbf{E}^1$ is defined as the kernel of the norm map $N : \mathrm{Res}_{E/F}(\mathbf{G}_m) \longrightarrow \mathbf{G}_m$. Let K_1/F be a quadratic extension with $K_1 \neq E$ and let $K = K_1E$, so that $\mathfrak{g}(K/F) = (\mathbf{Z}/2)^2$. Let $\sigma_1, \tau_1 \in \mathfrak{g}(K/F)$ be such that K_1 is the fixed field of σ_1 and E is the fixed field of τ_1. Define a two-dimensional torus T_{K_1} over F by the exact sequence:

$$1 \longrightarrow T_{K_1} \longrightarrow \mathrm{Res}_{K/F}(\mathbb{G}_m) \xrightarrow{N_{K/K_1}} \mathrm{Res}_{K_1 F}(\mathbb{G}_m) \longrightarrow 1$$

where N_{K/K_1} is the map $(1+\sigma_1)$.

If L/E is a cubic extension with an automorphism $\tilde{\sigma} \in \mathrm{Aut}(L)$ of order two whose restriction to E is σ, let $L^{\tilde{\sigma}}$ be the fixed field of $\tilde{\sigma}$ and define T_L by the exact sequence

$$1 \longrightarrow T_L \longrightarrow \mathrm{Res}_{L/F}(\mathbb{G}_m) \xrightarrow{\tilde{N}} \mathrm{Res}_{L^{\tilde{\sigma}}/F}(\mathbb{G}_m) \longrightarrow 1$$

where $\tilde{N}$ is the map $(1+\tilde{\sigma})$. Then T_L is a torus of dimension three over F.

<u>Proposition 8.4</u>: Let T be a CSG of G. Then T is isomorphic to one of the following types:

(0) $A = \mathrm{Res}_{E/F}(\mathbb{G}_m) \times \mathbb{E}^1$ (the CSG contained in B)

(1) $\mathbb{E}^1 \times \mathbb{E}^1 \times \mathbb{E}^1$

(2) $T_{K_1} \times \mathbb{E}^1$ where K_1/F is quadratic with $K_1 \neq E$.

(3) T_L where L/E is a cubic extension with an automorphism $\tilde{\sigma} \in \mathrm{Aut}(L)$ of order two whose restriction to E is σ.

Furthermore, in cases (0), (1), and (2), the stable conjugacy class of T is determined by the isomorphism class of T as a torus over F.

<u>Proof</u>: By Lemma 8.3, the stable conjugacy classes of CSG's are parametrized by $H^1(\mathfrak{G}, \bar{W})$. Let T be a CSG of G and let L be the centralizer of T in $M_3(E)$. We consider three cases separately.

<u>Case (i)</u>: $L = E \oplus E \oplus E$. Then T splits over E and $\{T\}_{st}$ is determined

by a cocycle in $H^1(\mathcal{G}(E/F),W)$, i.e., by an element $a_\sigma \in \bar{W}$ such that $a_\sigma \cdot \sigma(a_\sigma) = 1$. The possibilities are $a_\sigma = 1$, (123), (132), or (13) and since $\sigma((12))(123)(12) = \sigma((23))(132)(23) = 1$, the choices $a_\sigma = (123)$ or (132) are cohomologous to $a_\sigma = 1$. Hence we may assume that $a_\sigma = 1$ or $a_\sigma = (13)$. If $a_\sigma = 1$, then $\{T\}_{st} = \{A\}$ and if $a_\sigma = (13)$, then the twisted action of σ on $X_*(A)$ is the multiplication by -1. This is clear since the map $a_\sigma \cdot \sigma$ on the diagonal subgroup is $g \longmapsto g^{-1}$. So in this case, $T \xrightarrow{\sim} \mathbb{E}^1 \times \mathbb{E}^1 \times \mathbb{E}^1$.

<u>Lemma 8.5</u>: Let T be a CSG of G and let K/E be the splitting field of T. Let K' be the Galois closure of K over E. Then K is Galois over F.

<u>Proof</u>: The involution $g \longmapsto \Phi^{-1}\,{}^t\sigma(g)\Phi$ stabilizes $T(F)$, hence L, and induces an automorphism σ' of K whose restriction to E is σ. It follows that K' is stable under $\overline{\mathcal{G}}$.

<u>Case (ii)</u>: In this case, T splits over K with K/E quadratic. By Lemma 8.4, K/F is Galois and hence $\mathcal{G}(K/F) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$ or $\mathbf{Z}/4$. We first show that $\mathbf{Z}/4$ cannot occur. If it did and if τ_0 were a generator of $\mathcal{G}(K/F)$, then τ_0^2 would act trivially on E. The cocycle $\{a_\tau\} \in H^1(\mathcal{G}(K/F),\bar{W})$ associated to T would satisfy $(a_{\tau_0^2})^2 = (\tau_0(a_{\tau_0})a_{\tau_0})^2 = 1$ which implies that $a_{\tau_0} = 1$ or (13) (the cases (123) and (132) are cohomologous to $a_{\tau_0} = 1$ as in Case (i)). Hence $a_{\tau_0}^2 = 1$ and T splits over E, which is Case (i). Hence $\mathcal{G}(K/F) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$. Let σ_1 and τ_1 generate $\mathcal{G}(K/F)$ with $\sigma_1|_E = \sigma$ and $\tau_1|_E = 1$. Then

$$a_{\sigma_1\tau_1} = a_{\tau_1\sigma_1} = \sigma_1(a_{\tau_1})a_{\sigma_1} = a_{\sigma_1}a_{\tau_1} \quad \text{or}$$

$$a_{\sigma_1}^{-1}(\sigma_1(a_{\tau_1}))a_{\sigma_1} = a_{\tau_1} \; .$$

Up to coboundaries, the possibilities are

(1) $a_{\sigma_1} = 1$ $\quad a_{\tau_1} = (13)$ $\quad a_{\sigma_1\tau_1} = (13)$

(2) $a_{\sigma_1} = (13$ $\quad a_{\tau_1} = (13)$ $\quad a_{\sigma_1\tau_1} = 1$

and since σ_1 and $\sigma_1\tau_1$ both induce σ on E, their roles may be interchanged and we may assume the cocycle has the form (2). Let K_1 be the fixed field of σ_1 in K. It is easy to check that T is isomorphic to $T_{K_1} \times \mathbb{E}^1$.

Case (3) is clear since the involution $g \longrightarrow \Phi^{-1}\,{}^t\sigma(g)\Phi$ induces an automorphism $\tilde{\sigma}$ of order two on L whose restriction to $\mathbb{E}$ is σ.