AFTERNOON SEMINAR

Orbital integrals of spherical functions

J. Rogawski

Let E/F be an unramified quadratic extension of p-adic fields. Let G be the special unitary group in 3 variables with respect to E/F defined by the form $\Phi = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$. Let K be the maximal compact subgroup of integer matrices in G. Let

$$H = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \in G \right\}$$

and set $K_H = H \cap K$. The map

$$\varphi : {}^L H \hookrightarrow {}^L G$$

gives a homomorphism $\varphi^* : \mathcal{H}_G \to \mathcal{H}_H$ where $\mathcal{H}_G$ and $\mathcal{H}_H$ are the Hecke algebras of G and H with respect to K and K_H respectively.

There is a unique stable class $\{T\}_{st}$ of CSG's of G which consists of elliptic tori that split over E. In these notes, we show that the transfer factors $\Delta_G(\gamma)$, $\Delta_H(\gamma)$, $\tau(\gamma)$ defined in previous notes do in fact give a transfer of orbital integrals for spherical functions and the class $\{T\}_{st}$. The remaining classes of elliptic CSG's will be dealt with in a lecture of Kottwitz.

The conjugacy classes within $\{T\}_{st}$ are parametrized by $H^1(\mathfrak{g},T)$, where $\mathfrak{g} = \{1,\sigma\}$ is the Galois group of E/F. Let A be the diagonal subgroup of G and let π be a fixed prime element in F. We have $H^1(\mathfrak{g},T) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and the four conjugacy classes within $\{T\}_{st}$ are defined by the four cocycles with values in the normalizer $N(A)$ of A in $G(\overline{F})$:

$$a_{\sigma,0} = w \qquad a_{\sigma,1} = w\begin{pmatrix}\pi^{-1} & & \\ & \pi & \\ & & 1\end{pmatrix} \qquad a_{\sigma,2} = w\begin{pmatrix}1 & & \\ & \pi^{-1} & \\ & & \pi\end{pmatrix} \qquad a_{\sigma,3} = w\begin{pmatrix}\pi^{-1} & & \\ & 1 & \\ & & \pi\end{pmatrix}.$$

To obtain a representative for the conjugacy class defined by a cocycle a_σ, we write $a_\sigma = g^{-1}\sigma(g)$ for some $g \in G(\overline{F})$ and take $T = gAg^{-1}$ (T is defined over F, although $\gamma \to g\gamma g^{-1}$, $\gamma \in A$, is not defined over F).

Let T_i be a representative for the class defined by the cocycle $a_{\sigma,j}$. We choose T_0 and T_3 inside H (this is possible since $a_{\sigma,0}$ and $a_{\sigma,3}$ lie in H). Let $\delta_i \in G(\overline{F})$ be such that $\delta_i T_0 \delta_i^{-1} = T_i$. We define the unstable orbital integral for $\gamma \in T_0'$:

$$\Phi^{T/k}(\gamma,f) = \Phi(\gamma,f) - \Phi(\gamma^{\delta_1},f) - \Phi(\gamma^{\delta_2},f) + \Phi(\gamma^{\delta_3},f)$$

where $\Phi(x,f) = \int_G f(g^{-1}xg)dg$ for x elliptic regular and dg is chosen so that $\text{meas}(K) = 1$. For functions on H, we define

$$\Phi_H^{T/1}(\gamma,f) = \Phi_H(\gamma,f) + \Phi_H(\gamma^{\delta_3},f)$$

where the measure on H is normalized so that $\text{meas}(K_H) = 1$. We want to prove the following.

<u>Theorem</u>: For all $\gamma \in T_0'$ and $f \in \mathcal{H}_G$,

$$\tau(\gamma)\Delta_G(\gamma)\Phi^{T/\kappa}(\gamma,f) = \Delta_H(\gamma)\Phi_H^{T/1}(\gamma,\varphi^*(f)).$$

The map φ^* is defined in terms of the Satake transform. For $\lambda \in \mathbb{Z}$, set

$$a(\lambda) = \begin{pmatrix} \pi^{\lambda} & & \\ & 1 & \\ & & \pi^{-\lambda} \end{pmatrix}$$

$$F_f(\lambda) = \Delta(a(\lambda)) \int_{A\backslash G} f(g^{-1}a(\lambda)g)dg.$$

The Satake transform $f^{\vee}$ of $f \in \mathcal{H}_G$ may be viewed as a function on elements $\gamma \times \mathrm{Fr} \in {}^L A$ where $\gamma \in {}^L A^0$ and Fr is a Frobenius element. As such, it can be written in the form

$$f^{\vee}(\gamma \times \mathrm{Fr}) = \sum_{\lambda \in \mathbb{Z}} F_f(\lambda)\lambda(\gamma)$$

where $\lambda(\gamma)$ means $\lambda\alpha_3^{\vee}(\gamma)$ and $\alpha_3^{\vee}$ is the co-root corresponding to the root α_3. Let $\eta_j = m\alpha_3^{\vee} + m(\alpha_3^{\vee})^{-1}$. Since F_f is invariant under the Weyl group, $f^{\vee}$ is a finite linear combination of the η_j $(j=0,1,\ldots)$ and $\mathcal{H}_G$ is isomorphic to the algebra spanned by the η_j. Let ${}^H\eta_j$ denote η_j regarded as an element in the range of the Satake transform of $\mathcal{H}_H$. By definition of φ, we have:

$$\varphi\left(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} \times \mathrm{Fr}\right) = \begin{pmatrix} a & & \\ & 1 & \\ & & -a^{-1} \end{pmatrix} \times \mathrm{Fr}$$

from which it follows that $\varphi^*(\eta_j) = (-1)^j\, {}^H\eta_j$. Set

$$f_\lambda = \text{char. fn. of } Ka(\lambda)K$$
$${}^H f_\lambda = \text{char. fn. of } K_H a(\lambda) K_H.$$

The next lemma follows easily, either by direct calculation or by MacDonald's formula.

<u>Lemma</u> 1:

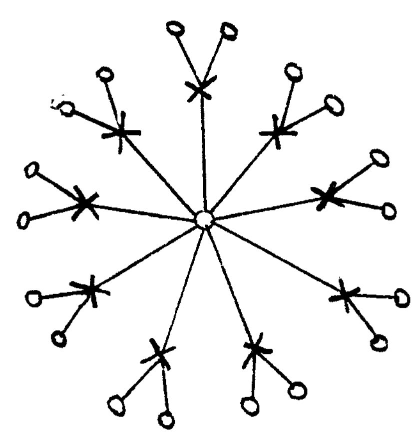
1) $$f^{\vee}_{\lambda} = q^{2\lambda}\eta_{\lambda} + q^{2\lambda}(1-\frac{1}{q^3}) \sum_{\substack{j\equiv\lambda(2)\\ 0\leq j<\lambda}} \eta_j + q^{2\lambda-1}(1-\frac{1}{q}) \sum_{\substack{j\not\equiv\lambda(2)\\ 0\leq j<\lambda}} \eta_j \qquad (\lambda > 0)$$

2) $${}^{H}f^{\vee}_{\lambda} = q^{\lambda}\eta_{\lambda} + q^{\lambda}(1-\frac{1}{q}) \sum_{j=0}^{\lambda-1} \eta_j \qquad (\lambda>0)$$

3) $$\varphi^{*}(f_{\lambda}) = (-1)^{\lambda}q^{\lambda}\, {}^{H}f_{\lambda} + (1-\frac{1}{q^2}) \sum_{j=0}^{\lambda-1} (-1)^{j}q^{2\lambda-j}\, {}^{H}f_{j}. \qquad (\lambda>0).$$

To prove the theorem, we shall calculate $\Phi(\gamma,f_{\lambda})$, $\Phi(\gamma^{\delta_j},f_{\lambda})$ for all λ and check that the values we obtain compare favorably with the known values of $\Phi_H^{T/1}(\gamma,{}^{H}f_{\lambda})$ through the formula 3) above.

As explained in Kottwitz's lecture, $\Phi(\gamma,f_{\lambda})$ is given combinatorially in terms of the Bruhat-Tits building associating to G. The building X is a tree with two G-conjugacy classes of vertices. We call the vertices conjugate to the vertex defined by K hs-vertices and call the remaining vertices s-vertices. Each hs-vertex has q^3+1 s-neighbors and each s-vertex has $q+1$ hs-neighbors. Thus, for $q=2$, the tree is as follows:



where the $\circ$ are hs-vertices and the x are s-vertices. For vertices $p,q \in X$, let $d(p,q)$ denote one-half the number of chambers that lie on a geodesic joining p and q; a chamber is an edge $\circ$—x. The next lemma is a trivial calculation.

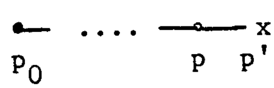
<u>Lemma</u>: For γ elliptic regular,

$$\Phi(\gamma, f_\lambda) = F_\lambda(\gamma)$$

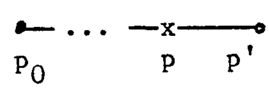
where $F_\lambda(\gamma)$ is the number of hs-vertices $p \in X$ such that $d(\gamma p, p) = \lambda$.

Let T be an elliptic Cartan subgroup. Then T is contained in some maximal compact subgroup of G and thus T fixes some vertex $p_0 \in X$. The set $\mathrm{Fix}(\gamma)$ of fixed points of an element $\gamma \in T$ is a connected neighborhood of p_0. If γ is regular, $\mathrm{Fix}(\gamma)$ is finite and $F_0(\gamma) = \Phi(\gamma, f_0)$.

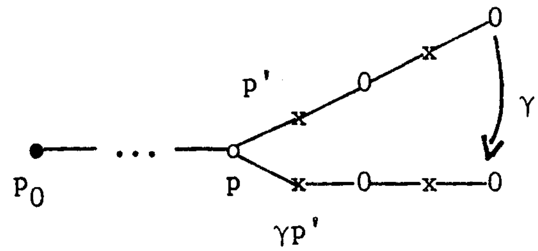
Let $N_{hs}(\gamma)$ be the number of paths in X of the following type:



The path begins at p_0, ends at an s-vertex p' which is not fixed by γ, but $p \in \mathrm{Fix}(\gamma)$. Similarly, let $N_s(\gamma)$ be the number of paths



with $p \in \mathrm{Fix}(\gamma)$ and $p' \notin \mathrm{Fix}(\gamma)$. As the following diagram indicates:



$F_\lambda(\gamma)$ is determined by $N_{hs}(\gamma)$ for λ even, $\lambda \neq 0$ and by $N_s(\gamma)$ for λ odd. The next lemma follows easily from the structure of the tree.

<u>Lemma</u> 2:

$$F_\lambda(\gamma) = \begin{cases} q^{2\lambda-3} N_{hs}(\gamma) & \lambda \text{ even} \\ q^{2\lambda-2} N_s(\gamma) & \lambda \text{ odd.} \end{cases}$$

We first calculate $F_0(\gamma)$. Let $\gamma_1, \gamma_2, \gamma_3$ be the eigenvalues of $\gamma \in T_0$. Since $T_0 \subset H$, we may choose γ_2 so that

$$\gamma = \begin{pmatrix} * & 0 & * \\ 0 & \gamma_2 & 0 \\ * & 0 & * \end{pmatrix}.$$

Let $\text{val} : E^* \to \mathbb{Z}$ be the valuation map and set $m_j = \text{val}(1-\alpha_j^{-1}(\gamma))$ where $\alpha_1(\gamma) = \gamma_1/\gamma_2$, $\alpha_2(\gamma) = \gamma_2/\gamma_3$, and $\alpha_3(\gamma) = \gamma_1/\gamma_3$ (the ambiguity in the labelling of γ_1 and γ_3 will not matter).

<u>Proposition</u> 1: Let $\gamma \in T_0$ be regular. Set $m = \min(m_1, m_2, m_3)$, $n = \max(m_1, m_2, m_3)$, and let

$$\delta(\gamma) = \begin{cases} 1 & \text{if } m \equiv n \quad (2) \\ 0 & \text{if } m \not\equiv n \quad (2). \end{cases}$$

a) $F_0(\gamma)$ is equal to

$$\frac{-(q+1)}{q^4-1} - q^{4[\frac{m}{2}]} \frac{q^2(q+1)}{q^4-1} - (-1)^{m+n} q^{m+n} \frac{1}{q-1} + \delta(\gamma) \frac{q^{m_1+m_2+m_3}(q+1)}{q-1} .$$

b) $F_0(\gamma^{\delta_3})$ is given by the following formulas

(i) If $m_1 \neq m_2$, then $F_0(\gamma^{\delta_3}) = (q^{4[\frac{m-1}{2}]+4} - 1)\frac{q+1}{q^4-1}$

(ii) If $m_1 = m_2$, then $F_0(\gamma^{\delta_3})$ is equal to

$$\frac{-(q+1)}{q^4-1} - q^{4[\frac{m}{2}]} \frac{q^2(q+1)}{q^4-1} + (-1)^{m+n} q^{m+n} \frac{1}{q-1} + (1-\delta(\gamma)) \frac{q^{m_1+m_2+m_3}(q+1)}{q-1} .$$

With this proposition, we can compute $\Phi^{T/\kappa}(\gamma, f_0)$. If m_1, m_2, m_3 are the integers attached to γ, set $F_j(m_1,m_2,m_3) = F_0(\gamma^{\delta_j})$, where we let $\delta_0 = 1$. The proof of the proposition shows that $F_2(m_1,m_2,m_3) = F_3(m_3,m_2,m_1)$ and $F_1(m_1,m_2,m_3) = F_3(m_1,m_3,m_2)$. We must calculate

$$F_0(m_1,m_2,m_3) - F_3(m_3,m_2,m_1) - F_3(m_1,m_3,m_2) + F_3(m_1,m_2,m_3).$$

The three possibilities are

(a) $m_1 > m_2 = m_3$ (b) $m_3 > m_1 = m_2$ (c) $m_2 \geq m_1 = m_3$.

Case (a): We are reduced to $F_0(m_1,m_2,m_3) - F_3(m_3,m_2,m_1)$. After cancelling off equal terms, we get

$$(-1)^{m_1+m_2} q^{m_1+m_2+m_3}\left(\frac{q+1}{q-1}\right) - 2(-1)^{m_1+m_2} q^{m_1+m_2}\left(\frac{1}{q-1}\right).$$

Case (b): The constant terms cancel and the highest order term is $q^{m_1+m_2+m_3}(\frac{q+1}{q-1})$. The intermediate terms are as follows:

$$F_0(m_1,m_2,m_3) \; : \; - q^{4[\frac{m}{2}]}\left(\frac{q^2(q+1)}{q^4-1}\right) - (-1)^{m+n} q^{m+n}\left(\frac{1}{q-1}\right)$$

$$- F_3(m_3,m_2,m_1) \; : \; - q^{4[\frac{m-1}{2}]+4}\left(\frac{q+1}{q^4-1}\right)$$

$$- F_3(m_1,m_3,m_2) \; : \; - q^{4[\frac{m-1}{2}]+4}\left(\frac{q+1}{q^4-1}\right)$$

$$F_3(m_1,m_2,m_3) \; : \; - q^{4[\frac{m}{2}]}\left(\frac{q^2(q+1)}{q^4-1}\right) + (-1)^{m+n} q^{m+n}\left(\frac{1}{q-1}\right).$$

The sum of these intermediate terms is

$$\frac{-2(q+1)}{q^4-1}\left(q^{4[\frac{m}{2}]+2} + q^{4[\frac{m-1}{2}]+4}\right) = \frac{-2q^{2m}(q+1)}{q-1} = \frac{-2q^{m_1+m_2}(q+1)}{q-1}.$$

Case (c): We are reduced to $F_0(m_1,m_2,m_3) - F_3(m_1,m_3,m_2)$ and the answer is the same as case (a). To summarize, we have the next proposition.

Proposition 2: For $\gamma \in T_0$ regular,

$$\Phi^{T/\kappa}(\gamma, f_0) = (-1)^{m_1+m_2}\left(\frac{1}{q-1}\right)\left(q^{m_1+m_2+m_3}(q+1) - 2q^{m_1+m_2}\right).$$

This Proposition gives the theorem for $f_0 = f$; it is clear that $\tau(\gamma) = (-1)^{m_1+m_2}$ and it is known that $\Phi_H^{T/1}(\gamma, {}^H f_0)$ is equal to $(q-1)^{-1}(q^{m_3}(q+1)-2)$.

Proof of Proposition 1: Since T_0 splits over E, we may choose $g \in SL_3(\mathcal{O}_E)$ so that $gA(E)g^{-1} = T_0(E)$, where $\mathcal{O}_E$ is the ring of integers in E. Suppose that $\gamma \in A(E)$ and $g\gamma g^{-1} \in T_0$. Then a vertex p is fixed by γ if and only if $g\gamma g^{-1}$ fixes gp. Let p_0 be the vertex associated to K and assume that $T_0 \subset K$, which we may up to conjugacy. An hs-vertex of X is of the form gxp_0 for some $x \in SL_3(E)$; gxp_0 lies in X if and only if $\sigma(gxp_0) = gxp_0$. Now, $\sigma(gxp_0) = gxp_0$ if and only if $x^{-1}(g^{-1}\sigma(g))\sigma(x) \in SL_3(\mathcal{O}_E)$ and since $g^{-1}\sigma(g) = w$, this is the condition

$$(*) \qquad {}^t\bar{x}x \in SL_3(\mathcal{O}_E).$$

By the Iwasawa decomposition, we may choose x upper-triangular. Let

$$x = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} a,b,c \in E^* \\ x,y,z \in E. \end{matrix}$$

A calculation shows that this x satisfies $(*)$ if the following six quantities lie in $\mathcal{O}_E$:

1) $a\bar{a}$
2) $a\bar{a}x$
3) $a\bar{a}y$
4) $b\bar{b} + a\bar{a}x\bar{x}$
5) $b\bar{b}z + a\bar{a}y\bar{x}$
6) $c\bar{c} + b\bar{b}z\bar{z} + a\bar{a}y\bar{y}$.

Now let

$$x = \begin{pmatrix} \pi^d & & \\ & 1 & \\ & & \pi^{-d} \end{pmatrix} \begin{pmatrix} 1 & \pi^{-j}\varepsilon_1 & \pi^{-2d}\varepsilon_2 \\ 0 & 1 & \pi^{-j}\varepsilon_3 \\ 0 & 0 & 1 \end{pmatrix}$$

where d = a non-negative integer, $0 \leq j \leq d$, and the ε_j are zero or lie in $\mathcal{O}_E^*$. Then x satisfies (*) if and only if the two conditions

1) $\pi^{-j}(\varepsilon_3+\bar{\varepsilon}_1\varepsilon_2) \in \mathcal{O}_E$

2) $\pi^{-2d}(1+\pi^{2d-2j}N(\varepsilon_3)+N(\varepsilon_2)) \in \mathcal{O}_E$

are satisfied, and if x,x' are of this form, $xp_0 = x'p_0$ only when

$$\varepsilon_1 = \varepsilon_1' \ (\pi^j) \qquad \varepsilon_3 \equiv \varepsilon_3' \ (\pi^j) \qquad \varepsilon_2' - \varepsilon_2 \equiv \pi^{2(d-j)}\varepsilon_1(\varepsilon_3' - \varepsilon_3) \quad (\pi^{2d})$$

($N(a)$ denotes the norm from E to F). Furthermore, for x of this form, $d(xp_0,p_0) = d$.

We will now describe the $q^{4d}(1+\frac{1}{q^3})$ points at a distance d from p_0 in X by writing them as $g \times p_0$ with x in the above form.

a) $j = 0$; take $\varepsilon_1 = \varepsilon_3 = 0$ and $N(\varepsilon_2) \equiv -1 \ (\pi^{2d})$. We obtain $q^{2d}(1+\frac{1}{q})$ points this way.

b) $j = d$; take $\varepsilon_1 = (\overline{-\varepsilon_3/\varepsilon_2})$ so that 1) is satisfied and choose ε_2 so that $N(\varepsilon_2) \not\equiv -1 \ (\pi)$;

then ε_3 is taken to be a unit satisfying 2). There are $q^{4d}(1-\frac{1}{q^2}) - q^{4d-2}(1+\frac{1}{q}) = q^{4d}(1-\frac{1}{q}-\frac{2}{q^2})$ choices for ε_2 and $q^{2d}(1+\frac{1}{q})$ choices for ε_3. Dividing by q^{2d} to account for redundancies, we obtained $q^{4d}(1-\frac{3}{q^2}-\frac{2}{q^3})$ points in this way.

c) $j = 1,\dots,d-1$; take $\varepsilon_1 = (\overline{-\varepsilon_3/\varepsilon_2})$ so that 1) is satisfied and pick ε_2 and ε_3 so that $1 + \pi^{2(d-j)}N(\varepsilon_3) + N(\varepsilon_2) \equiv 0\ (\pi^{2d})$. We must require that $N(\varepsilon_2) \equiv -1\ (\pi^{2(d-j)})$ and $N(\varepsilon_2) \not\equiv -1\ (\pi^{2(d-j)+1})$. Then 2) is satisfied by $q^{2(d-j)}(1+\frac{1}{q})(q^{4j}-q^{4j-1})$ choices of ε_2 and $q^{2j}(1+\frac{1}{q})$ choices of ε_3. Dividing by q^{2j} to account for redundancies, we obtain $q^{2d+2j}(1+\frac{1}{q})^2(1-\frac{1}{q})$ $q^{2d+2j}(1-\frac{1}{q^2})(1+\frac{1}{q})$ points. Summing over j gives $(1+\frac{1}{q})(q^{2d-2}-1)q^{2d}$ points.

For each of the points of the form $p = xp_0$ with x in the above form, we may consider the points $w_i p$ where

$$w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In cases a) and c) this leads to new points (by a lengthy but straightforward calculation) and $gw_i xp_0$ is F-rational if gxp_0 is because ${}^t w_i w_i = 1$. Hence we have described a total of

$$q^{4d}\left(1-\frac{3}{q^2}-\frac{2}{q^3}\right) + 3q^{2d}\left(1+\frac{1}{q}\right) + 3\left(1+\frac{1}{q}\right)(q^{2d-2}-1)q^{2d} = q^{4d}\left(1+\frac{1}{q^3}\right)$$

points of the form $p = xp_0$ or $w_j xp_0$ such that gxp_0 is an F-rational point of X and $d(gxp_0, p_0) = d$. This must therefore be all $p \in X$ such that $d(gxp_0, p_0) = d$.

Let $\gamma = \begin{pmatrix} \gamma_1 & & \\ & \gamma_2 & \\ & & \gamma_3 \end{pmatrix}$ and let $\alpha_1, \alpha_2, \alpha_3$ be the standard roots. If $n = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$, then

$$\gamma^{-1}n^{-1}\gamma n = \begin{pmatrix} 1 & (1-\alpha_1^{-1}(\gamma))x & (1-\alpha_3^{-1}(\gamma))y + (\alpha_3^{-1}(\gamma) - \alpha_1^{-1}(\gamma))xz \\ 0 & 1 & (1-\alpha_2^{-1}(\gamma))z \\ 0 & 0 & 1 \end{pmatrix}$$

Assume that $g\gamma g^{-1} \in T_0$ and let

$$(1-\alpha_j^{-1}(\gamma)) = \pi^{m_j}\eta_j \qquad \eta_j \in \mathcal{O}_E^*.$$

If $x = an$, $a \in A$, $n \in N$, then γ fixes xp_0 if and only if $\gamma^{-1}n^{-1}\gamma n \in K \cap N$. Suppose that

$$a = \begin{pmatrix} \pi^d & & \\ & 1 & \\ & & \pi^{-d} \end{pmatrix} \qquad n = \begin{pmatrix} 1 & \pi^{-j}\varepsilon_1 & \pi^{-2d}\varepsilon_2 \\ 0 & 1 & \pi^{-j}\varepsilon_3 \\ 0 & 0 & 1 \end{pmatrix}$$

with n of the type described above. Then if γ fixes anp_0, the following inequality holds:

$$(**) \qquad \mathrm{val}(N(\varepsilon_2)+\pi^{2(d-j)}N(\varepsilon_3)-\pi^{2(d-j)+m_1-m_3}\eta_1\eta_3^{-1}N(\varepsilon_3)) \geq 2d - m_3.$$

Furthermore, it is easily checked that

i) $m_1 > m_3 \Rightarrow m_3 = m_2$

ii) $m_1 < m_3 \Rightarrow m_1 = m_2$

iii) $m_1 = m_3 \Rightarrow m_2 \geq m_1 = m_3$.

We consider the cases separately.

1) $m_1 > m_3$: then γ fixes anp_0 if and only if $2d < m_3$ and $0 \leq j \leq m_2$.

2) $m_1 < m_3$: then γ fixes anp_0 if and only if either

i) $2d - 2j + m_1 - m_3 \geq 2d - m_3$, $j \leq \frac{m_1}{2}$, $2d \leq m_3$, and $0 \leq j \leq m_2$, or

ii) $2d > m_3$, $2d - 2j + m_1 - m_3 = 0$ and $N(\varepsilon_2) + \pi^{m_3-m_1}N(\varepsilon_3) - \eta_1\eta_3^{-1}N(\varepsilon_3) \equiv 0 \ (\pi^{2d-m_3})$.

This occurs only when $m_1 \equiv m_3$ (2) and $d = [\frac{m_3}{2}] + 1, \ldots, \frac{m_1+m_3}{2}$.

3) $m_1 = m_3$: then γ fixes anp_0 if and only if either

i) $2d \leq m_3$ and $0 \leq j \leq m_1$ or

ii) $0 \leq d = j \leq m_1$ and $val(N(\varepsilon_2)+(1-\eta_1\eta_3^{-1}N(\varepsilon_3)) \geq 2d - m_3$.

When $d \neq j$, we have to consider the points $w_j x p_0$ for $j = 1,2$ also; this amounts to replacing γ by $w_j \gamma w_j^{-1}$ and using the above conditions.

Suppose first that $m_1 \neq m_3$ and let $m = \min(m_1, m_3)$, $n = \max(m_1, m_3)$. Then the number of fixed points is equal to

$$1 + \sum_{d=1}^{[\frac{m}{2}]} q^{4d}(1+\frac{1}{q^3}) + \sum_{d=[\frac{m}{2}]+1}^{[\frac{n}{2}]} q^{2d}\left(\sum_{j=1}^{[\frac{m}{2}]} q^{2j}(1-\frac{1}{q^2})(1+\frac{1}{q}) + (1+\frac{1}{q})\right) \qquad (\alpha)$$

We obtain:

$$F_0(\gamma) = \frac{-(q+1)-q^{4[\frac{m}{2}]+2}(q+1)}{q^4-1} + \frac{q^{m_1+m_3}}{q-1} \quad \text{if} \quad m_2 \not\equiv m_3 \qquad (2).$$

If $m_1 \equiv m_3$ (2) and $m_2 = m$, we obtain the number (α) plus an additional

$$(1+\frac{1}{q})^2 q^m \sum_{d=[\frac{n}{2}]+1}^{\frac{m+n}{2}} q^{2d} = (\frac{q+1}{q-1})(q^{n+2m}-q^{m+2[\frac{n}{2}]})$$

points. We obtain

$$F_0(\gamma) = \frac{-(q+1)-q^{4[\frac{m}{2}]+2}(q+1)}{q^4-1} - \frac{q^{m_1+m_3}}{q-1} + \frac{q^{m_1+m_2+m_3}(q+1)}{q-1} \quad \text{if} \quad m_1 \equiv m_3,\ m_2 = m.$$

The only remaining case is $m_2 > m_1 = m_3$. We obtain

$$\frac{-(q+1)-q^{4[\frac{m}{2}]+2}(q+1)}{q^4-1} + \frac{q^{m_1+m_2}}{q-1} \qquad \text{if} \quad m_2 \not\equiv m \quad (2)$$

$$\frac{-(q+1)-q^{4[\frac{m}{2}]+2}(q+1)}{q^4-1} - \frac{q^{m_1+m_2}}{q-1} + \frac{q^{m_1+m_2+m_3}(q+1)}{q-1} \quad \text{if} \quad m_2 \equiv m \quad (2).$$

The result is as stated in part a) of Proposition 1.

Now we have to compute the fixed points for $\gamma \in T_j'$, $j = 1,2,3$. We will see that the cases T_1 and T_2 can be easily reduced to the case of T_3, so we deal now with this case where we may assume that T_3 is associated to the cocycle

$a_\sigma = w\begin{pmatrix}\pi^{-1} & & \\ & 1 & \\ & & \pi\end{pmatrix}$ and that $gAg^{-1} = T$. Let $\mathfrak{A}_A$ be the apartment in X associated to A, and let $p_{1/2}$ be the unique s-vertex fixed by a_σ (we may assume that T stabilizes $p_{1/2}$). As in the previous case, we will describe the hs-vertices p such that gp is F-rational in the form xp_0 and then compute the fixed points of $g\gamma g^{-1}$ by determining whether or not γ fixes xp_0.

$$\text{Let}\quad a = \begin{pmatrix}\pi^{d} & & \\ & 1 & \\ & & \pi^{-d}\end{pmatrix} \quad\text{and}\quad n = \begin{pmatrix}1 & \pi^{-j}\varepsilon_1 & \pi^{-2d-1}\varepsilon_2 \\ 0 & 1 & \pi^{-j}\varepsilon_3 \\ 0 & 0 & 1\end{pmatrix}, \quad j = 0,1,\dots,d,$$

where $\varepsilon_j \in O_E^\times$ or $\varepsilon_j = 0$. The point $ganp_0$ is F-rational if and only if ${}^t(\overline{an})\begin{pmatrix} & & \pi \\ & 1 & \\ \pi^{-1} & & \end{pmatrix}(an) \in SL_3(O_E)$ which is equivalent to the requirements

1) $\pi^{-j}(\varepsilon_3+\bar{\varepsilon}_1\varepsilon_2) \in O_E$.

2) $\pi^{-2d-1}(1+\pi^{2d-2j+1}N(\varepsilon_3)+N(\varepsilon_2)) \in O_E$.

We always take $\varepsilon_1 = -(\overline{\varepsilon_3/\varepsilon_2})$.

a) $j = 0$: take $\varepsilon_1 = \varepsilon_3 = 0$ and $N(\varepsilon_2) \equiv -1(\pi^{2d+1})$. We obtain $q^{2d}(q+1)$ points in this way.

b) $0 < j \leq d$: choose ε_2 such that $N(\varepsilon_2)\ \ -1(\pi^{2d-2j+1})$ and $N(\varepsilon_2) \not\equiv -1(\pi^{2d-2j+2})$, and ε_3 satisfying 2). We obtain a total of $q^{2d+2j+1}(1-\frac{1}{q^2})(1+\frac{1}{q})$ points in this way.

Adding cases a) and b), we obtain a description for $q^{2d+1}(1+\frac{1}{q}) + q^{2d+1}(1+\frac{1}{q})(1-\frac{1}{q^2})\sum_{j=0}^{d} q^{2j} = q^{4d}(q+1)$ points. Furthermore, all points obtained in this way satisfy $d(p,p_{1/2}) = d + \frac{1}{2}$ (where $d(p,q)$ = half the number of chambers separating p and q). Since exactly $q^{4d}(q+1)$

points satisfy $d(p,p_{1/2}) = d + \frac{1}{2}$, this must be all of them.

To reduce the cases of T_1 and T_2 to T_3, we remark that γ fixes xp_0 if and only if $w_j\gamma w_j^{-1}$ fixes $w_j x p_0$. Choose g_j $(j=1,2)$ so that $g_j A g_j^{-1} = T_j$. Then the number of fixed points of $g_1\gamma g_1^{-1}$ equals the number for $g w_1 \gamma w_1^{-1} g^{-1}$ and the number for $g_2\gamma g_2^{-1}$ equals the number for $g w_2 \gamma w_2^{-1} g^{-1}$.

Suppose $g\gamma g^{-1} \in T_3$, and let m_1, m_2, m_3 be as before. Then if $g\gamma g^{-1}$ fixes anp_0 with an of the usual form so that $ganp_0$ is F-rational, the following inequality holds:

$$(*) \quad \mathrm{val}(N(\varepsilon_2)+\pi^{2d-2j+1}N(\varepsilon_3)-\eta_1\eta_3^{-1}\pi^{2d-2j+1+m_1-m_3}N(\varepsilon_3)) \geq 2d + 1 - m_3.$$

There are two cases to consider:

a) $m_1 \geq m_3$: then $2d \leq m_3 - 1$ and $0 \leq j \leq \min(m_1,m_2)$.

b) $m_1 < m_3$: (i) $0 \leq j \leq \min(m_1,m_2)$, $d \leq \frac{m_3-1}{2}$ and $j \leq \frac{m_1}{2}$ or

(ii) $2d + 1 > m_3$, $0 \leq j \leq \min(m_1,m_2)$, $2(d-j) + 1 = m_3 - m_1$ and $v(N\varepsilon_2+\pi^{2d-2k+1}N\varepsilon_3-\eta_1\eta_3^{-1}N\varepsilon_3) \geq 2d + 1 - m_3$

((ii) occurs only when $m_1 \not\equiv m_3(2)$.)

The number of fixed points in the two cases is

$$\text{a)} \quad \sum_{d=0}^{[\frac{m_3-1}{2}]} q^{4d}(q+1) = \frac{q+1}{q^4-1}\left(q^{4[\frac{m_3-1}{2}]+4}-1\right).$$

$$\text{b)} \quad \sum_{d=0}^{[\frac{m_1}{2}]} q^{4d}(q+1) + \sum_{d=[\frac{m_1}{2}]+1}^{[\frac{m_3-1}{2}]} q^{2d}((q+1) + \sum_{j=0}^{[\frac{m_1}{2}]} q^{2j}(q+1)(1-\frac{1}{q^2})) +$$

$$+\ (1+\frac{1}{q})^2 \sum_{d=[\frac{m_3-1}{2}]+1}^{\frac{m_1+m_3-1}{2}} q^{m_1+1+2d} \qquad (\text{if } m_1 \not\equiv m_3(2))$$

$$= \begin{cases} \dfrac{-(q+1)-q^2(q+1q^{4[\frac{m_1}{2}]}}{q^4-1} + \dfrac{q^{m_1+m_3}}{q-1} & \text{if } m_1 \equiv m_3(2) \\[2ex] \dfrac{-(q+1)-q^2(q+1)q^{4[\frac{m_1}{2}]}}{q^4-1} - \dfrac{q^{m_1+m_3}}{q-1} + \dfrac{q+1}{q-1}\, q^{2m_1+m_3} & \text{if } m_1 \not\equiv m_3(2). \end{cases}$$

To obtain the theorem for f_λ, $\lambda \neq 0$, it is necessary to compute $N_{hs}(\gamma)$ and $N_s(\gamma)$. It is known that

$$\Phi_H^{T/1}(\gamma, {}^H f_\lambda) = q^{\lambda+m_3}(1+\frac{1}{q}).$$

The next lemma follows from Lemma 1.

<u>Lemma</u> 3:

$$\Phi_H^{T/1}(\gamma, \boldsymbol{\varphi}^*(f_\lambda)) = \begin{cases} q^{2\lambda+m_3}(1+\frac{1}{q})((1+\frac{1}{q}+\frac{1}{q^2})-2q^{-(m_3+1)}) & \lambda \text{ even} \\[1ex] q^{2\lambda+m_3-1}(1+\frac{1}{q})(1-2q^{-m_3}) & \lambda \text{ odd}. \end{cases}$$

For $\gamma \in T_0$, regular, let $N_{hs}^{\kappa}(\gamma) = N_{hs}(\gamma) - N_{hs}(\gamma^{\delta_1}) - N_{hs}(\gamma^{\delta_2}) + N_{hs}(\gamma^{\delta_3})$ and $N_s^{\kappa}(\gamma) = N_s(\gamma) - N_s(\gamma^{\delta_1}) - N_s(\gamma^{\delta_2}) + N_s(\gamma^{\delta_3})$. The result is equivalent to the next proposition.

<u>Proposition</u> 3: Let $\gamma \in T_0$ be regular.

a) $N^K_{hs}(\gamma) = (-1)^{m_1+m_2} q^{m_1+m_2+m_3}(q^3+q^2)((1+\frac{1}{q}+\frac{1}{q^2})-2q^{-(m_3+1)})$

b) $N^K_s(\gamma) = (-1)^{m_1+m_2} q^{m_1+m_2+m_3}(q+1)(1-2q^{-m_3})$.

The computations required to verify this proposition can be carried out using the description of the fixed points of γ given in the proof of Proposition. According to Kottwitz's lecture, we can alternatively compute the number of s-vertices fixed by the stable conjugates of γ, for a simple argument shows that N_{hs} and N_s can be expressed in terms of the numbers of fixed s and hs-vertices. A computation of the number of fixed s-vertices will be given in subsequent notes.