Afternoon Seminar

THE STRUCTURE OF TRACE FORMULAS AND THEIR COMPARISON

R. Langlands

1. Credo. There are now many results on the trace formula and many ideas, so that we can now begin to look beyond the analytic difficulties and attempt to put it in a form suitable for applications. Since obstacles remain, some of the ideas are only tentative. One purpose of the afternoon seminar is to test them in particular cases where the difficulties can be overcome.

In this part of the seminar I want to review the general ideas, at first briefly; state the result towards which it seems all efforts are tending; and then, in the context of $U(3)$ or $SU(3)$, explain the ideas in more detail, and at least partially justify them. However much will be left for later.

I begin by recalling what we have available in the way of results and ideas, introducing some catchwords whose meaning it is one of my purposes to explain.

1. First and foremost, the trace formula of Arthur, both the ordinary and the twisted forms.

2. The observation that the twisted formula can be used for the transfer of automorphic representations from one group to another, due to Saito-Shintani for base change and to Jacquet for the Gelbart-Jacquet transfer from $SL(2)$ to $PGL(3)$.

3. Stabilization.

4. Shelstad's formalism of endoscopic groups in the twisted case.

5. For (3) and (4) one needs the transfer of orbital integrals to endoscopic groups and the fundamental lemma. These represent perhaps the major obstacle at present, but results of Shelstad, Kottwitz, Rogawski, and Kazhdan permit considerable confidence that the transfer is possible and the lemma valid.

6. Hierarchical structure of the trace formula and decomposition of measures.

pairing?

7. This hierarchical structure will be obtained by paring off the contributions from proper parabolic subgroups by a procedure that I refer to as Flicker's trick. It is the necessity of utilizing this device, whose value was first emphasized by Flicker, that forces us to modify the basic identity, using ${}_{\varepsilon}\sigma_1^2$ rather than σ_1^2.

8. The principle of cancellation of singularities. This is a suggestion of Arthur, who may feel that its elevation to the status of a principle is premature.

Recall that to obtain the trace formula we start from the basic identity (modified) and integrate both sides over $G\backslash\mathbb{G}^1$, obtaining on the left $J^T(\phi)$ and on the right $\Theta^T(\phi)$. They both depend on the parameter T and are both distributions in ϕ, in general non-invariant.

The fine $\mathfrak{o}$-expansion will - it is hoped - allow us to decompose $J^T(\phi)$ as a sum

$$J^T(\phi) = \sum_M J_M^T(\phi) \quad ,$$

the sum being over conjugacy classes of Levi subgroups of ε-invariant standard parabolics, and thus over associate classes of ε-invariant parabolics.

We will go into this decomposition in more detail later. For now there are only two points to remark:

(a) Both $J^T(\phi)$ and all $J^T_M(\phi)$ are polynomials in T. The degree of $J^T_M(\phi)$ is $\dim \mathfrak{a}^\varepsilon_M/\mathfrak{a}^\varepsilon_G$. In particular $J_G(\phi) = J^T_G(\phi)$ is independent of T.

(b) The larger M is the closer J^T_M is to being ε-invariant. In particular J_G is ε-invariant.

The distribution J_G will have a simple form. To describe it, it may be best to fix once and for all a finite set of places S containing all infinite places and all places ramified for G and to assume that outside of S, ϕ_v is the characteristic function of K_v divided by the measure of K_v. Thus we are assuming that

$$\phi(g) = \prod_v \phi_v(g_v)$$

and ϕ is determined by

$$\phi_S = \prod_{v \in S} \phi_v \quad .$$

Let $\mathcal{O} = \mathcal{O}_S$ be the set of conjugacy classes in $G(\mathbb{A}_S)$ with elliptic representatives in $G(\mathbb{Q})$. Then

$$J_G(\phi) = \sum_{\mathfrak{o}} c_{\mathfrak{o}} \int_{G_\gamma(\mathbb{A}_S \backslash G(\mathbb{A}_S)} \phi_S(g^{-1}\gamma g)\,dg \quad ,$$

γ in $G(\mathbb{Q})$ being a representative of the class $\mathcal{O}$.

There will be a similar decomposition

$$\Theta^T(\phi) = \sum_M \Theta_M^T(\phi)$$

and conditions (a) and (b) will be satisfied. To describe the form of $\Theta_G = \Theta_G^T$ we need some simple definitions.

Recalling the presence of ω in the definition of $R(\theta)$ (I now shift to the better afternoon notation replacing ε by θ) we agree to call an automorphic representation θ-invariant if it satisfies

$$(h \longrightarrow \pi(h)) \sim (h \longrightarrow \omega(\theta^{-1}(h))\pi(\theta^{-1}(h))$$

and of type ξ if

$$\pi(z) = \xi(z)I , \qquad z \in \mathbb{Z}_0 .$$

Two θ-invariant automorphic representations π, π' of type ξ will be called projectively equivalent if

$$\pi' = \pi_\chi = \pi \otimes \chi ,$$

χ being a character of G trivial on $\mathbb{Z}_0$ and satisfying

$$\chi(h) = \chi(\theta^{-1}(h)) .$$

Denote the set of such characters by $\mathfrak{X}(\theta, \mathbb{Z}_0) = \mathfrak{X}$.

There will be a countable set Y of projective equivalence classes such that

(c) $$\Theta_G(\phi) = \sum_{y \in Y} d(y) \int_{\mathfrak{X}} \operatorname{tr}(\pi_\chi(\phi)\pi_\chi(\theta))\,d\chi$$

π denoting some arbitrary element of y. The meaning of $\pi_\chi(\theta)$ will be explained later. It should also be observed that the numbers $d(y)$ may not be positive and, indeed, may not even be real.

As observed in Shelstad's lectures we proceed now in two steps. We first stabilize the ordinary trace formula for quasi-split groups inductively. This is going to lead us from the formula

$$\sum_M J_M^T(\phi) = \sum_M \Theta_M^T(\phi)$$

to a formula

$$\sum_M SJ_M^T(\phi) = \sum_M S\Theta_M^T(\phi) \quad .$$

If H is a cuspidal endoscopic group for G then, G being quasi-split every associate class $\mathfrak{P}$ for H determines one for G. Namely an element of the class has a Levi factor M and the center of M has a maximal split torus A which can be transferred to a torus A' in G. The centralizer of A' is the Levi factor M' of a parabolic whose class $\mathfrak{P}'$ is the image of $\mathfrak{P}$. We write $\mathfrak{P} = \mathfrak{P}_H$ or $M = M_H$ and $\mathfrak{P}' = \mathfrak{P}_G$ or $M' = M_H$ and write $\mathfrak{P}_H \longrightarrow \mathfrak{P}_G$ or $M_H \longrightarrow M_G$.

We set

$$SJ_{M_G}^T(\phi) = J_{M_G}^T(\phi) - {\sum_H}' \iota(G,H) \sum_{M_H \to M_G} SJ_{M_H}^T(\phi^H) \quad ,$$

the prime indicating that we sum over all cuspidal endoscopic groups (or better data) except G itself and ϕ^H being a function associated to ϕ by transfer of orbital integrals.

In the same way we define

$$S\Theta^T_{M_G}(\phi) = \Theta^T_{M_G}(\phi) - \sum_H{}' \iota(G,H)\sum_{M_H\to M_G} S\Theta^T_{M_H}(\phi^H) \quad .$$

Since all the H are cuspidal they have the same split center as G. Thus (c) will continue to hold for $S\Theta_G(\phi) = S\Theta^T_G(\phi)$ and (a) and (b) will hold for the $SJ^T_M(\phi)$ and the $S\Theta^T_M(\phi)$. Moreover the inductive definition has been so made that

$$\sum_M SJ^T_M(\phi) = \sum S\Theta^T_M(\phi)$$

holds.

We are interested in the distribution $S\Theta_G$ and we would like to know in particular that it is stable. In this case $G = G^*$ is already quasi-split, but $\phi^* = \phi^{G^*}$ is not necessarily equal to ϕ. It need only have the same stable orbital integrals as ϕ, and the assertion that $S\Theta_G$ is stable is the assertion that

$$S\Theta_G(\phi^*) = S\Theta_G(\phi)$$

for all choices of ϕ^*.

The plan of attack, and we will see how it works out in particular cases, is to show that SJ_G, which is a sum of orbital integrals, is in fact a sum of stable orbital integrals and thus that

$$SJ_G(\phi^*) = SJ_G(\phi) \quad .$$

This leaves us with the equality of

$$\text{(A)} \qquad \sum_{M \neq G} (SJ_M^T(\phi) - SJ_M^T(\phi^*)) - \sum_{M \neq G} (S\Theta_M^T(\phi) - S\Theta_M^T(\phi^*))$$

and

$$\text{(B)} \qquad S\Theta_G(\phi) - S\Theta_G(\phi^*) \quad .$$

The idea is to add one unramified place v_0 to S, working then with $S' = S \cup \{v_0\}$ rather than S, and to take $\phi_{v_0} = \phi^*_{v_0}$ to be an element of the Hecke algebra $\mathcal{H} = \mathcal{H}(G, \mathbb{Q}_v)$. We then prove the equality of (A) and (B) by treating them as linear forms on $\mathcal{H}$, substituting finally the identity of the Hecke algebra for ϕ_{v_0} to obtain the identity desired.

This is an argument already used to prove base change for $GL(2)$. The Hecke algebra has an involution $\phi_{v_0} \longrightarrow \tilde{\phi}_{v_0} : g \longrightarrow \bar{\phi}_{v_0}(g^{-1})$. The linear forms (A) and (B) may be represented by measures on the set $\mathfrak{A}$ of homomorphisms λ of the Hecke algebra into $\mathbb{C}$ satisfying

$$\lambda(\phi_{v_0}) = \lambda(\tilde{\phi}_{v_0})$$

for all ϕ_{v_0}.

It follows from (c), applied to $S\Theta_G$, that the measure attached to (B) is of Lebesgue type and dimension equal to $\dim \mathfrak{X}$, which is often zero, whereas one can expect to prove that (A) is a sum of measures of Lebesgue type and dimension between $\dim \mathfrak{X}+1$ and $\dim \mathfrak{X} + \dim \mathfrak{a}_{M_0}/\mathfrak{a}_G$. The conclusion must be that (A) and (B) are separately 0.

At the moment I only have a clear idea how to do this when G is of rational rank 1 and quasi-split, but this will do for the purposes of this seminar. In this case, as we shall see

$$SJ^T_{M_0}(\phi) = SJ_{M_0}(\phi^T_{M_0})$$

where $\phi^T_{M_0}$ is a function on $\mathbb{M}_0$. In the same way

$$SJ^T_{M_0}(\phi^*) = SJ_{M_0}(\phi^{*T}_{M_0}) \quad .$$

Neither ϕ^T_M nor ϕ^{*T}_M will be smooth in general. However the difference $\phi^T_{M_0} - \phi^{*T}_{M_0}$ will be smooth (cancellation of singularities). So we can apply the trace formula on M_0 to the difference obtaining

$$SJ^T_{M_0}(\phi) - SJ^T_{M_0}(\phi^*) = S\Theta_{M_0}(\phi^T_{M_0} - \phi^{*T}_{M_0}) \quad .$$

Then it will be easy to show that all three linear forms

$$\phi_{v_0} \longrightarrow S\Theta_{M_0}(\phi^T_{M_0} - \phi^{*T}_{M_0})$$
$$\phi_{v_0} \longrightarrow S\Theta^T_{M_0}(\phi)$$
$$\phi_{v_0} \longrightarrow S\Theta^T_{M_0}(\phi^*)$$

are given by measures of Lebesgue type and dimension equal to $\dim \mathfrak{X}+1$.

The stable trace $S\Theta_G(\phi)$ once defined we can at least state what appears to be our final goal. Thus for any group G and any θ we want to show that

$$* \qquad \boxed{\Theta_G(\phi) = \sum_H \iota(G,\theta,H)S\Theta_H(\phi^H)} \quad .$$

The sum is over all cuspidal endoscopic groups for the pair (G, θ).

The proof will of course be about the same. One will show directly, or almost directly, that

$$\text{(C)} \qquad J_G(\phi) = \sum_H \iota(G,\theta,H)SJ_H(\phi^H)$$

and then apply cancellation of singularities and decomposition of measures.

At least one extra difficulty will arise. For example, for the ordinary trace formula the term

$$\operatorname{meas}(G\backslash\mathbb{G}^1)\phi(1)$$

will occur on the left side of (C) and the term

$$\operatorname{meas}(G^*\backslash\mathbb{G}^{*1})\phi^{G^*}(1)$$

will occur on the right, G^* being the quasi-split form of G. The relation between $\phi(1)$ and $\phi^{G^*}(1)$ will be simple, presumably

$$\phi^{G^*}(1) = \phi(1) \quad .$$

Thus to achieve cancellation we will need to show that

$$\mathrm{meas}(G\backslash\mathbb{G}^1) = \mathrm{meas}(G^*\backslash\mathbb{G}^{*1}) \quad .$$

In some cases this will be known from results on Tamagawa numbers, but it will be preferable to derive it from the trace formula itself, by an elaboration of the measure-theoretic arguments.

Shelstad has explained in her lectures the meaning of the identity * for the twisted trace formula arising from base change for U(1). In this case there are several endoscopic groups, all isomorphic to U(1). Some other cases of the identity are implicit in the literature. If G is the multiplicative group of a quaternion algebra the only cuspidal endoscopic group is GL(2). So the identity is quite simple, and was used in effect in §16 of Jacquet-Langlands. In general if G is the multiplicative group of a division algebra of degree n^2 then there is only one cuspidal endoscopic group, namely GL(n) and weak forms of the identity have been used by Deligne-Kazhdan and Rogawski. For SL(2) there are many cuspidal endoscopic groups, SL(2) itself and all anisotropic tori. For base change for GL(n) there is only one cuspidal endoscopic group and that is GL(n). For U(3) or SU(3) there are, as we have seen, more than one cuspidal endoscopic group. The consequences of this are, as we shall see, quite fascinating.

There are two papers which explore, in a somewhat tentative way but for general groups, the consequences and meaning of * for the ordinary trace formula:

1. J. Arthur, On some problems suggested by the trace formula.

2. R. Kottwitz, Stable trace formula: cuspidal tempered terms.

Our purpose at the moment is however to concentrate on U(3) and SU(3) and to see whether the plan of attack outlined here is feasible or nothing but a pipe dream.

AFTERNOON SEMINAR

Orbital integrals of spherical functions

J. Rogawski

Let E/F be an unramified quadratic extension of p-adic fields. Let G be the special unitary group in 3 variables with respect to E/F defined by the form $\Phi = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$. Let K be the maximal compact subgroup of integer matrices in G. Let

$$H = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \in G \right\}$$

and set $K_H = H \cap K$. The map

$$\varphi : {}^L H \hookrightarrow {}^L G$$

gives a homomorphism $\varphi^* : \mathcal{H}_G \to \mathcal{H}_H$ where $\mathcal{H}_G$ and $\mathcal{H}_H$ are the Hecke algebras of G and H with respect to K and K_H respectively.

There is a unique stable class $\{T\}_{st}$ of CSG's of G which consists of elliptic tori that split over E. In these notes, we show that the transfer factors $\Delta_G(\gamma)$, $\Delta_H(\gamma)$, $\tau(\gamma)$ defined in previous notes do in fact give a transfer of orbital integrals for spherical functions and the class $\{T\}_{st}$. The remaining classes of elliptic CSG's will be dealt with in a lecture of Kottwitz.

The conjugacy classes within $\{T\}_{st}$ are parametrized by $H^1(\mathfrak{g},T)$, where $\mathfrak{g} = \{1,\sigma\}$ is the Galois group of E/F. Let A be the diagonal subgroup of G and let π be a fixed prime element in F. We have $H^1(\mathfrak{g},T) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and the four conjugacy classes within $\{T\}_{st}$ are defined by the four cocycles with values in the normalizer $N(A)$ of A in $G(\overline{F})$:

$$a_{\sigma,0} = w \qquad a_{\sigma,1} = w\begin{pmatrix}\pi^{-1} & & \\ & \pi & \\ & & 1\end{pmatrix} \qquad a_{\sigma,2} = w\begin{pmatrix}1 & & \\ & \pi^{-1} & \\ & & \pi\end{pmatrix} \qquad a_{\sigma,3} = w\begin{pmatrix}\pi^{-1} & & \\ & 1 & \\ & & \pi\end{pmatrix}.$$

To obtain a representative for the conjugacy class defined by a cocycle a_σ, we write $a_\sigma = g^{-1}\sigma(g)$ for some $g \in G(\overline{F})$ and take $T = gAg^{-1}$ (T is defined over F, although $\gamma \to g\gamma g^{-1}$, $\gamma \in A$, is not defined over F).

Let T_i be a representative for the class defined by the cocycle $a_{\sigma,j}$. We choose T_0 and T_3 inside H (this is possible since $a_{\sigma,0}$ and $a_{\sigma,3}$ lie in H). Let $\delta_i \in G(\overline{F})$ be such that $\delta_i T_0 \delta_i^{-1} = T_i$. We define the unstable orbital integral for $\gamma \in T_0'$:

$$\Phi^{T/k}(\gamma,f) = \Phi(\gamma,f) - \Phi(\gamma^{\delta_1},f) - \Phi(\gamma^{\delta_2},f) + \Phi(\gamma^{\delta_3},f)$$

where $\Phi(x,f) = \int_G f(g^{-1}xg)dg$ for x elliptic regular and dg is chosen so that $\mathrm{meas}(K) = 1$. For functions on H, we define

$$\Phi_H^{T/1}(\gamma,f) = \Phi_H(\gamma,f) + \Phi_H(\gamma^{\delta_3},f)$$

where the measure on H is normalized so that $\mathrm{meas}(K_H) = 1$. We want to prove the following.

<u>Theorem</u>: For all $\gamma \in T_0'$ and $f \in \mathcal{H}_G$,

$$\tau(\gamma)\Delta_G(\gamma)\Phi^{T/\kappa}(\gamma,f) = \Delta_H(\gamma)\Phi_H^{T/1}(\gamma,\varphi^*(f)).$$

The map φ^* is defined in terms of the Satake transform. For $\lambda \in \mathbb{Z}$, set

$$a(\lambda) = \begin{pmatrix} \pi^{\lambda} & & \\ & 1 & \\ & & \pi^{-\lambda} \end{pmatrix}$$

$$F_f(\lambda) = \Delta(a(\lambda)) \int_{A\backslash G} f(g^{-1}a(\lambda)g)dg.$$

The Satake transform $f^{\vee}$ of $f \in \mathcal{H}_G$ may be viewed as a function on elements $\gamma \times Fr \in {}^L A$ where $\gamma \in {}^L A^0$ and Fr is a Frobenius element. As such, it can be written in the form

$$f^{\vee}(\gamma \times Fr) = \sum_{\lambda \in \mathbb{Z}} F_f(\lambda)\lambda(\gamma)$$

where $\lambda(\gamma)$ means $\lambda\alpha_3^{\vee}(\gamma)$ and $\alpha_3^{\vee}$ is the co-root corresponding to the root α_3. Let $\eta_j = m\alpha_3^{\vee} + m(\alpha_3^{\vee})^{-1}$. Since F_f is invariant under the Weyl group, $f^{\vee}$ is a finite linear combination of the η_j $(j=0,1,\ldots)$ and $\mathcal{H}_G$ is isomorphic to the algebra spanned by the η_j. Let ${}^H\eta_j$ denote η_j regarded as an element in the range of the Satake transform of $\mathcal{H}_H$. By definition of φ, we have:

$$\varphi\left(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} \times Fr\right) = \begin{pmatrix} a & & \\ & 1 & \\ & & -a^{-1} \end{pmatrix} \times Fr$$

from which it follows that $\varphi^*(\eta_j) = (-1)^j\, {}^H\eta_j$. Set

$$f_\lambda = \text{char. fn. of } Ka(\lambda)K$$

$${}^H f_\lambda = \text{char. fn. of } K_H a(\lambda) K_H.$$

The next lemma follows easily, either by direct calculation or by MacDonald's formula.

<u>Lemma</u> 1:

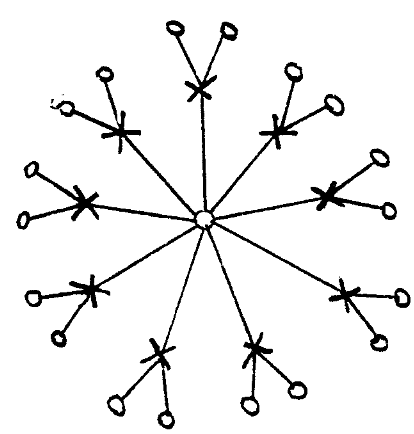
1) $$f^{v}_{\lambda} = q^{2\lambda}\eta_{\lambda} + q^{2\lambda}(1-\frac{1}{q^3}) \sum_{\substack{j\equiv\lambda(2)\\ 0\le j<\lambda}} \eta_j + q^{2\lambda-1}(1-\frac{1}{q}) \sum_{\substack{j\not\equiv\lambda(2)\\ 0\le j<\lambda}} \eta_j \qquad (\lambda>0)$$

2) $${}^{H}f^{v}_{\lambda} = q^{\lambda}\eta_{\lambda} + q^{\lambda}(1-\frac{1}{q}) \sum_{j=0}^{\lambda-1} \eta_j \qquad (\lambda>0)$$

3) $$\varphi^{*}(f_{\lambda}) = (-1)^{\lambda}q^{\lambda}\, {}^{H}f_{\lambda} + (1-\frac{1}{q^2}) \sum_{j=0}^{\lambda-1} (-1)^{j}q^{2\lambda-j}\, {}^{H}f_{j}. \qquad (\lambda>0).$$

To prove the theorem, we shall calculate $\Phi(\gamma,f_{\lambda})$, $\Phi(\gamma^{\delta_j},f_{\lambda})$ for all λ and check that the values we obtain compare favorably with the known values of $\Phi^{T/1}_{H}(\gamma,{}^{H}f_{\lambda})$ through the formula 3) above.

As explained in Kottwitz's lecture, $\Phi(\gamma,f_{\lambda})$ is given combinatorially in terms of the Bruhat-Tits building associating to G. The building X is a tree with two G-conjugacy classes of vertices. We call the vertices conjugate to the vertex defined by K hs-vertices and call the remaining vertices s-vertices. Each hs-vertex has q^3+1 s-neighbors and each s-vertex has $q+1$ hs-neighbors. Thus, for $q=2$, the tree is as follows:



where the $\circ$ are hs-vertices and the x are s-vertices. For vertices $p,q \in X$, let $d(p,q)$ denote one-half the number of chambers that lie on a geodesic joining p and q; a chamber is an edge $\circ$—x. The next lemma is a trivial calculation.

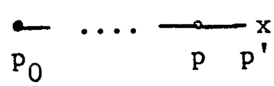
<u>Lemma</u>: For γ elliptic regular,

$$\Phi(\gamma,f_\lambda) = F_\lambda(\gamma)$$

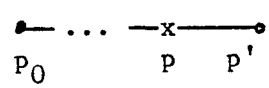
where $F_\lambda(\gamma)$ is the number of hs-vertices $p \in X$ such that $d(\gamma p,p) = \lambda$.

Let T be an elliptic Cartan subgroup. Then T is contained in some maximal compact subgroup of G and thus T fixes some vertex $p_0 \in X$. The set $\mathrm{Fix}(\gamma)$ of fixed points of an element $\gamma \in T$ is a connected neighborhood of p_0. If γ is regular, $\mathrm{Fix}(\gamma)$ is finite and $F_0(\gamma) = \Phi(\gamma,f_0)$.

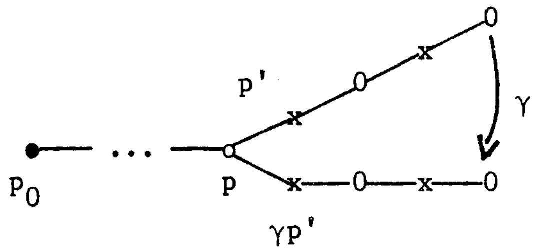
Let $N_{hs}(\gamma)$ be the number of paths in X of the following type:



The path begins at p_0, ends at an s-vertex p' which is not fixed by γ, but $p \in \mathrm{Fix}(\gamma)$. Similarly, let $N_s(\gamma)$ be the number of paths



with $p \in \mathrm{Fix}(\gamma)$ and $p' \notin \mathrm{Fix}(\gamma)$. As the following diagram indicates:



$F_\lambda(\gamma)$ is determined by $N_{hs}(\gamma)$ for λ even, $\lambda \neq 0$ and by $N_s(\gamma)$ for λ odd. The next lemma follows easily from the structure of the tree.

Lemma 2:

$$F_\lambda(\gamma) = \begin{cases} q^{2\lambda-3} N_{hs}(\gamma) & \lambda \text{ even} \\ q^{2\lambda-2} N_s(\gamma) & \lambda \text{ odd.} \end{cases}$$

We first calculate $F_0(\gamma)$. Let $\gamma_1,\gamma_2,\gamma_3$ be the eigenvalues of $\gamma \in T_0$. Since $T_0 \subset H$, we may choose γ_2 so that

$$\gamma = \begin{pmatrix} * & 0 & * \\ 0 & \gamma_2 & 0 \\ * & 0 & * \end{pmatrix} .$$

Let $\mathrm{val} : E^* \to \mathbb{Z}$ be the valuation map and set $m_j = \mathrm{val}(1-\alpha_j^{-1}(\gamma))$ where $\alpha_1(\gamma) = \gamma_1/\gamma_2$, $\alpha_2(\gamma) = \gamma_2/\gamma_3$, and $\alpha_3(\gamma) = \gamma_1/\gamma_3$ (the ambiguity in the labelling of γ_1 and γ_3 will not matter).

Proposition 1: Let $\gamma \in T_0$ be regular. Set $m = \min(m_1,m_2,m_3)$, $n = \max(m_1,m_2,m_3)$, and let

$$\delta(\gamma) = \begin{cases} 1 & \text{if } m \equiv n \quad (2) \\ 0 & \text{if } m \not\equiv n \quad (2). \end{cases}$$

a) $F_0(\gamma)$ is equal to

$$\frac{-(q+1)}{q^4-1} - q^{4[\frac{m}{2}]}\frac{q^2(q+1)}{q^4-1} - (-1)^{m+n}q^{m+n}\frac{1}{q-1} + \delta(\gamma)\frac{q^{m_1+m_2+m_3}(q+1)}{q-1}.$$

b) $F_0(\gamma^{\delta_3})$ is given by the following formulas

(i) If $m_1 \neq m_2$, then $F_0(\gamma^{\delta_3}) = (q^{4[\frac{m-1}{2}]+4}-1)\frac{q+1}{q^4-1}$

(ii) If $m_1 = m_2$, then $F_0(\gamma^{\delta_3})$ is equal to

$$\frac{-(q+1)}{q^4-1} - q^{4[\frac{m}{2}]}\frac{q^2(q+1)}{q^4-1} + (-1)^{m+n}q^{m+n}\frac{1}{q-1} + (1-\delta(\gamma))\frac{q^{m_1+m_2+m_3}(q+1)}{q-1}.$$

With this proposition, we can compute $\Phi^{T/\kappa}(\gamma,f_0)$. If m_1,m_2,m_3 are the integers attached to γ, set $F_j(m_1,m_2,m_3) = F_0(\gamma^{\delta_j})$, where we let $\delta_0 = 1$. The proof of the proposition shows that $F_2(m_1,m_2,m_3) = F_3(m_3,m_2,m_1)$ and $F_1(m_1,m_2,m_3) = F_3(m_1,m_3,m_2)$. We must calculate

$$F_0(m_1,m_2,m_3) - F_3(m_3,m_2,m_1) - F_3(m_1,m_3,m_2) + F_3(m_1,m_2,m_3).$$

The three possibilities are

(a) $m_1 > m_2 = m_3$ (b) $m_3 > m_1 = m_2$ (c) $m_2 \geq m_1 = m_3$.

Case (a): We are reduced to $F_0(m_1,m_2,m_3) - F_3(m_3,m_2,m_1)$. After cancelling off equal terms, we get

$$(-1)^{m_1+m_2} q^{m_1+m_2+m_3}\left(\frac{q+1}{q-1}\right) - 2(-1)^{m_1+m_2} q^{m_1+m_2}\left(\frac{1}{q-1}\right).$$

Case (b): The constant terms cancel and the highest order term is $q^{m_1+m_2+m_3}(\frac{q+1}{q-1})$. The intermediate terms are as follows:

$$F_0(m_1,m_2,m_3) \; : \; - q^{4[\frac{m}{2}]}\left(\frac{q^2(q+1)}{q^4-1}\right) - (-1)^{m+n} q^{m+n}\left(\frac{1}{q-1}\right)$$

$$- F_3(m_3,m_2,m_1) \; : \; - q^{4[\frac{m-1}{2}]+4}\left(\frac{q+1}{q^4-1}\right)$$

$$- F_3(m_1,m_3,m_2) \; : \; - q^{4[\frac{m-1}{2}]+4}\left(\frac{q+1}{q^4-1}\right)$$

$$F_3(m_1,m_2,m_3) \; : \; - q^{4[\frac{m}{2}]}\left(\frac{q^2(q+1)}{q^4-1}\right) + (-1)^{m+n} q^{m+n}\left(\frac{1}{q-1}\right).$$

The sum of these intermediate terms is

$$\frac{-2(q+1)}{q^4-1}\left(q^{4[\frac{m}{2}]+2} + q^{4[\frac{m-1}{2}]+4}\right) = \frac{-2q^{2m}(q+1)}{q-1} = \frac{-2q^{m_1+m_2}(q+1)}{q-1}.$$

Case (c): We are reduced to $F_0(m_1,m_2,m_3) - F_3(m_1,m_3,m_2)$ and the answer is the same as case (a). To summarize, we have the next proposition.

Proposition 2: For $\gamma \in T_0$ regular,

$$\Phi^{T/\kappa}(\gamma,f_0) = (-1)^{m_1+m_2}\left(\frac{1}{q-1}\right)\left(q^{m_1+m_2+m_3}(q+1) - 2q^{m_1+m_2}\right).$$

This Proposition gives the theorem for $f_0 = f$; it is clear that $\tau(\gamma) = (-1)^{m_1+m_2}$ and it is known that $\Phi_H^{T/1}(\gamma, {}^H f_0)$ is equal to $(q-1)^{-1}(q^{m_3}(q+1)-2)$.

<u>Proof of Proposition</u> 1: Since T_0 splits over E, we may choose $g \in SL_3(\mathcal{O}_E)$ so that $gA(E)g^{-1} = T_0(E)$, where $\mathcal{O}_E$ is the ring of integers in E. Suppose that $\gamma \in A(E)$ and $g\gamma g^{-1} \in T_0$. Then a vertex p is fixed by γ if and only if $g\gamma g^{-1}$ fixes gp. Let p_0 be the vertex associated to K and assume that $T_0 \subset K$, which we may up to conjugacy. An hs-vertex of X is of the form gxp_0 for some $x \in SL_3(E)$; gxp_0 lies in X if and only if $\sigma(gxp_0) = gxp_0$. Now, $\sigma(gxp_0) = gxp_0$ if and only if $x^{-1}(g^{-1}\sigma(g))\sigma(x) \in SL_3(\mathcal{O}_E)$ and since $g^{-1}\sigma(g) = w$, this is the condition

$$(*) \qquad {}^t\bar{x}x \in SL_3(\mathcal{O}_E).$$

By the Iwasawa decomposition, we may choose x upper-triangular. Let

$$x = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} a,b,c \in E^* \\ x,y,z \in E. \end{matrix}$$

A calculation shows that this x satisfies $(*)$ if the following six quantities lie in $\mathcal{O}_E$:

1) $a\bar{a}$
2) $a\bar{a}x$
3) $a\bar{a}y$
4) $b\bar{b} + a\bar{a}x\bar{x}$
5) $b\bar{b}z + a\bar{a}y\bar{x}$
6) $c\bar{c} + b\bar{b}z\bar{z} + a\bar{a}y\bar{y}$.

Now let

$$x = \begin{pmatrix} \pi^{d} & & \\ & 1 & \\ & & \pi^{-d} \end{pmatrix} \begin{pmatrix} 1 & \pi^{-j}\varepsilon_1 & \pi^{-2d}\varepsilon_2 \\ 0 & 1 & \pi^{-j}\varepsilon_3 \\ 0 & 0 & 1 \end{pmatrix}$$

where d = a non-negative integer, $0 \leq j \leq d$, and the ε_j are zero or lie in $\mathcal{O}_E^*$. Then x satisfies (*) if and only if the two conditions

1) $\pi^{-j}(\varepsilon_3+\bar{\varepsilon}_1\varepsilon_2) \in \mathcal{O}_E$

2) $\pi^{-2d}(1+\pi^{2d-2j}N(\varepsilon_3)+N(\varepsilon_2)) \in \mathcal{O}_E$

are satisfied, and if x,x' are of this form, $xp_0 = x'p_0$ only when

$$\varepsilon_1 = \varepsilon_1' \ (\pi^j) \qquad \varepsilon_3 \equiv \varepsilon_3' \ (\pi^j) \qquad \varepsilon_2' - \varepsilon_2 \equiv \pi^{2(d-j)}\varepsilon_1(\varepsilon_3' - \varepsilon_3) \quad (\pi^{2d})$$

($N(a)$ denotes the norm from E to F). Furthermore, for x of this form, $d(xp_0,p_0) = d$.

We will now describe the $q^{4d}(1+\frac{1}{q^3})$ points at a distance d from p_0 in X by writing them as $g \times p_0$ with x in the above form.

a) $j = 0$; take $\varepsilon_1 = \varepsilon_3 = 0$ and $N(\varepsilon_2) \equiv -1 \ (\pi^{2d})$. We obtain $q^{2d}(1+\frac{1}{q})$ points this way.

b) $j = d$; take $\varepsilon_1 = (\overline{-\varepsilon_3/\varepsilon_2})$ so that 1) is satisfied and choose ε_2 so that $N(\varepsilon_2) \not\equiv -1 \ (\pi)$;

then ε_3 is taken to be a unit satisfying 2). There are $q^{4d}(1-\frac{1}{q^2}) - q^{4d-2}(1+\frac{1}{q}) = q^{4d}(1-\frac{1}{q}-\frac{2}{q^2})$ choices for ε_2 and $q^{2d}(1+\frac{1}{q})$ choices for ε_3. Dividing by q^{2d} to account for redundancies, we obtained $q^{4d}(1-\frac{3}{q^2}-\frac{2}{q^3})$ points in this way.

c) $j = 1,\ldots,d-1$; take $\varepsilon_1 = (\overline{-\varepsilon_3/\varepsilon_2})$ so that 1) is satisfied and pick ε_2 and ε_3 so that $1 + \pi^{2(d-j)}N(\varepsilon_3) + N(\varepsilon_2) \equiv 0\ (\pi^{2d})$. We must require that $N(\varepsilon_2) \equiv -1\ (\pi^{2(d-j)})$ and $N(\varepsilon_2) \not\equiv -1\ (\pi^{2(d-j)+1})$. Then 2) is satisfied by $q^{2(d-j)}(1+\frac{1}{q})(q^{4j}-q^{4j-1})$ choices of ε_2 and $q^{2j}(1+\frac{1}{q})$ choices of ε_3. Dividing by q^{2j} to account for redundancies, we obtain $q^{2d+2j}(1+\frac{1}{q})^2(1-\frac{1}{q})$ $q^{2d+2j}(1-\frac{1}{q^2})(1+\frac{1}{q})$ points. Summing over j gives $(1+\frac{1}{q})(q^{2d-2}-1)q^{2d}$ points.

For each of the points of the form $p = xp_0$ with x in the above form, we may consider the points $w_i p$ where

$$w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In cases a) and c) this leads to new points (by a lengthy but straightforward calculation) and $gw_i xp_0$ is F-rational if gxp_0 is because ${}^t w_i w_i = 1$. Hence we have described a total of

$$q^{4d}(1-\frac{3}{q^2}-\frac{2}{q^3}) + 3q^{2d}(1+\frac{1}{q}) + 3(1+\frac{1}{q})(q^{2d-2}-1)q^{2d} = q^{4d}(1+\frac{1}{q^3})$$

points of the form $p = xp_0$ or $w_j xp_0$ such that gxp_0 is an F-rational point of X and $d(gxp_0,p_0) = d$. This must therefore be all $p \in X$ such that $d(gxp_0,p_0) = d$.

Let $\gamma = \begin{pmatrix} \gamma_1 & & \\ & \gamma_2 & \\ & & \gamma_3 \end{pmatrix}$ and let $\alpha_1,\alpha_2,\alpha_3$ be the standard roots. If $n = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$, then

$$\gamma^{-1}n^{-1}\gamma n = \begin{pmatrix} 1 & (1-\alpha_1^{-1}(\gamma))x & (1-\alpha_3^{-1}(\gamma))y + (\alpha_3^{-1}(\gamma) - \alpha_1^{-1}(\gamma))xz \\ 0 & 1 & (1-\alpha_2^{-1}(\gamma))z \\ 0 & 0 & 1 \end{pmatrix}$$

Assume that $g\gamma g^{-1} \in T_0$ and let

$$(1-\alpha_j^{-1}(\gamma)) = \pi^{m_j}\eta_j \qquad \eta_j \in \mathcal{O}_E^*.$$

If $x = an$, $a \in A$, $n \in N$, then γ fixes xp_0 if and only if $\gamma^{-1}n^{-1}\gamma n \in K \cap N$. Suppose that

$$a = \begin{pmatrix} \pi^d & & \\ & 1 & \\ & & \pi^{-d} \end{pmatrix} \qquad n = \begin{pmatrix} 1 & \pi^{-j}\varepsilon_1 & \pi^{-2d}\varepsilon_2 \\ 0 & 1 & \pi^{-j}\varepsilon_3 \\ 0 & 0 & 1 \end{pmatrix}$$

with n of the type described above. Then if γ fixes anp_0, the following inequality holds:

$$(**) \qquad \mathrm{val}(N(\varepsilon_2)+\pi^{2(d-j)}N(\varepsilon_3)-\pi^{2(d-j)+m_1-m_3}\eta_1\eta_3^{-1}N(\varepsilon_3)) \geq 2d - m_3.$$

Furthermore, it is easily checked that

i) $m_1 > m_3 \Rightarrow m_3 = m_2$

ii) $m_1 < m_3 \Rightarrow m_1 = m_2$

iii) $m_1 = m_3 \Rightarrow m_2 \geq m_1 = m_3$.

We consider the cases separately.

1) $m_1 > m_3$: then γ fixes anp_0 if and only if $2d < m_3$ and $0 \leq j \leq m_2$.

2) $m_1 < m_3$: then γ fixes anp_0 if and only if either

i) $2d - 2j + m_1 - m_3 \geq 2d - m_3$, $j \leq \frac{m_1}{2}$, $2d \leq m_3$, and $0 \leq j \leq m_2$, or

ii) $2d > m_3$, $2d - 2j + m_1 - m_3 = 0$ and $N(\varepsilon_2) + \pi^{m_3-m_1}N(\varepsilon_3) - \eta_1\eta_3^{-1}N(\varepsilon_3) \equiv 0\ (\pi^{2d-m_3})$.

This occurs only when $m_1 \equiv m_3\ (2)$ and $d = [\frac{m_3}{2}] + 1, \ldots, \frac{m_1+m_3}{2}$.

3) $m_1 = m_3$: then γ fixes anp_0 if and only if either

i) $2d \leq m_3$ and $0 \leq j \leq m_1$ or

ii) $0 \leq d = j \leq m_1$ and $val(N(\varepsilon_2)+(1-\eta_1\eta_3^{-1}N(\varepsilon_3)) \geq 2d - m_3$.

When $d \neq j$, we have to consider the points $w_j x p_0$ for $j = 1,2$ also; this amounts to replacing γ by $w_j \gamma w_j^{-1}$ and using the above conditions.

Suppose first that $m_1 \neq m_3$ and let $m = \min(m_1,m_3)$, $n = \max(m_1,m_3)$. Then the number of fixed points is equal to

$$1 + \sum_{d=1}^{[\frac{m}{2}]} q^{4d}(1+\frac{1}{q^3}) + \sum_{d=[\frac{m}{2}]+1}^{[\frac{n}{2}]} q^{2d}\left(\sum_{j=1}^{[\frac{m}{2}]} q^{2j}(1-\frac{1}{q^2})(1+\frac{1}{q}) + (1+\frac{1}{q})\right) \qquad (\alpha)$$

We obtain:

$$F_0(\gamma) = \frac{-(q+1)-q^{4[\frac{m}{2}]+2}(q+1)}{q^4-1} + \frac{q^{m_1+m_3}}{q-1} \quad \text{if} \quad m_2 \not\equiv m_3 \qquad (2).$$

If $m_1 \equiv m_3$ (2) and $m_2 = m$, we obtain the number (α) plus an additional

$$(1+\frac{1}{q})^2 q^m \sum_{d=[\frac{n}{2}]+1}^{\frac{m+n}{2}} q^{2d} = (\frac{q+1}{q-1})(q^{n+2m}-q^{m+2[\frac{n}{2}]})$$

points. We obtain

$$F_0(\gamma) = \frac{-(q+1)-q^{4[\frac{m}{2}]+2}(q+1)}{q^4-1} - \frac{q^{m_1+m_3}}{q-1} + \frac{q^{m_1+m_2+m_3}(q+1)}{q-1} \quad \text{if} \quad m_1 \equiv m_3,\ m_2 = m.$$

The only remaining case is $m_2 > m_1 = m_3$. We obtain

$$\frac{-(q+1)-q^{4[\frac{m}{2}]+2}(q+1)}{q^4-1} + \frac{q^{m_1+m_2}}{q-1} \qquad \text{if} \quad m_2 \not\equiv m \quad (2)$$

$$\frac{-(q+1)-q^{4[\frac{m}{2}]+2}(q+1)}{q^4-1} - \frac{q^{m_1+m_2}}{q-1} + \frac{q^{m_1+m_2+m_3}(q+1)}{q-1} \quad \text{if} \quad m_2 \equiv m \quad (2).$$

The result is as stated in part a) of Proposition 1.

Now we have to compute the fixed points for $\gamma \in T_j'$, $j = 1,2,3$. We will see that the cases T_1 and T_2 can be easily reduced to the case of T_3, so we deal now with this case where we may assume that T_3 is associated to the cocycle

$a_\sigma = w\begin{pmatrix} \pi^{-1} & & \\ & 1 & \\ & & \pi \end{pmatrix}$ and that $gAg^{-1} = T$. Let $\mathfrak{A}_A$ be the apartment in X associated to A, and let $p_{1/2}$ be the unique s-vertex fixed by a_σ (we may assume that T stabilizes $p_{1/2}$). As in the previous case, we will describe the hs-vertices p such that gp is F-rational in the form xp_0 and then compute the fixed points of $g\gamma g^{-1}$ by determining whether or not γ fixes xp_0.

$$\text{Let } a = \begin{pmatrix} \pi^d & & \\ & 1 & \\ & & \pi^{-d} \end{pmatrix} \text{ and } n = \begin{pmatrix} 1 & \pi^{-j}\varepsilon_1 & \pi^{-2d-1}\varepsilon_2 \\ 0 & 1 & \pi^{-j}\varepsilon_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad j = 0,1,\ldots,d,$$

where $\varepsilon_j \in 0_E^\times$ or $\varepsilon_j = 0$. The point $ganp_0$ is F-rational if and only if $^t(\overline{an})\begin{pmatrix} \pi & & \\ & 1 & \\ & & \pi^{-1} \end{pmatrix}(an) \in SL_3(0_E)$ which is equivalent to the requirements

1) $\pi^{-j}(\varepsilon_3+\bar{\varepsilon}_1\varepsilon_2) \in 0_E$.

2) $\pi^{-2d-1}(1+\pi^{2d-2j+1}N(\varepsilon_3)+N(\varepsilon_2)) \in 0_E$.

We always take $\varepsilon_1 = -(\overline{\varepsilon_3/\varepsilon_2})$.

a) $j = 0$: take $\varepsilon_1 = \varepsilon_3 = 0$ and $N(\varepsilon_2) \equiv -1(\pi^{2d+1})$. We obtain $q^{2d}(q+1)$ points in this way.

b) $0 < j \leq d$: choose ε_2 such that $N(\varepsilon_2) \quad -1(\pi^{2d-2j+1})$ and $N(\varepsilon_2) \not\equiv -1(\pi^{2d-2j+2})$, and ε_3 satisfying 2). We obtain a total of $q^{2d+2j+1}(1-\frac{1}{q^2})(1+\frac{1}{q})$ points in this way.

Adding cases a) and b), we obtain a description for $q^{2d+1}(1+\frac{1}{q}) + q^{2d+1}(1+\frac{1}{q})(1-\frac{1}{q^2})\sum_{j=0}^{d} q^{2j} = q^{4d}(q+1)$ points. Furthermore, all points obtained in this way satisfy $d(p,p_{1/2}) = d + \frac{1}{2}$ (where $d(p,q)$ = half the number of chambers separating p and q). Since exactly $q^{4d}(q+1)$

points satisfy $d(p,p_{1/2}) = d + \frac{1}{2}$, this must be all of them.

To reduce the cases of T_1 and T_2 to T_3, we remark that γ fixes xp_0 if and only if $w_j\gamma w_j^{-1}$ fixes $w_j xp_0$. Choose g_j $(j=1,2)$ so that $g_j A g_j^{-1} = T_j$. Then the number of fixed points of $g_1\gamma g_1^{-1}$ equals the number for $gw_1\gamma w_1^{-1}g^{-1}$ and the number for $g_2\gamma g_2^{-1}$ equals the number for $gw_2\gamma w_2^{-1}g^{-1}$.

Suppose $g\gamma g^{-1} \in T_3$, and let m_1, m_2, m_3 be as before. Then if $g\gamma g^{-1}$ fixes anp_0 with an of the usual form so that $ganp_0$ is F-rational, the following inequality holds:

$$(*) \quad \mathrm{val}(N(\varepsilon_2)+\pi^{2d-2j+1}N(\varepsilon_3)-\eta_1\eta_3^{-1}\pi^{2d-2j+1+m_1-m_3}N(\varepsilon_3)) \geq 2d + 1 - m_3.$$

There are two cases to consider:

a) $m_1 \geq m_3$: then $2d \leq m_3 - 1$ and $0 \leq j \leq \min(m_1,m_2)$.

b) $m_1 < m_3$: (i) $0 \leq j \leq \min(m_1,m_2)$, $d \leq \frac{m_3-1}{2}$ and $j \leq \frac{m_1}{2}$ or

(ii) $2d + 1 > m_3$, $0 \leq j \leq \min(m_1,m_2)$, $2(d-j) + 1 = m_3 - m_1$ and $v(N\varepsilon_2+\pi^{2d-2k+1}N\varepsilon_3-\eta_1\eta_3^{-1}N\varepsilon_3) \geq 2d + 1 - m_3$

((ii) occurs only when $m_1 \not\equiv m_3(2)$.)

The number of fixed points in the two cases is

$$\text{a)} \quad \sum_{d=0}^{[\frac{m_3-1}{2}]} q^{4d}(q+1) = \frac{q+1}{q^4-1}\,(q^{4[\frac{m_3-1}{2}]+4}-1).$$

$$\text{b)} \quad \sum_{d=0}^{[\frac{m_1}{2}]} q^{4d}(q+1) + \sum_{d=[\frac{m_1}{2}]+1}^{[\frac{m_3-1}{2}]} q^{2d}((q+1) + \sum_{j=0}^{[\frac{m_1}{2}]} q^{2j}(q+1)(1-\frac{1}{q^2})) +$$

$$+ \left(1+\frac{1}{q}\right)^2 \sum_{d=\left[\frac{m_3-1}{2}\right]+1}^{\frac{m_1+m_3-1}{2}} q^{m_1+1+2d} \qquad (\text{if } m_1 \not\equiv m_3(2))$$

$$= \begin{cases} \dfrac{-(q+1)-q^2(q+1q^{4[\frac{m_1}{2}]}}{q^4-1} + \dfrac{q^{m_1+m_3}}{q-1} & \text{if } m_1 \equiv m_3(2) \\[2ex] \dfrac{-(q+1)-q^2(q+1)q^{4[\frac{m_1}{2}]}}{q^4-1} - \dfrac{q^{m_1+m_3}}{q-1} + \dfrac{q+1}{q-1}\, q^{2m_1+m_3} & \text{if } m_1 \not\equiv m_3(2). \end{cases}$$

To obtain the theorem for f_λ, $\lambda \neq 0$, it is necessary to compute $N_{hs}(\gamma)$ and $N_s(\gamma)$. It is known that

$$\Phi_H^{T/1}(\gamma, {}^H f_\lambda) = q^{\lambda+m_3}\left(1+\frac{1}{q}\right).$$

The next lemma follows from Lemma 1.

<u>Lemma</u> 3:

$$\Phi_H^{T/1}(\gamma, \boldsymbol{\varphi}^*(f_\lambda)) = \begin{cases} q^{2\lambda+m_3}\left(1+\frac{1}{q}\right)\left(\left(1+\frac{1}{q}+\frac{1}{q^2}\right)-2q^{-(m_3+1)}\right) & \lambda \text{ even} \\[1ex] q^{2\lambda+m_3-1}\left(1+\frac{1}{q}\right)\left(1-2q^{-m_3}\right) & \lambda \text{ odd.} \end{cases}$$

For $\gamma \in T_0$, regular, let $N_{hs}^{\kappa}(\gamma) = N_{hs}(\gamma) - N_{hs}(\gamma^{\delta_1}) - N_{hs}(\gamma^{\delta_2}) + N_{hs}(\gamma^{\delta_3})$ and $N_s^{\kappa}(\gamma) = N_s(\gamma) - N_s(\gamma^{\delta_1}) - N_s(\gamma^{\delta_2}) + N_s(\gamma^{\delta_3})$. The result is equivalent to the next proposition.

Proposition 3: Let $\gamma \in T_0$ be regular.

a) $N^K_{hs}(\gamma) = (-1)^{m_1+m_2} q^{m_1+m_2+m_3}(q^3+q^2)((1+\frac{1}{q}+\frac{1}{q^2})-2q^{-(m_3+1)})$

b) $N^K_s(\gamma) = (-1)^{m_1+m_2} q^{m_1+m_2+m_3}(q+1)(1-2q^{-m_3})$.

The computations required to verify this proposition can be carried out using the description of the fixed points of γ given in the proof of Proposition. According to Kottwitz's lecture, we can alternatively compute the number of s-vertices fixed by the stable conjugates of γ, for a simple argument shows that N_{hs} and N_s can be expressed in terms of the numbers of fixed s and hs-vertices. A computation of the number of fixed s-vertices will be given in subsequent notes.

Notes I for the seminar "Analytical Aspects of the Trace Formula II"

Preliminary Facts About Unitary Groups in Three Variables

J. Rogawski

§1. Definitions. Let E/F be a separable quadratic extension of fields and let $\mathfrak{G}(E/F) = \{1, \sigma\}$ be the Galois group of E/F. Let D be a simple algebra which is central over E. An involution of the second kind τ is an anti-automorphism of order two of D such that the restriction of τ to the center E of D coincides with σ. If A is a commutative F-algebra, then σ extends to an automorphism of $E \otimes_F A$ and τ extends to an involution of $D \otimes_F A$ whose restriction to $E \otimes_F A$ is σ. Given a pair (D, τ), we obtain an algebraic group U_τ defined over F such that for every F-algebra A, the group of A-rational points is given by:

$$U_\tau(A) = \{g \in (D \otimes_F A)^* : \tau(g)g = 1\} .$$

We call U_τ the unitary group defined by (D, τ). We also obtain the groups

$$SU_\tau(A) = \{g \in (D \otimes_F A)^* : \tau(g)g = 1, \ Nm(g) = 1\}$$
$$GU_\tau(A) = \{g \in (D \otimes_F A)^* : \tau(g)g \in (E \otimes_F A)^*\}$$

where Nm is the reduced norm map.

Let $M_n(R)$ be the algebra of $n \times n$ matrices over R for any ring R. If $D = M_n(E)$, an involution of the second kind τ is of the form $\tau(x) = \Phi^{-1}\sigma({}^t x)\Phi$ for $x \in M_n(E)$ and $\Phi \in GL_n(E)$ is Hermitian, i.e., ${}^t\Phi = \sigma(\Phi)$. In this case, U_τ is the unitary group attached to the Hermitian form $\langle v_1, v_2 \rangle = {}^t\sigma(v_1)\Phi v_2$ where v_1 and v_2 are column vectors in E^n.

Let

$$\Phi_n = \begin{pmatrix} 0 & & & 1 & 1 \\ & & \cdot & & \\ & \cdot & & & \\ 1 & 1 & & & 0 \end{pmatrix} \in GL_n(E)$$

and let U_n denote the unitary group with respect to E/F defined by the Hermitian form Φ_n. Then U_n is quasi-split (a reductive group over a field F is called quasi-split if it contains a Borel subgroup over F) since the subgroup of upper-triangular matrices in U_n is a Borel subgroup over F.

We recall the definition of an inner form of an algebraic group. Let G and G' be algebraic groups over a field F and suppose that there is an isomorphism $\varphi : G \longrightarrow G'$ defined over a Galois extension E/F with Galois group $\mathcal{G}(E/F)$. For $\sigma \in \mathcal{G}(E/F)$, $a_\sigma = \varphi^{-1} \circ {}^\sigma\varphi$ is an automorphism of G over E and $a_{\sigma\tau} = a_\sigma \circ \sigma(a_\tau)$ for all $\tau \in \mathcal{G}(E/F)$. Hence $\{a_\sigma\} \in H^1(\mathcal{G}(E/F), \mathrm{Aut}(G))$ and if a_σ is an inner automorphism for all $\sigma \in \mathcal{G}(E/F)$, then G' is called an inner form of G. Every connected reductive group is an inner form of a unique quasi-split reductive group. Hence all unitary groups are inner forms of the groups U_n defined above.

§2. Unitary groups in three variables. In this section, suppose that (D, τ) is a pair as in §1 where $\dim_E D = 9$. The group U_τ is called a unitary group in three variables. We list some facts about unitary groups in three variables.

Fact (i): If $F = \mathbb{R}$ and $E = \mathbb{C}$, all unitary groups in three variables are isomorphic to either the quasi-split form U_3 or the compact form defined by the Hermitian form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Fact (ii): If F is a p-adic field, all unitary groups in three variables are quasi-split, hence isomorphic to U_3.

Now let E/F be a quadratic extension of number fields and let v be a place of F. Let U_τ be a unitary group in three variables with respect to a pair (D, τ) for the extension E/F. From the definition of U_τ and the above facts, we get:

Fact (iii): a) If v is infinite, then $U_\tau(F_v)$ is isomorphic to $U_3(\mathbb{R})$ or its compact form if v is ramified and it is isomorphic to $GL_3(\mathbb{R})$ or $GL_3(\mathbb{C})$ if v is unramified and $F_v = \mathbb{R}$ or $F_v = \mathbb{C}$, respectively.

b) If v is finite and does not split in E, then $U_\tau(F_v)$ is isomorphic to $U_3(F_v)$.

c) If v is finite and splits in E, then $U_\tau(F_v)$ is isomorphic to $(D \otimes_E E_w)^*$ where w is a place of E lying above v (the groups for the two different places are isomorphic).

Fact (iv): The isomorphism class of U_τ is determined by the isomorphism classes of the groups $U_\tau(F_v)$ for all places v of F, i.e., U_{τ_1} is isomorphic to U_{τ_2} if and only if $U_{\tau_1}(F_v)$ is isomorphic to $U_{\tau_2}(F_v)$ for all places v of F, where (D_1, τ_1) and (D_2, τ_2) are pairs as in §1.

§3. L-groups. The L-group of a connected reductive group G over a field F is a complex Lie group of the form ${}^LG = {}^LG^o \rtimes W_{E/F}$ where E/F is a Galois extension over which G splits (we may take E/F as large as desired, or may take E to be an algebraic closure of F) and $W_{E/F}$ is the Weil group of E/F. The L-group depends only on the quasi-split group G^* of which G is an inner form: ${}^LG^o$ is the group of complex points of the connected reductive group over $\mathbb{C}$ whose root data is inverse to that of G^* and the action of $W_{E/F}$ on ${}^LG^o$ is defined through the action of $\mathfrak{G}(E/F)$ on the root data defined by a Borel subgroup of G^* defined over F. We refer to Borel's article in [ ] for precise definitions. We simply state what is needed for this seminar.

(i) Let E/F be a separable quadratic extension and let $G = U_\tau$ be a unitary group in n-variables with respect to E/F. Then ${}^LG^o = GL_n(\mathbb{C})$ and ${}^LG = {}^LG^o \rtimes W_{E/F}$ where $W_{E/F}$ acts on ${}^LG^o$ through its projection onto $\mathfrak{G}(E/F) = \{1, \sigma\}$ and σ acts on ${}^LG^o$ by the automorphism $g \longmapsto J_n^{-1}\, {}^tg^{-1}\, J_n$ with

$$J_n = \begin{pmatrix} 0 & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ (-1)^{n-1} & & & & 0 \end{pmatrix}$$

(ii) With G, E/F as in (i), let $\tilde{G} = \mathrm{Res}_{E/F}(G)$ where $\mathrm{Res}_{E/F}$ denotes restriction of scalars. Then

$$ {}^L\tilde{G}^o = GL_n(\mathbb{C}) \times GL_n(\mathbb{C}), \quad {}^L\tilde{G} = {}^L\tilde{G}^o \rtimes W_{E/F} $$

where $W_{E/F}$ acts on ${}^L\tilde{G}^o$ through its projection onto $\mathfrak{G}(E/F)$ and σ acts on ${}^L\tilde{G}^o$ by the automorphism:

$$ (g_1, g_2) \longmapsto (J_n^{-1}\, {}^t g_2^{-1} J_n,\ J_n^{-1}\, {}^t g_1^{-1} J_n) $$

for $(g_1, g_2) \in {}^L\tilde{G}^o$.

There is a homomorphism ${}^LG \xrightarrow{\psi_G} {}^L\tilde{G}$ defined by $\psi_G(g \times w) = (g, g) \times w$ for $g \times w \in {}^LG^o \rtimes W_{E/F}$.

§4. Unramified Representations.

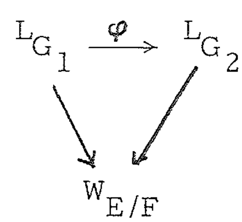
Let G be a connected reductive group over a p-adic field F. It is called unramified if G is quasi-split over F and splits over an unramified extension E/F. If G is unramified, it contains a "hyper-special" maximal compact subgroup K (see [ ]) and an irreducible admissible representation π of $G(F)$ is called unramified if the space π^K of vectors in π which are fixed by some such K is non-zero.

Let G be unramified with E/F as above and let ${}^LG = {}^LG^o \rtimes \mathfrak{G}(E/F)$. Let $\Phi \in \mathfrak{G}(E/F)$ be the Frobenius element. Then the set $\prod^{un}(G)$ of unramified representations of G is parametrized by the set of ${}^LG^o$-conjugacy classes in ${}^LG$ which contain an element of the form $\gamma \times \Phi$ with $\gamma \in {}^LG^o$ semisimple.

If π is unramified and irreducible, then $\dim \pi^K = 1$ and the Hecke algebra $\mathcal{H}_K$ of bi-K-invariant, compactly supported functions on G(F) acts on π^K through a character $\lambda : \mathcal{H}_K \longrightarrow \mathbb{C}$. The above assertion amounts to an identification of the characters of the commutative algebra $\mathcal{H}_K$ with the conjugacy classes $\{\gamma \times \Phi\}$. See Borel's article in [ ] for details.

§5. Functoriality in the unramified case.

If G_1 and G_2 are connected reductive groups over F (local or global) with L-groups ${}^LG_1 = {}^LG_1^o \rtimes W_{E/F}$ and ${}^LG_2 = {}^LG_2^o \rtimes W_{E/F}$ respectively (we take E large enough to define both L-groups), then a map of L-groups is a homomorphism:



such that the above diagram commutes, where the maps to $W_{E/F}$ are the projections on the second factor.

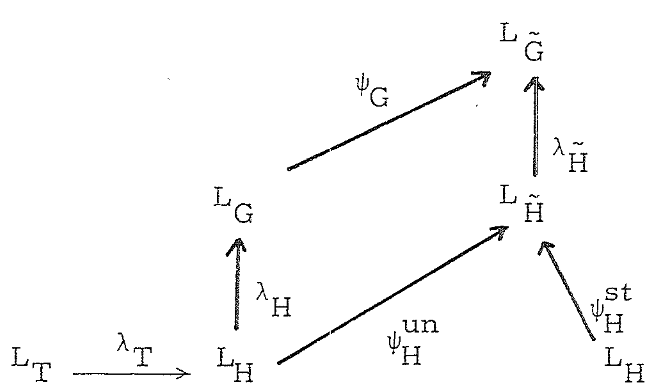
Now assume that F is p-adic and that G_1 and G_2 are unramified. Take E to be an unramified extension of F over which both G_1 and G_2 split. By §4, an L-group map $\varphi : {}^LG_1 \longrightarrow {}^LG_2$ gives a map $\prod^{un}(G_1) \longrightarrow \prod^{un}(G_2)$ by associating the unramified representation corresponding to the ${}^LG_1^o$-conjugacy class of $\gamma \times \Phi$ to the one corresponding to the ${}^LG_2^o$-conjugacy class of $\varphi(\gamma \times \Phi)$.

§6. The basic diagram of L-group homomorphisms.

Let E/F be a quadratic extension of number fields with Galois group $\mathcal{G}(E/F) = \{1, \sigma\}$. We define the following five groups:

$$G = U_3 \text{ with respect to } E/F$$
$$H = U_2 \text{ with respect to } E/F$$
$$\tilde{G} = \mathrm{Res}_{E/F}(G)$$
$$\tilde{H} = \mathrm{Res}_{E/F}(H)$$
$$T = \mathbb{E}^1 \times \mathbb{E}^1$$

where $\mathbb{E}^1$ is the algebraic group over F defined by the group $E^1 = \{x \in E : N_{E/F}(x) = 1\}$ of norm one elements in E. Thus, for any field extension K/F, $\mathbb{E}^1(K) = \{x \in E \otimes_F K : N_{E\otimes_F K}(x) = 1\}$. We will define a diagram of L-groups:



such that $\psi_G \circ \lambda_H = \lambda_{\tilde{H}} \circ \psi_H^{un}$.

For any number field k, let $\mathbb{A}_k$, $\mathbb{A}_k^*$, and C_k denote the adeles, ideles, and idele classes of k, respectively. Recall that there is an exact sequence $1 \longrightarrow C_E \longrightarrow W_{E/F} \longrightarrow \mathcal{G}(E/F) \longrightarrow 1$. We regard C_F as contained in C_E in a natural way and $N_{E/F}(C_E) \subset C_F$.

All L-groups occurring in the basic diagram have been defined in §3 except ${}^LT$. We have ${}^LT = {}^LT^o \rtimes W_{E/F}$ where

$$ {}^LT^o = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in GL_2(\mathbb{C}) \right\} $$

and $W_{E/F}$ acts on ${}^LT^o$ through its projection onto $\mathfrak{g}(E/F)$, and σ acts on ${}^LT^o$ through the automorphism

$$ \sigma\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} . $$

The maps in the basic diagram are defined as follows.

(i) ψ_G is the map defined in §3.

(ii) ψ_H^{st} is the map ψ_H defined in §3.

Let $\alpha : GL_2(\mathbb{C}) \longrightarrow GL_3(\mathbb{C})$ be the map

$$ \alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & 0 & b \\ 0 & \delta & 0 \\ c & 0 & d \end{pmatrix} \text{ where } \delta^{-1} = ad-bc . $$

We fix an element $w_\sigma \in W_{E/F}$ which projects to $\sigma \in \mathfrak{g}(E/F)$ and a character μ of C_E whose restriction to C_F is the character of order two associated to E/F by class field theory. We have $w_\sigma^2 \in C_F - N_{E/F}(C_E)$ and hence $\mu(w_\sigma^2) = -1$.

(iii) λ_H maps $h \times 1 \in {}^LH$ to $\alpha(h) \times 1 \in {}^LG$,

$$\lambda_H(1\times z) = \begin{pmatrix} \mu(z) & & \\ & \mu(z)^{-2} & \\ & & \mu(z) \end{pmatrix} \times z \in {}^L G \quad \text{for} \quad z \in C_E$$

$$\lambda_H(1\times w_\sigma) = \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix} \times w_\sigma \in {}^L G \ .$$

Then λ_H defines a map of L-groups

(iv) $\lambda_{\tilde H}$ maps $(h_1, h_2, z) \in {}^L\tilde{H}^o \times C_E$ to $(\alpha(h_1), \alpha(h_2), z) \in {}^L\tilde{G}$ and

$$\lambda_{\tilde H}((1, 1) \times w_\sigma) = \left(\begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}\right) \times w_\sigma \in {}^L\tilde{G} \ .$$

(v) ψ_H^{un} maps $h \times 1 \in {}^L H$ to $(h, h) \times 1 \in {}^L\tilde{H}$ and

$$\psi_H^{un}(1\times z) = \left(\begin{pmatrix} \mu(z) & \\ & \mu(z) \end{pmatrix}, \begin{pmatrix} \mu(z) & \\ & \mu(z) \end{pmatrix}\right) \times z \in {}^L\tilde{H}$$

$$\psi_H^{un}(1\times w_\sigma) = \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right) \times w_\sigma \in {}^L\tilde{H} \ .$$

(vi) ψ_T maps $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times z \in {}^L T^o \rtimes C_E$ to $\begin{pmatrix} a\mu^{-1}(z) & \\ & b\mu^{-1}(z) \end{pmatrix} \times z \in {}^L H^o$

$$\psi_T(1\times w_\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times w_\sigma \ .$$

§7. What to expect.

One would like to understand functoriality for the basic diagram as completely as possible. The general problem and techniques for solving it (stable, twisted trace formula and matching orbital integrals) were outlined by Langlands. Contributions (in chronological order) are as follows:

Rogawski: existence of quasi-lifts for λ_H and matching orbital integrals for λ_H (part of thesis, 1980, and unpublished manuscript, 1981)

Kottwitz: matching orbital integrals for ψ_G (unpublished manuscript, 1981)

Flicker: discussion of ψ_H^{un} and ψ_H^{st} (Duke J. Math. 49, no. 3, 1982) and preliminary draft dealing with ψ_G

Further references are:

Labesse-Langlands: L-indistinguishability for SL(2), Can. J. Math. 31 (1979). (This paper treats the analogue of the map λ_T for SL(2) and applies, with minor modifications, to λ_T. It was a motivating example for subsequent work on L-indistinguishability.)

Langlands: Les débuts d'une formule des traces stable, Pub. Math. Paris VII, 1982.

D. Shelstad has an extensive bibliography of papers dealing with L-indistinguishability for real groups. See Langlands Paris VII notes and further notes of this seminar for references.

What follows is a sketch of results which one would like to prove and questions one would like to answer. Some are suggested by the unpublished manuscript of Flicker cited above. In some cases, certain facts are already known and one wants to understand the compatibility of what is known with the formalism of L-groups.

For the definition of an automorphic representation, we refer to the article of Borel and Jacquet in [ ]. An automorphic representation will be called cuspidal, discrete residual, or Eisenstein according as it occurs in the space of cusp forms, residues of Eisenstein series, or in the orthogonal complement of the discrete spectrum, respectively. It occurs discretely if it is cuspidal or residual.

If G is a connected reductive group over a local (resp. global) field F, $\prod(G)$ will denote the set of irreducible, admissible (resp. automorphic) representations of $G(F)$ (resp. $G(\mathbb{A})$). If F is global $\prod_v(G)$ will denote $\prod(G/F_v)$ and $\prod_v^{un}(G)$ will denote the set $\prod^{un}(G/F_v)$.

If G_1 and G_2 are groups over a global field and $\varphi : {}^LG_1 \longrightarrow {}^LG_2$ is a map of L-groups, we obtain maps $\varphi : {}^LG_{1v} \longrightarrow {}^LG_{2v}$ by restriction, where ${}^LG_v$ is the L-group of G/F_v. Hence φ gives rise to maps $\varphi : \prod_v^{un}(G_1) \longrightarrow \prod_v^{un}(G_2)$ for all places v at which G_1 and G_2 are unramified as in §5.

Definition: With φ, G_1, G_2 as above, let $\pi^1 = \otimes \pi_v^1 \in \prod(G_1)$ and $\pi^2 = \otimes \pi_v^2 \in \prod(G_2)$. Then π^2 is called a quasi-transfer of π^1 (with respect to φ) if $\pi_v^2 = \varphi(\pi_v^1)$ for almost all v at which G_1, G_2 and π_v are unramified.

The following is a list of questions:

Q1. Do quasi-transfers exist for all maps in the basic diagram? This is known so far for all of the maps except ψ_G (for $\lambda_{\tilde{H}}$, it follows from the theory of Eisenstein series and for the other maps from the references cited above).

We want to define transfers locally and globally and there are two points to understand beforehand. First, in general it makes no sense to transfer a representation to an individual representation because, for example, in the local tempered case, transfers should satisfy certain character identities and these identities can only be formulated in terms of stable conjugacy. Postponing details for the time being, it suffices to say that one can only compare certain linear combinations of characters of irreducible π with such linear combinations on other groups and not the characters themselves. The linear combinations are built out of elements in finite sets of representations called L-packets, in the local case. Second, transfers and L-packets for non-tempered representations will generally not obey the same formalism as that which one hopes for in the tempered case. This comes under the rubric of "anomalous" representations discussed below.

To define L-packets for G locally means to partition $\Pi_v(G)$ into finite sets Π_v - it will consist either entirely of tempered or entirely of non-tempered representations. Tempered Π_v should satisfy character identities defined by functoriality and be compatible with global transfer. The definition for non-tempered Π_v is reduced to the tempered case via the Langlands classification.

Suppose local L-packets for a group G have been defined. A global L-packet $\Pi = \otimes \Pi_v$ is then, by definition, obtained by choosing a local L-packet Π_v for all v such that Π_v contains an unramified representation π_v^o for almost all v and setting

$$\Pi = \otimes \Pi_v = \{\otimes \pi_v : \pi_v \in \Pi_v \quad \text{for all} \quad v, \ \pi_v \quad \text{unramified for} \quad \text{a.a.v.}\} \ .$$

It should be stressed that apart from the case $F = \mathbb{R}$ or $\mathbb{C}$, there is no general definition of local L-packets. See [Langlands, Paris VII notes] for a description of the formalism one would like L-packets to satisfy. For

$GL(n)$, all L-packets have one element (locally and, hence, globally). For $SL(n)$, local L-packets consist of the sets of representations which are equivalent under conjugation by $GL(n)$, and one may use the same definition globally. For a torus, L-packets consist of one element. These are the only cases in which L-packets have been defined for all places. In particular, this defines L-packets for the groups $\tilde{G}$, $\tilde{H}$, and T in the basic diagram since $\tilde{G}(F) = GL_3(E)$, $\tilde{H} = GL_2(E)$, and T is a torus.

For the group H, L-packets can also be defined because the derived group H^d of H is $SL(2)$. Hence $GL(2)$ acts on H by conjugation and L-packets are defined via conjugation by $GL(2)$ as in the case of $SL(2)$.

To formulate transfer results, it remains to define L-packets for G. We give an ad hoc definition - the trace formula can be expected to show that this definition has the properties wanted. Fix a Borel subgroup $B = AN$ of G and for χ a character of $A(F_v)$, let $i(\chi) = \mathrm{ind}_{B(F_v)}^{G(F_v)} \chi$ (unitary induction).

(i) For χ a character of $A(F_v)$, define an L-packet

$$\prod(i(\chi)) = \begin{cases} i(\chi) & \text{if } i(\chi) \text{ is irreducible} \\ \{\text{constituents of } i(\chi)\} & \text{if } \chi \text{ is unitary} \end{cases} .$$

(ii) If $\pi \in \prod_v(G)$ is non-tempered and is not of the form $i(\chi)$, π is in an L-packet by itself.

By the Langlands classification, the only $\pi \in \prod_v(G)$ not covered by (i) and (ii) are square-integrable. To define the L-packet $\prod(\pi)$ of π

for π square integrable, we shall assume that there exists an element $\pi^o = \otimes \pi^o_w \in \prod(G)$ such that $\pi^o_v = \pi$ and such that there exists a cuspidal quasi-transfer $\tilde{\pi} = \otimes \tilde{\pi}_w$ with respect to ψ_G (see Q1). In this case, we call $\tilde{\pi}_v$ a transfer (via ψ_G) of π_v. The trace formula should imply that $\tilde{\pi}$ exists and that $\tilde{\pi}_v$ is uniquely determined by π_v.

(iii) For square-integrable $\tilde{\pi}$ lifting to $\tilde{\pi}_v$ as above, set

$$\prod(\pi) = \{\pi' \in \prod_v(G) : \pi' \text{ transfers to } \tilde{\pi}_v\} .$$

A global L-packet $\prod = \otimes \prod_v$ is called automorphic if some $\pi \in \prod$ is automorphic.

Q2. Let $\varphi : {}^LG_1 \longrightarrow {}^LG_2$ be one of the maps in the basic diagram and let $\prod$ be a global L-packet for G_1. Set

$$\varphi(\prod) = \{\tilde{\pi} \in \prod(G_2) : \tilde{\pi} \text{ is the quasi-transfer of some } \pi \in \prod \text{ via } \varphi\} .$$

If $\prod$ is tempered, is $\varphi(\prod)$ a global L-packet for G_2? One expects so in all cases and it is known for all maps except λ_H and ψ_G. If it is true, say $\varphi(\prod) = \tilde{\prod} = \otimes \tilde{\prod}_v$, then it is reasonable to set $\varphi(\prod_v) = \tilde{\prod}_v$ and thereby obtain a map for local tempered L-packets. The φ-transfer of non-tempered L-packets is defined in all cases by using the Langlands classification.

For each character μ_v of $E^1(F_v)$, let $St(\mu_v)$ denote the associated Steinberg representation in $\prod_v(H)$. Then $\{St(\mu_v)\}$ is a local L-packet for H. Flicker's paper suggests the following question.

Q3. Is $\lambda_H(\{St(\mu_v)\}) = \{\pi^+_\mu, \pi^-_\mu\}$ where π^+_μ is square-integrable but not supercuspidal and π^-_μ is supercuspidal?

Q4. Is π^-_μ unipotent in the sense of Lusztig? Does π^+_μ have Iwahori-fixed vectors and if so, are π^-_μ and π^+_μ related as in Lusztig's conjectural classification of representations with Iwahori-fixed vectors for Chevalley groups?

Q5. Is the set of local and global L-packets for G with more than one element equal to the image under λ_H of local and global L-packets on H.

Q6. Suppose $\Pi \subset \Pi(G)$ is a discretely-occurring global L-packet for G such that $\psi_G(\Pi)$ does not occur discretely. Is $\Pi = \lambda_H(\Pi_H)$ for some discretely-occurring global $\Pi_H \subset \Pi(H)$?

Q7. Suppose $\Pi \subset \Pi(H)$ is a discretely-occurring global L-packet for H such that $\lambda_H(\Pi)$ does not occur discretely. Is $\Pi = \lambda_T(\Pi_T)$ for some $\Pi_T \subset \Pi(T)$?

Recall that $\tilde{G}(F) = GL_3(E)$. Let $\pi = \otimes \pi_v$ be a cuspidal automorphic representation of $M(\mathbb{A}_E)$, where M is the Levi factor of a parabolic subgroup P of GL_3. Let ω be the central character of π and suppose that $Re(|\omega|) \in X_*(A_P) \otimes \mathbb{R}$ lies in the closure of the positive Weyl chamber defined by P, where A_P is the center of M. For all v, the induced representation $I(\pi_v)$ has a unique irreducible quotient π'_v and the global representation $I(\pi)$ has $\pi' = \otimes \pi'_v$ as a quotient. All automorphic representations π' obtained in this way are called <u>isobaric</u>; all other non-cuspidal automorphic representations are called anomalous.

Let $\Pi = \otimes \Pi_v$ be a global automorphic L-packet for G and let $\tilde{\Pi} = \otimes \tilde{\Pi}_v$ where $\tilde{\Pi}_v = \psi_G(\Pi_v)$. If $\tilde{\Pi}$ is anomalous, we call the L-packet Π anomalous (note that $\tilde{\Pi}$ is a single representation).

<u>Q8</u>. Are all discretely-occurring anomalous representations of G obtained as the ψ_H-transfer of a one-dimensional automorphic representation of H?

<u>Q9</u>. Are the tempered components of a discretely-occurring anomalous representation of G of the form $\pi_{\bar{\mu}_v}$?

There are a number of questions regarding multiplicities in the discrete spectrum for G which are suggested by the results for SL(2) of Langlands-Labesse. Let $\Pi_d(G) = \{\pi \in \Pi(G) : \pi \text{ occurs discretely}\}$. If $\Pi \subset \Pi(G)$ is a global L-packet such that $\Pi \cap \Pi_d(G) \neq 0$ we will say that Π occurs discretely. For $\pi \in \Pi_d(G)$, let $m(\pi)$ be the multiplicity of π in the discrete spectrum of G.

<u>Q10</u>. Is $m(\pi) = 1$ for all $\pi \in \Pi_d(G)$?

<u>Q11</u>. If $\Pi \subset \Pi(G)$ occurs discretely and is not in the image of λ_H, is $m(\pi)$ the same for all $\pi \in \Pi$?

<u>Q12</u>. Let $\Pi \subset \Pi(G)$ be a tempered, global L-packet which occurs discretely and suppose that $\Pi = \lambda_H(\Pi_H)$ for some $\Pi_H \subset \Pi(H)$. Is it possible to define a positive rational number $n(\Pi)$ and maps

$$\varepsilon_{iv} : \Pi_v \longrightarrow \mathbf{C}^* \qquad \text{for } i = 1, \ldots, N-1$$

such that for all $\pi = \otimes \pi_v \in \prod_v$,

$$m(\pi) = \frac{1}{N}(n(\prod) + \sum_{i=1}^{N-1} \varepsilon_i(\pi))$$

where $\varepsilon_i(\pi) = \prod_v \varepsilon_{iv}(\pi_v)$ and $\varepsilon_{iv}(\pi_v) = 1$ for a.a.v.?

Q13. If $\prod$ and $\prod'$ are global, isobaric, discretely-occurring L-packets for G such that $\prod_v = \prod'_v$ for a.a.v., is $\prod_v = \prod'_v$ for all v?

One would like to know the image of ψ_G. If π is a local or global admissible representation of $\tilde{G}$, define π^σ by the formula $\pi^\sigma(g) = \pi(\sigma(g))$ and call π σ-invariant if $\pi \approx \pi^\sigma$.

Q14. Does the image of ψ_G consist of σ-invariant representations? If $\tilde{\pi}$ is σ-invariant and not in the image of ψ_G, is it in the image of $\lambda_{\tilde{H}} \circ \psi_H^{st}$?

If $\prod(G)$ can be analyzed, one can attempt to compare G with its inner forms, the other unitary groups in three variables. One question is:

Q15. Do anomalous representations occur discretely for an inner form G' of G if and only if G' is not defined by a pair (D, τ) where D is a division algebra?

Finally, one would like to understand any possible relations between the residual spectrum and the cuspidal spectrum of G. For example, do analogues of the "CAP" representations for Sp(4) constructed by Piatetski-Shapiro occur for G? The residual spectrum, which is spanned by the residues of Eisenstein series, is determined by the behavior of certain L-functions as follows.

Let $B = AN$ be the Borel subgroup of upper-triangular matrices in G, where A is the diagonal subgroup of G and N is the unipotent radical of B. Then

$$A(F) = \left\{ \begin{pmatrix} \alpha\beta & & \\ & \beta & \\ & & \bar{\alpha}\beta^{-1} \end{pmatrix} : \alpha \in E^*, \ \beta \in E' \right\}$$

and A is isomorphic to $(\mathrm{Res}_{E/F}(\mathbb{G}_m)) \times \mathbb{E}^1$. Let χ be a character of $A(F)\backslash A(\mathbb{A})$ and let $I(\chi) = \mathrm{ind}_{G(\mathbb{A})}^{G(\mathbb{A})} \chi$. A constituent of $I(\chi)$ is of the form $\otimes \pi_v$ with π_v a constituent of $i(\chi_v)$ for all v and π_v unramified for almost all π_v.

Q16. Which constituents of $I(\chi)$ occur in the cuspidal spectrum of G?

Suppose that χ is unitary and consider the Eisenstein series $E(s)$ associated to the character $\chi\alpha_s$ where

$$\alpha_s \left(\begin{pmatrix} \alpha\beta & & \\ & \beta & \\ & & \bar{\alpha}_\beta^{-1} \end{pmatrix} \right) = |\alpha|^s \quad \text{for } s \in \mathbb{C}, \ \alpha \in \mathbb{A}_E \ .$$

The poles of $E(s)$ are the same as those of the function

$$\xi(s) = \frac{L(s, \chi_1)L(2s, \chi_2\omega)}{L(s+1, \chi_1)L(2s+1, \chi_2\omega)}$$

where χ_1 is the character of C_E obtained by restricting χ to the image of the map

$$\alpha \longmapsto \begin{pmatrix} \alpha & & \\ & 1 & \\ & & \bar{\alpha}^{-1} \end{pmatrix} \qquad \alpha \in \mathbb{A}_E^* \ ,$$

χ_2 is the character of C_F obtained by restricting χ_1 to the image of C_F in C_E, and ω is the character of order two of C_F associated to E/F by class field theory.

The residues of $E(s)$ for $s \in [0. 1]$ contribute to the discrete spectrum. If χ_1 is trivial, $\xi(s)$ has a simple pole at $s = 1$. For χ such that χ_1 is non-trivial, $\xi(s)$ has a pole for $s \in [0, 1]$ if and only if $\chi_2\omega$ is trivial and $L(\frac{1}{2}, \chi_1) \neq 0$, in which case the pole occurs at $s = \frac{1}{2}$ and is simple.

<u>Q17</u>. If $\chi_2\omega$ is trivial and $L(\frac{1}{2}, \chi_1) = 0$, so that $\xi(s)$ is finite at $s = \frac{1}{2}$, do constituents of $I(\chi)$ occur in the cuspidal spectrum of G?

Notes II for the seminar "Analytical Aspects of the Trace Formula II"

Stable Conjugacy, Stable Trace Formula   J. Rogawski

§8. Stable conjugacy, twisted conjugacy

[1] Langlands, Les débuts d'une formule des traces stable
Pub. Math., Paris VII

[2] Langlands, Stable conjugacy, definitions and lemmas
Can. J., 31 (1979)

[3] Kottwitz, Rational conjugacy classes in reductive groups
Duke J., 49, No. 4 (1982)

In order to compare the trace formulas of different groups, we need a way of matching conjugacy classes in different groups.

Example: Let G and G^* be connected reductive groups over F and let $\mathfrak{G} = \mathrm{Gal}(\bar{F}/F)$. Then $\mathfrak{G}$ acts on $\mathrm{Hom}_{\bar{F}}(G,G^*)$. Let $G \xrightarrow{\psi} G^*$ be an isomorphism defined over $\bar{F}$ and suppose that $\sigma\psi = \psi\circ\mathrm{ad}(g_\sigma)$ for some $g_\sigma \in G(\bar{F})$, for all $\sigma \in \mathfrak{G}$. Then G is an inner form of G^* and for all $\gamma \in G(F)$, $\psi(g_\sigma\gamma g_\sigma^{-1}) = \sigma\psi(\gamma) = \sigma(\psi(\gamma))$ and hence $\psi(\gamma)$ and $\sigma(\psi(\gamma))$ are conjugate in $G^*(\bar{F})$. In other words, the conjugacy class of $\psi(\gamma)$ is defined over F and we have a map

$$\{\text{Conjugacy classes in } G(F)\} \xrightarrow{\psi} \{\text{Conjugacy classes in } G^*(\bar{F}) \text{ which are defined over } F\} .$$

If $\{\gamma\},\{\gamma'\}$ are conjugacy classes in $G(F)$, then $\psi(\{\gamma\}) = \psi(\{\gamma'\})$ if and only if γ and γ' are conjugate in $G(\bar{F})$.

Kottwitz's Theorem ([3]): If G is connected, reductive and quasi-split over F, and if the derived group of G is simply-connected, then every conjugacy class in $G(\bar{F})$ which is defined over F intersects $G(F)$.

Definition: Let G be connected, reductive, quasi-split over F with simply connected derived group. We say that γ and γ' in $G(F)$ are stably conjugate if they are conjugate in $G(\bar{F})$ (abbreviation: st-conjugate).

In the above example $G \xrightarrow{\psi} G^*$ with $\psi^{-1} \circ \sigma\psi$ an inner automorphism for all $\sigma \in \overline{\mathcal{G}}$, Kottwitz's theorem shows that if the derived group of G^* is simply-connected and G^* is quasi-split, then we obtain an injection

$$\{\text{stable conj. classes in } G\} \hookrightarrow \{\text{stable conj. classes in } G^*\}$$

$$G(F) \cap \{g^{-1}\gamma g : g \in G(\bar{F})\} \longmapsto G^*(F) \cap \{g^{-1}\psi(\gamma) g : g \in G^*(\bar{F})\}$$

Parametrization of conjugacy classes in a stable class: If $\gamma, \gamma' \in G(\bar{F})$ and $g^{-1}\gamma g = \gamma'$ for some $g \in G(\bar{F})$, then by applying $\sigma \in \overline{\mathcal{G}}$ to this equality we see that $\sigma(g)g^{-1} \in G_\gamma(\bar{F})$ where G_γ is the centralizer of γ, and $\{a_\sigma = \sigma(g)g^{-1}\}$ is a cocycle in $H^1(\overline{\mathcal{G}}, G_\gamma)$ whose image in $H^1(\overline{\mathcal{G}}, G)$ is trivial. The next lemma is easy to check.

Lemma 8.1: Let $\gamma \in G(F)$. The set of $G(F)$-conjugacy classes within the stable conjugacy class of γ is parametrized by the set:

$$\mathcal{D}(\gamma/F) = \operatorname{Ker}\{H^1(\overline{\mathcal{G}}, G_\gamma) \longrightarrow H^1(\overline{\mathcal{G}}, G)\}$$

Let G^{sc} denote the simply-connected covering group of the derived group G^{der} of G and for any $\gamma \in G$, let G^{sc}_{γ} denote the centralizer in G^{sc} of γ under the map $G^{sc} \longrightarrow Ad(G)$. Then $G^{sc}_{\gamma} \longrightarrow G_{\gamma}$ by restricting the map $G^{sc} \longrightarrow G^{der}$. Let

$$\mathcal{E}(\gamma/F) = \text{Image of } H^1(\overline{\mathfrak{G}}, G^{sc}_{\gamma}) \text{ in } H^1(\overline{\mathfrak{G}}, G_{\gamma}) .$$

Then it is easy to see that $\mathcal{D}(\gamma/F) \subset \mathcal{E}(\gamma/F)$: the images of $G^{sc}(\bar{F})$ and $G(\bar{F})$ in the adjoint group coincide.

From now on we use CSG to abbreviate "Cartan subgroup of G defined over F."

<u>Definition</u>: Let T_1 and T_2 be CSG's of G. We say that T_1 and T_2 are stably conjugate if there is a $g \in G(\bar{F})$ such that $T_2 = g^{-1}T_1g$ and the map $t \longmapsto g^{-1}tg$ is defined over F (equivalently, T_1 and T_2 are stably conjugate if some regular element of $T_1(F)$ is stably conjugate to an element of $T_2(F)$).

For T a CSG of G, set

$$\mathcal{O}\mathcal{l}(T) = \{g \in G(\bar{F}) : g^{-1}Tg \text{ and } t \longrightarrow g^{-1}tg \text{ are defined over } F\} .$$

Set: $\mathcal{D}(T/F) = T(\bar{F})\backslash \mathcal{O}\mathcal{l}(T)/G(F)$. It is easy to check that the map

$$\mathcal{D}(T/F) \longrightarrow \text{Ker}\{H^1(\overline{\mathfrak{G}}, T) \longrightarrow H^1(\overline{\mathfrak{G}}, G)\}$$

$$g \longmapsto \{a_\sigma = \sigma(g)g^{-1}\}$$

is a bijection and identifies $\mathcal{D}(T/F)$ with $\mathcal{D}(\gamma/F)$ for any regular $\gamma \in T(F)$. Let

$$\mathcal{E}(T/F) = \text{Image of } H^1(\overline{\mathfrak{g}}, T^{sc}) \text{ in } H^1(\overline{\mathfrak{g}}, T)$$

where T^{sc} is the inverse image of $T^{sc} \cap G^{der}$ in G^{sc}. Then $\mathcal{D}(T/F) \subset \mathcal{E}(T/F)$.

Stabilization of the elliptic regular terms: Let $f = \prod_v f_v$ be a function on $G(\mathbb{A})$ of the usual type to which one applies the trace formula. Assume that $f(zg) = \xi^{-1}(z)f(g)$ for all $z \in Z(\mathbb{A})$, where Z is the center of G and ξ is a character of $Z(F)\backslash Z(\mathbb{A})$. The elliptic regular term of the trace formula is:

$$E(f) = \sum_{\{\gamma\}\text{ elliptic}}' \delta(\gamma)^{-1} \operatorname{meas}(Z(\mathbb{A})G_\gamma(F)\backslash G_\gamma(\mathbb{A}))\Phi(\gamma, f)$$

where Σ' denotes a sum of regular elements, $\{\gamma\}$ ranges over the elliptic regular conjugacy classes in $Z(F)\backslash G(F)$, and $\delta(\gamma)$ is the index of $Z(F)\backslash G_\gamma(F)$ in the centralizer of γ in $Z(F)\backslash G(F)$. Set:

$$\Omega_F^\circ(T,G) = \text{the Weyl group of } T \text{ in } G(F)$$

$$\Omega_{\bar{F}}(T,G) = \text{the Weyl group of } T \text{ in } G(\bar{F})$$

$$\Omega_F(T,G) = \{w \in \Omega_{\bar{F}}(T,G) : t \longmapsto w^{-1}tw \text{ is defined over } F\}$$

$$\Omega_F^\circ(T,G)_\gamma = \{w \in \Omega_F^\circ(T,G) : w^{-1}\gamma w\gamma^{-1} \in Z(F)\}$$

for γ regular in $T(F)$.

For $\gamma \in T'(F)$, $\delta(\gamma) = |\Omega_F^\circ(T,G)_\gamma|$ and the conjugacy class of γ modulo $Z(F)$ intersects $Z(F)\backslash T(F)$ in $|\Omega_F^\circ(T,G)|\delta(\gamma)^{-1}$ points. Hence, if $\mathcal{J}_G$ is a set of representatives for the conjugacy classes of CSG's in G, we have

$$E(f) = \sum_{T \in \mathcal{J}_G} |\Omega_F^\circ(T,G)|^{-1} \mathrm{meas}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A})) \sideset{}{'}\sum_{\gamma \in Z(F)\backslash T(F)} \Phi(\gamma,f) .$$

where Σ' means sum over regular elements. For $\delta \in \mathcal{D}(T/F)$ and $\gamma \in T(F)$, let T^δ and γ^δ denote $h^{-1}Th$ and $h^{-1}\gamma h$ where $h \in \mathcal{A}(T)$ is any element representing δ. It suffices that T^δ and γ^δ are defined up to $G(F)$-conjugacy because the orbital integral

$$\Phi(\gamma^\delta,f) = \int_{h^{-1}T(\mathbb{A})h\backslash G(\mathbb{A})} f(g^{-1}h^{-1}\gamma hg)dg$$

depends only on δ.

Given a stable conjugacy class $\{T\}_{st}$ of CSG's and $T_0 \in \{T\}_{st}$, the number of T_0-conjugates of the form T^δ for $\delta \in \mathcal{D}(T/F)$ is equal to

$$\frac{|\Omega_F(T,G)|}{|\Omega_F^\circ(T,G)|}$$

and hence we may write:

$$E(f) = \sum_{T \in \mathcal{J}_G^{st}} |\Omega_F(T,G)|^{-1} \mathrm{meas}(Z(\mathbb{A})T(F)\backslash G(\mathbb{A})) \sideset{}{'}\sum_{\gamma \in Z(F)\backslash T(F)} \sum_{\delta \in \mathcal{D}(T/F)} \Phi(\gamma^\delta,f)$$

where $\mathcal{T}_G^{st}$ is a set of representatives for the stable conjugacy classes of CSG's of G.

Now fix $T \in \mathcal{T}_G^{st}$ and $\gamma \in Z(F)\backslash T(F)$ and consider the sum

$$\sum_{\delta \in \mathcal{D}(T/F)} \Phi(\gamma^{\delta}, f) \quad .$$

Set:

$$\mathcal{D}(T/\mathbb{A}) = \bigoplus_v \mathcal{D}(T/F_v)$$

$$\mathcal{E}(T/\mathbb{A}) = \bigoplus_v \mathcal{E}(T/F_v) \quad .$$

We have:

$$\begin{array}{ccc} \mathcal{D}(T/F) & \longrightarrow & \mathcal{D}(T/\mathbb{A}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{E}(T/F) & \xrightarrow{\psi_T} & \mathcal{E}(T/\mathbb{A}) \end{array}$$

and since $\Phi(\gamma^{\delta}, f) = \prod_v \Phi(\gamma^{\delta}, f_v)$, it is clear that $\Phi(\gamma^{\delta}, f)$ depends only on the image of δ under ψ_T.

Let $\mathfrak{K}(T/F)$ be the set of characters of $\mathcal{E}(T/\mathbb{A})$ which are trivial on $\psi_T(\mathcal{D}(T/F))$. By Tate-Nakayama duality, $|\text{Ker } \psi_T| < \infty$ and $[\mathcal{E}(T/\mathbb{A}) : \psi_T(\mathcal{E}(T/F)) < \infty$. The following two lemmas are proved in [1].

<u>Lemma</u> <u>8.2</u>: Let $\delta \in \mathcal{E}(T/F)$ and suppose that $\psi_T(\delta) \in \mathcal{D}(T/\mathbb{A})$. Then $\delta \in \mathcal{D}(T/F)$.

<u>Lemma</u> <u>8.3</u>: The set of places v of F such that $\Phi(\gamma^{\delta_v}, f_v) \neq 0$ for some

$\delta_v \in \mathcal{D}(T/F_v)$ with $\delta_v \neq 1$ is finite.

For $\delta \in \mathcal{E}(T/F) - \mathcal{D}(T/F)$ (resp. $\delta \in \mathcal{E}(T/F_v) - \mathcal{D}(T/F_v)$), set $\Phi(\gamma^\delta, f) = 0$ (resp. $\Phi(\gamma^\delta, f_v) = 0$). The orthogonality relations for finite abelian groups give:

$$\sum_{\delta \in \mathrm{Im}(\psi_T)} \Phi(\gamma^\delta, f) = \sum_{\kappa \in \mathfrak{K}(T/F)} \sum_{\delta \in \mathcal{E}(T/\mathbb{A})} \kappa(\delta)\Phi(\gamma^\delta, f) ,$$

where convergence of the right-hand side is assured by Lemma 8.3. By Lemma 8.2, $\Phi(\gamma^\delta, f)$ depends only on $\psi_T(\delta)$ for $\delta \in \mathcal{E}(T/F)$. Hence

$$\sum_{\delta \in \mathcal{E}(T/F)} \Phi(\gamma^\delta, f) = |\mathrm{Ker}\, \psi_T| \sum_{\kappa \in \mathfrak{K}(T/F)} \sum_{\delta \in \mathcal{E}(T/\mathbb{A})} \kappa(\delta)\Phi(\gamma^\delta, f) .$$

For v a place of F and κ_v a character of $\mathcal{E}(T/F_v)$, set

$$\Phi^{T/\kappa_v}(\gamma, f) = \sum_{\delta \in \mathcal{E}(T/F_v)} \kappa(\delta)\Phi(\gamma^\delta, f) .$$

For $\kappa \in \mathfrak{K}(T/F)$, let κ_v be the restriction of κ to $\mathcal{E}(T/F_v)$. Then Lemma 8.3 gives:

$$\Phi^{T/\kappa}(\gamma, f) = \sum_{\delta \in \mathcal{E}(T/\mathbb{A})} \kappa(\delta)\Phi(\gamma^\delta, f)$$

where $\Phi^{T/\kappa}(\gamma, f) = \prod_v \Phi^{T/\kappa_v}(\gamma, f_v)$. This proves the next proposition.

<u>Proposition</u> <u>8.4</u>:

$$E(f) = \sum_{T \in \mathcal{T}_G^{st}} \frac{|\mathrm{Ker}\, \psi_T|\, \mathrm{meas}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A}))}{|\Omega_F(T,G)| \cdot |\mathfrak{K}(T/F)|} \sum_{\gamma \in Z(F)\backslash T(F)}' \sum_{\kappa \in \mathfrak{K}(T/F)} \Phi^{T/\kappa}(\gamma, f) .$$

Twisted conjugacy:

Recall that $\tilde{G} = \mathrm{Res}_{E/F}(G)$. Since G is isomorphic to GL_3 over $\bar{F}$, $\tilde{G}$ is a twisted form of $GL_3 \times GL_3$. The action of $\mathcal{G}$ on $GL_3 \times GL_3$ which defines $\tilde{G}$ is: $((g_1,g_2) \in GL_3(\bar{F}) \times GL_3(\bar{F}))$

$$(g_1,g_2) \longmapsto \begin{cases} (\tau(g_1),\ \tau(g_2)) & \text{if } \tau|_E = 1 \\ (\tilde{\tau}(g_2),\ \tilde{\tau}(g_1)) & \text{if } \tau|_E = \sigma \end{cases}$$

where $\tilde{\tau}(g) = \Phi^{-1}\tau({}^tg^{-1})\Phi$ for $g \in GL_3(\bar{F})$ and $\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. For $g \in GL_3(F)$, $\tilde{\sigma}(g)$ will denote $\Phi^{-1}\sigma({}^tg^{-1})\Phi$. We have

$$\tilde{G}(F) = \{(g,\tilde{\sigma}(g)) : g \in GL_3(E)\} .$$

Let α be the automorphism of $\tilde{G}$ which interchanges the GL_3-factors: $\alpha((g_1,g_2)) = (g_2,g_1)$. Then G embeds in $\tilde{G}$ as the fixed point set of α, i.e.,

$$G = \{(g,g) \in \tilde{G}\} \qquad \text{and}$$

$$G(F) = \{(g,g) \in GL_3(E) \times GL_3(E) : \tilde{\sigma}(g) = g\} .$$

We define a norm map

$$N:\ \tilde{G} \longrightarrow \tilde{G}$$
$$g \longmapsto g\alpha(g) .$$

If we identify $\tilde{G}(F)$ with $GL_3(E)$ by projection onto the first factor, then

N on $GL_3(E)$ is the map $g \longmapsto g\tilde{\sigma}(g)$.

Lemma 8.5: Let $g = (g_1,\tilde{\sigma}(g_1)) \in \tilde{G}(F)$. Then the $\tilde{G}(F)$-conjugacy class of $N(g)$ intersects $G(F)$ in a unique stable conjugacy class.

Proof: $N(g) = (g_1\tilde{\sigma}(g_1),\tilde{\sigma}(g_1)g_1)$ and $g_1\tilde{\sigma}(g_1) \in GL_3(E)$. Since $\tilde{\sigma}(g_1\tilde{\sigma}(g_1)) = g_1^{-1}(g_1\tilde{\sigma}(g))g_1$, the conjugacy class of $g_1\tilde{\sigma}(g_1)$ in $G(E) = GL_3(E)$ is defined over F and hence Kottwitz's theorem implies that there is an $x \in GL_3(\bar{F})$ such that $x^{-1}(g_1\tilde{\sigma}(g_1))x \in G(F)$. Let $h = (x,\tilde{\sigma}(g_1)x)$. Then $h^{-1}N(g)h \in G(F) \subset \tilde{G}(F)$ and so the $\tilde{G}(F)$-conjugacy class of $N(g)$ intersects $G(F)$. If (y_1,y_1) and (y_2,y_2) in $G(F)$ are both $\tilde{G}(\bar{F})$-conjugate to $N(g)$, it is clear that y_1 and y_2 are $G(\bar{F}) = GL_3(\bar{F})$-conjugate and the lemma follows.

This lemma gives a map

$$\tilde{G}(F) \longrightarrow \{\text{stable conjugacy classes in } G(F)\}.$$

To describe the fibers of the map, we make the following definitions.

Definition: Let $\gamma_1, \gamma_2 \in \tilde{G}(F)$. We call γ_1 and γ_2 twisted conjugate (t-conjugate for short) if there is a $g \in \tilde{G}(F)$ such that $g^{-1}\gamma_1\alpha(g) = \gamma_2$ and twisted stably conjugate (tst-conjugate for short) if such a $g \in \tilde{G}(\bar{F})$ exists.

Since $N(x^{-1}\gamma\alpha(x)) = x^{-1}N(\gamma)x$, it is clear that the fibers of the above map are tst-conjugacy classes. Let

$$\mathcal{N} : \{\text{tst-conjugate classes in } \tilde{G}(\bar{F})\} \longrightarrow \{\text{st-conjugacy classes in } G(\bar{F})\}$$

be the resulting map. For $\gamma \in \tilde{G}(F)$, let $\{\gamma\}_{tst}$ denote the tst-conjugacy class of γ. We will write $\mathcal{N}(\gamma) = \gamma_0$ to indicate that $\mathcal{N}(\{\gamma\}_{tst}) = \{\gamma_0\}_{st}$.

For $\gamma = (\gamma_1, \gamma_2) \in \tilde{G}(F)$, set

$$\tilde{G}_{\gamma\alpha} = \{g \in \tilde{G} : g^{-1}\gamma\alpha(g) = \gamma\} .$$

The group $\tilde{G}_{\gamma\alpha}$ is defined over F since α is, and if $(g_1, g_2) \in \tilde{G}_{\alpha\gamma}$, then $g_1^{-1}N(\gamma)g_1 = N(\gamma)$. Projection onto the first factor gives an isomorphism $\tilde{G}_{\gamma\alpha} \to G_\gamma$ defined over $\bar{F}$.

If $\gamma_1, \gamma_2 \in \tilde{G}(F)$ and $g \in \tilde{G}(\bar{F})$ is such that $g^{-1}\gamma_1\alpha(g) = \gamma_2$, then

$$\{a_\alpha = \sigma(g)g^{-1}\} \in \mathrm{Ker}\{H^1(\overline{\mathfrak{G}}, \tilde{G}_{\gamma\alpha}) \longrightarrow H^1(\overline{\mathfrak{G}}, \tilde{G})\} .$$

Denote this kernel by $\mathcal{J}_\alpha(\gamma/F)$; it parametrizes the t-conjugacy classes within the tst-conjugacy class of γ.

Stabilization of the twisted elliptic term

For the purposes of the trace formula, we deal only with the F-points $\tilde{G}$ and it is convenient to deal instead with $G(E) = \tilde{G}(F) = GL_3(E)$. The center of $G(E)$ is $Z(E) = E^*$ and the norm map on Z is:

$$N : \quad Z(E) \longrightarrow Z(F)$$

$$z \longmapsto z/\bar{z} \quad .$$

We also have $N: Z(\mathbb{A}_E) \longrightarrow Z(\mathbb{A})$. Let $\tilde{\xi} = \xi \circ N$ where ξ is a character of $Z(F)\backslash Z(\mathbb{A})$ and let $\phi = \prod_v \phi v$ be a function on $G(\mathbb{A}_E)$ of the type to which we can apply the trace formula and assume that $\phi(zg) = \tilde{\xi}(z)^{-1}\phi(g)$.

Let $\mathcal{E}$ be a set of representatives for the t-conjugacy classes of $\gamma \in G(E)$ such that $\mathcal{N}(\gamma)$ is elliptic regular, taken module $Z(E)$.

We are interested first in the contribution from $\mathcal{E}$ to the trace formula applied to the kernel

$$\sum \phi(g^{-1}\gamma\alpha(h)) \quad .$$

As a function along the diagonal $g = h$, it is invariant under $Z(\mathbb{A}_E)$ since $\tilde{\xi}(z^{-1}\tilde{\sigma}(z)) = 1$ for all $z \in Z(\mathbb{A}_E)$. Set

$$\Phi_\alpha(\gamma,\phi) = \int_{Z(\mathbb{A}_E)\tilde{G}_{\alpha\gamma}(\mathbb{A})\backslash G(\mathbb{A}_E)} \phi(g^{-1}\gamma\tilde{\sigma}(g)dg$$

$$\Phi_\alpha(\gamma,\phi_v) = \int_{\tilde{Z}(F_v)\tilde{G}_{\alpha\gamma}(F_v)\backslash\tilde{G}(F_v)} \phi(g^{-1}\gamma\tilde{\sigma}(g)dg \quad .$$

Let $\delta_\alpha(\gamma)$ be the index of $Z(E)\tilde{G}_{\alpha\gamma}(F)$ in the α-centralizer of γ in $Z(E)\backslash G(F)$. Let

$$TE(\phi) = \sum_{\gamma\in\mathcal{E}} \delta_\alpha(\gamma)^{-1} \operatorname{meas}(Z(\mathbb{A})\tilde{G}_{\alpha\gamma}(F)\backslash\tilde{G}_{\alpha\gamma}(\mathbb{A}))\Phi_\alpha(\gamma,f) \quad .$$

This is the contribution of $\mathcal{E}$ to the twisted trace formula. For the next proposition, recall that $\mathcal{J}_G^{st}$ is a set of representatives for the st-conjugacy classes of CSG's of G.

Let $\tilde{F}^* = \{(z,1):\tilde{G}(\bar{F}): z \in F^*\}$. Then we have a map $\tilde{F}^* \longrightarrow \mathcal{D}_\alpha(\gamma/F)$ which sends $(z,1)$ to $\delta(z) = \{\sigma((z,1))(z,1)^{-1}\}$ and $\gamma^{\delta(z)} = z\gamma$. If $\delta \in \mathcal{D}_\alpha(\gamma/F)$ is represented by $\{\sigma(g)g^{-1}\}$, then $\delta(z)\delta$ is represented by $\sigma((z,1)g)$ and $\gamma^{\delta(z)\delta} = z\gamma^\delta$. Hence $\tilde{F}^*$ acts on $\mathcal{D}(\gamma/F)$. Let $\tilde{\mathcal{D}}_\alpha(\gamma/F)$ be the set of orbits of $\tilde{F}^*$ in $\mathcal{D}_\alpha(\gamma/F)$. Since $\tilde{\xi}$ is trivial on $\{z \in Z(E):N(z)=1\} = \{z \in Z(E) : z \in F^*\}$, the twisted orbital integral $\Phi_\alpha(\gamma^\delta,f)$ depends only on the image of δ in $\tilde{\mathcal{D}}_\alpha(\gamma/F)$.

Proposition 8.6:

$$TE(\phi) = \sum_{T \in \mathcal{T}_G^{st}} \sum'_{\gamma_0 \in Z(F)\backslash T(F)} |\Omega_F(T,G)|^{-1} \mathrm{meas}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A})) \sum_{\delta \in \tilde{\mathcal{D}}_\alpha(\gamma/F)} \Phi_\alpha(\gamma^\delta, f)$$

where the last sum is defined by any $\gamma \in \tilde{G}(F)$ such that $\eta(\gamma) = \gamma_0$ (it equals zero if $\{\gamma_0\}_{st}$ is not in the image of η).

Proof: For $\gamma \in \tilde{G}(F)$ and $\gamma_0 \in T(F)$ such that $\eta(\gamma) = \gamma_0$, set

$$\Omega_F(T,G)_{\gamma_0} = \{g \in G(\bar{F}) : g^{-1}\gamma_0 g \gamma_0^{-1} \in Z(F)\}/G_\gamma(\bar{F})$$

$$\Omega(\gamma) = \{g \in \tilde{G}(\bar{F}) : g^{-1}\gamma\alpha(g)\gamma^{-1} \in Z(E)\}/\tilde{F}^*\tilde{G}_{\gamma\alpha}(\bar{F}) .$$

Lemma 8.7: The map $\Omega(\gamma) \longrightarrow \Omega_F(T,G)_{\gamma_0}$ given by projecting $g = (g_1,g_2) \in \Omega(\gamma)$ to the first factor g_1 is an isomorphism.

Proof: If $g = (g_1,g_2)$ represents an element of $\Omega(\gamma)$ is such that $g^{-1}\gamma\alpha(g) = z\gamma$ for some $z \in Z(E)$, then $g_1^{-1}N(\gamma)g_1 = (z/\bar{z})N(\gamma)$. Hence $g_1 \in G_\gamma(\bar{F})$ if and only if $z \in F^* \subset z(E)$. If $g_1^{-1}N(\gamma)g_1 = (z/\bar{z})N(\gamma)$ (every element in $Z(F)$ is of the form $(z/\bar{z})$ for $z \in E^*$), then $g = (g_1,g_2)$ satisfies $g^{-1}\gamma\alpha(g) = z\gamma$, where $g_2 = \gamma_1^{-1}g_1 z\gamma_1$ and $\gamma = (\gamma_1,\tilde{\sigma}(\gamma_1)) \in \tilde{G}(F)$. To see that the map is an isomorphism, we have to show that if $g^{-1}\gamma\alpha(g) = z\gamma$ with $z \in F^*$, then $g \in \tilde{F}^* G_{\gamma\alpha}(\bar{F})$ and this is clear.

Now let

$$\Omega^0(\gamma) = \{g \in \tilde{G}(F) : g^{-1}\gamma\alpha(g)\gamma^{-1} \in Z(E)\}/Z(E)\tilde{G}_{\gamma\alpha}(F) .$$

It is clear that $|\Omega^0(\gamma)| = \delta_\alpha(\gamma)$. If $g \in \tilde{G}(F)$ is such that $g^{-1}\gamma\alpha(g) = z\gamma$ with $z \in F^*$, then $z \in N_{E/F}(E^*)$, as one sees by taking determinants: $z = N_{E/F}(\det(g))^{-1}z^{-2}$. Therefore, the obvious map $\Omega^0(\gamma) \longrightarrow \Omega(\gamma)$ is injective.

To prove the proposition, note that in the sum over $Z(F)\backslash T(F)$, a given stable conjugacy class $\{\gamma_0\}$ occurs $|\Omega_F(T,G)|\cdot|\Omega_F(T,G)_{\gamma_0}|^{-1}$-times. Let $\delta'_\alpha(\gamma^\delta)$ be the number of $\delta_1 \in \tilde{\mathcal{D}}_\alpha(\gamma/F)$ such that γ^{δ_1} is t-conjugate to $z\gamma^\delta$ for some $z \in Z(E)$. It will suffice to show that $|\Omega_F(T,G)_{\gamma_0}| = \delta_\alpha(\gamma^\delta)\delta'_\alpha(\gamma^\delta)$, or, by the above, that $\delta'_\alpha(\gamma) = [\Omega(\gamma) : \Omega^0(\gamma)]$ (we may take $\delta = 1$). This is clear from the definition of $\tilde{\mathcal{D}}_\alpha(\gamma/F)$.

§9. Conjugacy classes in G

For later use, it will be convenient to have a list of the stable conjugacy classes of CSG's in G. Let A be a fixed CSG of G; for the next Proposition, G can be any connected reductive group. If T is any other CSG, there is a $g \in G(\bar{F})$ such that $g^{-1}Ag = T$ (the map $t \longrightarrow g^{-1}tg$ is not, in general, defined over F). Hence $\{a_\sigma = \sigma(g)g^{-1}\} \in H^1(\bar{\mathfrak{g}},\bar{N})$ where $\bar{N}$ is the normalizer of A in $G(\bar{F})$. It is easy to check that $\{a_\alpha\}$ determines the G(F)-conjugacy class of T. Let $\bar{\Omega}$ be the Weyl group of A in $G(\bar{F})$ and let

$$\psi : H^1(\bar{\mathfrak{g}},\bar{N}) \longrightarrow H^1(\bar{\mathfrak{g}},\bar{\Omega})$$

be the natural map.

Both $\bar{\mathfrak{g}}$ and $\bar{\Omega}$ act on A and the character group $X^*(A) = \mathrm{Hom}(A,GL(1))$. A cocyle $\alpha = \{a_\sigma\} \in H^1(\bar{\mathfrak{g}}, \bar{W})$ defines a twisted action of $\bar{\mathfrak{g}}$ on $X^*(A)$:

$$\tilde{\sigma}(\alpha) = a_\sigma\cdot\sigma(\alpha) \qquad \text{for } \alpha \in X^*(A)$$

and hence $\alpha = \{a_\sigma\}$ defines a form A_α of A.

Proposition 9.1:

(a) Let T_1, T_2 be CSG's of G associated to cocycles $\alpha_1, \alpha_2 \in H^1(\bar{\mathfrak{G}}, \bar{N})$. Then T_1 and T_2 are st-conjugate if and only if $\psi(\alpha_1) = \psi(\alpha_2)$.

(b) If G is quasi-split, then every $\bar{\alpha} \in H^1(\bar{\mathfrak{G}}, \bar{W})$ is of the form $\psi(\alpha)$ where α arises from a CSG of G, i.e., $\bar{\alpha} = \psi(\alpha)$ for some $\alpha \in \mathrm{Ker}\{H^1(\bar{\mathfrak{G}}, \bar{N}) \longrightarrow H^1(\bar{\mathfrak{G}}, G)\}$.

Proof: Part (a) follows easily from the definitions. For (b), suppose $\bar{\alpha} \in H^1(\bar{\mathfrak{G}}, \bar{W})$ and let $A_{\bar{\alpha}}$ be the twisted form of A that it defines. Let $\gamma \in A_{\bar{\alpha}}(F)$ be regular. There is an isomorphism $\phi: A_{\bar{\alpha}}(\bar{F}) \longrightarrow A_\alpha(\bar{F})$ and $\phi(\gamma) \in A_\alpha(\bar{F})$ is regular. Furthermore, $\sigma(\phi(\gamma)) = a_\sigma \phi(\gamma) a_\sigma^{-1}$ for all $\sigma \in \bar{\mathfrak{G}}$, where $\alpha = \{a_\sigma\}$. Hence the conjugacy class of $\phi(\gamma)$ is defined over F. By Steinberg's theorem, $G(F)$ contains an element γ_0 in the $G(\bar{F})$-conjugacy of γ if G is quasi-split. The CSG G_{γ_0} then corresponds to the cocycle α.

We now consider the quasi-split unitary group in three variables $G = U_3$ with respect to a local or global quadratic extension E/F. We may assume that U_3 is the unitary group of the Hermitian form $\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (it is isomorphic to the unitary group of the form $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$). Let A be the diagonal subgroup of G. Then $\bar{W}$ is isomorphic to the symmetric group S_3. Let

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

The elements lie in $\bar{N}$ and we identify $\bar{W}$ with S_3 by mapping w, w_1, w_2 to the transpositions (13), (12), (23) respectively. The Galois group $\bar{\mathfrak{G}}$ acts on G as follows:

$$\tau : g \longmapsto \begin{cases} w^{-1}({}^{t}\tau(g))^{-1}w & \text{if } \tau|_E = \sigma \\ \tau(g) & \text{if } \tau|_E = 1 \end{cases}$$

where $\mathfrak{G}(E/F) = \{1,\sigma\}$. Under the identification of $\bar{W}$ with S_3, $\tau \in \bar{\mathfrak{G}}$ acts on S_3 trivially if $\tau|_E = 1$ and if $\tau|_E = \sigma$:

$$\tau : \begin{array}{l} (13) \longmapsto (13) \\ (12) \longmapsto (23) \\ (23) \longmapsto (12) \end{array} .$$

Let T be a CSG of G and let L be the centralizer of T in $M_3(E)$. Since L is a maximal, commutative, semi-simple subalgebra of $M_3(E)$, it is isomorphic to a direct sum of field extensions of E and the possibilities are:

(1) $L = E \oplus E \oplus E$

(2) $L = K \oplus E$ with K/E quadratic

(3) L is a cubic extension of E.

To state what we need about stable conjugacy classes of CSG's, we first define some tori. Recall that $\mathbf{E}^1$ is defined as the kernel of the norm map $N : \mathrm{Res}_{E/F}(\mathbf{G}_m) \longrightarrow \mathbf{G}_m$. Let K_1/F be a quadratic extension with $K_1 \neq E$ and let $K = K_1E$, so that $\mathfrak{G}(K/F) = (\mathbf{Z}/2)^2$. Let $\sigma_1, \tau_1 \in \mathfrak{G}(K/F)$ be such that K_1 is the fixed field of σ_1 and E is the fixed field of τ_1. Define a two-dimensional torus T_{K_1} over F by the exact sequence:

$$1 \longrightarrow T_{K_1} \longrightarrow \mathrm{Res}_{K/F}(\mathbb{G}_m) \xrightarrow{N_{K/K_1}} \mathrm{Res}_{K_1 F}(\mathbb{G}_m) \longrightarrow 1$$

where N_{K/K_1} is the map $(1+\sigma_1)$.

If L/E is a cubic extension with an automorphism $\tilde{\sigma} \in \mathrm{Aut}(L)$ of order two whose restriction to E is σ, let $L^{\tilde{\sigma}}$ be the fixed field of $\tilde{\sigma}$ and define T_L by the exact sequence

$$1 \longrightarrow T_L \longrightarrow \mathrm{Res}_{L/F}(\mathbb{G}_m) \xrightarrow{\tilde{N}} \mathrm{Res}_{L^{\tilde{\sigma}}/F}(\mathbb{G}_m) \longrightarrow 1$$

where $\tilde{N}$ is the map $(1+\tilde{\sigma})$. Then T_L is a torus of dimension three over F.

Proposition 8.4: Let T be a CSG of G. Then T is isomorphic to one of the following types:

(0) $A = \mathrm{Res}_{E/F}(\mathbb{G}_m) \times \mathbb{E}^1$ (the CSG contained in B)

(1) $\mathbb{E}^1 \times \mathbb{E}^1 \times \mathbb{E}^1$

(2) $T_{K_1} \times \mathbb{E}^1$ where K_1/F is quadratic with $K_1 \neq E$.

(3) T_L where L/E is a cubic extension with an automorphism $\tilde{\sigma} \in \mathrm{Aut}(L)$ of order two whose restriction to E is σ.

Furthermore, in cases (0), (1), and (2), the stable conjugacy class of T is determined by the isomorphism class of T as a torus over F.

Proof: By Lemma 8.3, the stable conjugacy classes of CSG's are parametrized by $H^1(\mathfrak{g},\bar{W})$. Let T be a CSG of G and let L be the centralizer of T in $M_3(E)$. We consider three cases separately.

Case (i): $L = E \oplus E \oplus E$. Then T splits over E and $\{T\}_{st}$ is determined by a cocycle in $H^1(\mathcal{G}(E/F),W)$, i.e., by an element $a_\sigma \in \bar{W}$ such that $a_\sigma \cdot \sigma(a_\sigma) = 1$. The possibilities are $a_\sigma = 1$, (123), (132), or (13) and since $\sigma((12))(123)(12) = \sigma((23))(132)(23) = 1$, the choices $a_\sigma = (123)$ or (132) are cohomologous to $a_\sigma = 1$. Hence we may assume that $a_\sigma = 1$ or $a_\sigma = (13)$. If $a_\sigma = 1$, then $\{T\}_{st} = \{A\}$ and if $a_\sigma = (13)$, then the twisted action of σ on $X_*(A)$ is the multiplication by -1. This is clear since the map $a_\sigma \cdot \sigma$ on the diagonal subgroup is $g \longmapsto g^{-1}$. So in this case, $T \xrightarrow{\sim} \mathbb{E}^1 \times \mathbb{E}^1 \times \mathbb{E}^1$.

<u>Lemma</u> <u>8.5</u>: Let T be a CSG of G and let K/E be the splitting field of T. Let K' be the Galois closure of K over E. Then K is Galois over F.

<u>Proof</u>: The involution $g \longmapsto \Phi^{-1}\,{}^t\sigma(g)\Phi$ stabilizes T(F), hence L, and induces an automorphism σ' of K whose restriction to E is σ. It follows that K' is stable under $\bar{\mathcal{G}}$.

<u>Case</u> <u>(ii)</u>: In this case, T splits over K with K/E quadratic. By Lemma 8.4, K/F is Galois and hence $\mathcal{G}(K/F) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$ or $\mathbf{Z}/4$. We first show that $\mathbf{Z}/4$ cannot occur. If it did and if τ_0 were a generator of $\mathcal{G}(K/F)$, then τ_0^2 would act trivially on E. The cocycle $\{a_\tau\} \in H^1(\mathcal{G}(K/F),\bar{W})$ associated to T would satisfy $(a_{\tau_0^2})^2 = (\tau_0(a_{\tau_0})a_{\tau_0})^2 = 1$ which implies that $a_{\tau_0} = 1$ or (13) (the cases (123) and (132) are cohomologous to $a_{\tau_0} = 1$ as in Case (i)). Hence $a_{\tau_0}^2 = 1$ and T splits over E, which is Case (i). Hence $\mathcal{G}(K/F) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$. Let σ_1 and τ_1 generate $\mathcal{G}(K/F)$ with $\sigma_1|_E = \sigma$ and $\tau_1|_E = 1$. Then

$$a_{\sigma_1\tau_1} = a_{\tau_1\sigma_1} = \sigma_1(a_{\tau_1})a_{\sigma_1} = a_{\sigma_1}a_{\tau_1} \quad \text{or}$$

$$a_{\sigma_1}^{-1}(\sigma_1(a_{\tau_1}))a_{\sigma_1} = a_{\tau_1} \quad .$$

Up to coboundaries, the possibilities are

(1) $a_{\sigma_1} = 1$ $a_{\tau_1} = (13)$ $a_{\sigma_1\tau_1} = (13)$

(2) $a_{\sigma_1} = (13$ $a_{\tau_1} = (13)$ $a_{\sigma_1\tau_1} = 1$

and since σ_1 and $\sigma_1\tau_1$ both induce σ on E, their roles may be interchanged and we may assume the cocycle has the form (2). Let K_1 be the fixed field of σ_1 in K. It is easy to check that T is isomorphic to $T_{K_1} \times \mathbb{E}^1$.

Case (3) is clear since the involution $g \longrightarrow \Phi^{-1}\,{}^t\sigma(g)\Phi$ induces an automorphism $\tilde{\sigma}$ of order two on L whose restriction to $\mathbb{E}$ is σ.

Thursday Morning Seminar

DIVISION ALGEBRAS I

R. Langlands

Introduction.

In the Friday afternoon seminar a method for comparing traces on different groups was described and applied to some low-dimensional groups of rank one. As a further test of its effectiveness we will consider here the comparison of the trace formula on $G = GL(n)$ and $G' = D^*$, D being a division algebra of degree n^2 over the global field F of characteristic zero.

It should be emphasized that the advantage of the method is that it does not require that the trace formula be made invariant, so that many problems in local harmonic analysis are avoided. On the other hand it cannot, so far as I can see, be applied when one is working, for whatever reason, with a single trace formula.

The procedure.

We suppose we are given a function $\phi' = \prod_v \phi'_v$ on G', smooth and of compact support, and a similar function $\phi = \prod \phi_v$ on G. We suppose that at each place v of F and for each regular γ in G_v the orbital integral $\Phi(\gamma, \phi_v)$ is equal to $\Phi(\gamma, \phi'_v)$ if γ occurs in G'_v and to 0 otherwise. We want to show that

$$\Theta_{G'}(\phi') = \Theta_G(\phi) \quad .$$

Observe that G and G' have no non-trivial cuspidal endoscopic groups, so that stabilization is superfluous.

The trace formula for G' reads simply

$$J_{G'}(\phi') = \Theta_{G'}(\phi') \quad .$$

The trace formula for G reads

$$\sum_M J_M^T(\phi) = \sum_M \Theta_M^T(\phi) \quad .$$

The sum over M runs over conjugacy classes of Levi subgroups of parabolic subgroups of G. They are indexed by unordered partitions of n. One lesson to be drawn from the following is that it is better to use a sum over $M \supseteq M_0$.

We take it for granted that $J_G^T(\phi) = J_G(\phi)$ is equal to $J_{G'}(\phi')$, obtaining an equality

$$(1) \qquad \Theta_G(\phi) - \Theta_{G'}(\phi') = \sum_{M\neq G} J_M^T(\phi) - \sum_{M\neq G} \Theta_M^T(\phi) \quad .$$

Before going on I underline a peculiar feature of the notation. Both J_M^T and Θ_M^T are distributions on $\mathbf{G}$ and thus depend on the pair M and G, the dependence on G being implicit in the function ϕ. If however we write $J_M^T(\psi_{M'})$ or $\Theta_M^T(\psi_{M'})$ where $M \subseteq M'$ and $\psi_{M'}$ is a function on $\mathbf{M}'$ then it is understood that the distributions involved are those for the pair M, M'. Since the T is of no concern to us we drop it from the notation.

What we want to do is show the existence of smooth, compactly supported functions ψ^M on M, $M \neq G$, such that

(2) $$\sum_{M \neq G} J_M(\phi) = \sum_{M'} \sum_M J_M(\psi^{M'}) \quad ,$$

the inner sum being taken over conjugacy classes of M in M'. The trace formula for M' yields

$$\sum_M J_M(\psi^{M'}) = \sum_M \Theta_M(\psi^{M'}) \quad ,$$

and the relation (1) becomes

$$\Theta_G(\phi) - \Theta_{G'}(\phi') = \sum_{M' \neq G} \sum_M \Theta_M(\psi^{M'}) - \sum_{M \neq G} \Theta_M(\phi) \quad .$$

This equality will allow us - provided the $\psi^{M'}$ satisfy a supplementary condition that is to be explained later - to proceed with the argument on decomposition of measures and to show not only that

$$\Theta_G(\phi) = \Theta_{G'}(\phi')$$

but also that for each $M \neq G$

$$\sum_{M'} \Theta_M(\psi^{M'}) = \Theta_M(\phi) \quad ,$$

appropriate care being taken with the range of summation on the left.

However, our concern at present is with (2) and indeed at first with a weaker statement. We anticipate that we will be provided with an expression for $J_M(\phi)$ of the following form,

$$J_M(\phi) = \frac{|\Omega^M|}{|\Omega^G|} \sum_{M'} \sum_{\sigma'} J_{M'}(\sigma', \phi) \quad ,$$

the outer sum running over all M' containing M_0 and conjugate to M and the inner sum over all elliptic conjugacy classes in M. Thus we may expect that (2) reduces to a collection of equalities, one for each $\sigma \subseteq G$,

$$(3) \quad \sum_{\substack{M \neq G \\ M_0 \subseteq M}} \frac{|\Omega^M|}{|\Omega^G|} \sum_{\sigma_M \subseteq \sigma} J_M(\sigma_M, \phi) = \sum_{M' \neq G} \sum_{M_0 \subseteq M \subseteq M'} \frac{|\Omega^M|}{|\Omega^{M'}|} \sum_{\sigma_M \subseteq \sigma} J_M(\sigma_M, \psi^{M'}) \quad .$$

It is thus convenient to fix a Levi subgroup M_0 of P_0 and to work only with M containing M_0. It will also be convenient to suppose that ψ^Q, a function on $\mathbf{M}_Q$, is defined for each Q containing M_0. So we are reformulating the problem, the functions we originally introduced being $m \longrightarrow \sum_{M'} \sum_{Q \in \mathcal{P}(H')} \psi^Q(g^{-1}mg)$, with g in the normalizer of M_0, $g^{-1}Mg = M'$, and M' running over the conjugates of M.

In this lecture we are concerned with (3) only for regular σ, and it is clear it would follow from

$$(5) \qquad J_M(\sigma, \phi) = \sum_{\substack{M \subseteq Q \\ Q \neq G}} J_M(\sigma, \psi^Q) \quad ,$$

σ now being a conjugacy class in M.

Let $\rho(M)$ be the dimension of the center of M minus the dimension of G and let $\rho(Q) = \rho(M_Q)$. The functions ψ^Q are to be defined inductively on $\rho(Q)$, starting with $\rho(Q) = 1$. The only condition is that (5) be satisfied at each stage.

Weighted orbital integrals.

Let S be a finite set of places containing all infinite places, all places at which D ramifies, and all places at which ϕ_v is not a spherical function. If $v \notin S$ and if M is the Levi factor of Q then

$$(6) \qquad \phi_{Q,v}(m) = \rho_Q(m) \int_{K_v} \int_{N_Q(F_v)} \phi_v(k^{-1}mnk)\,dn\,dk \quad ,$$

ρ_Q being the square root of the absolute value of the determinant of $\mathrm{ad}\, m \backslash \mathfrak{n}_Q$, is a spherical function on M_v and is independent of Q. So we sometimes denote it $\phi_v^{M_Q}$. We demand, and this is the supplementary condition mentioned above, that

$$\psi^Q = \psi_S^Q \prod_{v \notin S} \phi_v^{M_Q} \quad .$$

Since the function ϕ has compact support there will be only finitely many conjugacy classes $\mathfrak{o}$ for which $J_M(\mathfrak{o}_M, \phi) \neq 0$ for some M and some $\mathfrak{o}_M \subseteq \mathfrak{o}$. It is easily seen that we can choose the finite set of places $S' = S(\phi)$ to be so large that for each such $\mathfrak{o}$ and $\mathfrak{o}_M \subseteq \mathfrak{o}$ the class $\mathfrak{o}_M$ has a representation γ which for $v \notin S'$ lies in K_v and is such that $g^{-1}\gamma g \in K_v$, $g \in G_v$ implies that $g \in G_\gamma(F_v)K_v$. The group K_v is of course the standard maximal compact subgroup of G_v.

As a consequence we may replace $J_M(\mathfrak{o}_M, \phi)$ by

$$c(\mathfrak{o}_M) \int_{G_\gamma(\mathbb{A}_{S'}) \backslash G(\mathbb{A}_S)} \phi(g^{-1}\gamma g) v_M^G(g)\,dg$$

or

$$c(\mathfrak{o}_M)\int_{G_\gamma(\mathbf{A}_{S'})\backslash G(\mathbf{A}_S)} \phi_{S'}(g^{-1}\gamma g)V_M^G(\gamma, g)dg \quad .$$

Let

$$K^{S'} = \prod_{v \notin S} K_v$$

and let $\delta(\mathfrak{o}_M)$ be 1 or 0 according as $\mathfrak{o}_M$ does or does not meet

$$\{g^{-1}K^{S'}g | g \in G(\mathbf{A}^{S'})\} \quad .$$

Then

$$c(\mathfrak{o}_M) = \text{meas}(G_\gamma\backslash \mathbf{G}'_\gamma)\text{meas}(K^{S'} \cap \mathbf{G}_\gamma\backslash K^{S'})\delta(\mathfrak{o}_M) \quad .$$

The weight factor $v_M^G(g)$ is that given by the trace formula and $V_M^G(\gamma, g)$ that given by Flicker's trick. For the γ being studied at the moment they are equal.

We may disregard the factor $c(\mathfrak{o}_M)$, examining instead

$$J_M(\gamma, \phi) = |D^G(\gamma)|^{\frac{1}{2}}\int_{G_\gamma(\mathbf{A}_{S'})\backslash G(\mathbf{A}_{S'})} \phi_{S'}(g^{-1}\gamma g)V_M^G(\gamma, g)dg \quad .$$

The factor $|D^G(\gamma)|$ is that introduced on p. 30 of Arthur's Annals paper. It is 1 for regular semi-simple γ in G. We shall study $J_M^G(\gamma, \phi)$ for γ in $M(\mathbf{A}_{S'})$ and regular in G. We need to show, by the same inductive procedure, the existence of $\psi_{S'}^Q$ such that

$$(7) \qquad J_M(\gamma,\ \phi_{S'}) = \sum_{\substack{M\subset Q\\ Q\neq G}} J_M(\gamma,\ \psi^Q_{S'})$$

for every M. There is only one observation to make. We began by choosing $S' = S(\phi)$. It can always be made larger. As we construct the ψ^Q inductively we will introduce $S(\psi^Q)$. They may not be contained in S'. We simply enlarge S' at each stage to accommodate them.

This being understood we work entirely within the given set S', sometimes dropping it from the notation. All functions will be spherical outside of S.

Basic lemmas.

The distributions $\phi \longrightarrow J_M(\gamma,\ \phi) = J^G_M(\gamma,\ \phi)$ are very similar to the distributions $f \longrightarrow J^G_{M,\gamma}(f)$ studied in §8 of Arthur's Annals paper. Apart from the fact that our S' is his S the difference is that he works with v^G_M rather than V^G_M. The analogue of his Lemma 8.2 is valid and the proof is exactly the same.

LEMMA 1. _Let_ $L \supseteq M$ _be Levi factors of_ G _over_ F. _Let_ h _lie in_ $L(\mathbf{A}_{S'})$ _and let_ γ _on_ $M(\mathbf{A}_{S'})$ _be regular in_ $L(\mathbf{A}_{S'})$. If ψ _is smooth and of compact support on_ $L(\mathbf{A}_{S'})$ _then_

$$J_M(\gamma,\ \psi^h) = \sum_{Q\in \mathcal{F}(M)} J_M(\gamma,\ \psi_{Q,h}) \quad .$$

The sum runs over the parabolic subgroups of L over F which contain M. Moreover $\psi_{Q,h}$ is a function on $M_Q(\mathbf{A}_{S'})$. So in agreement with our notational conventions $J_M(\gamma,\ \psi_{Q,h})$ is $J^{M_Q}_M(\gamma,\ \psi_{Q,h})$. The function

$\psi_{Q,h}$ is defined on p. 20 of Arthur's paper and is smooth and of compact support on $M_Q(\mathbf{A}_{S'})$.

It is important to observe that - this one sees immediately from the definition - if $h \in L(\mathbf{A}_S)$ then $f = \psi_{Q,h}$ is a product,

$$f = f_S \cdot \prod_{v \in S'-S} \phi_v^{M_Q} \quad .$$

The function $\phi_v^{M_Q}$ is defined by (6) and $f_S = \psi_{S,Q,h}$.

For technical reasons it is convenient to fuse all infinite places into a single place, denoted ∞. With this convention F_∞ denotes $\prod_{v \in S_\infty} F_v$ and $M_\infty = \prod_{v \in S_\infty} M_v$. Moreover $S = S_{fin} \cup \{\infty\}$, $S' = S'_{fin} \cup \{\infty\}$.

Suppose that L is a Levi factor of G containing M_0 and that f_S is a smooth, compactly supported function on $L(\mathbf{A}_S)$. A collection of functions F_S^Q, Q lying in L and F_S^Q being a smooth, compactly supported function $M_Q(\mathbf{A}_S)$, will be said to be adapted to f_S if for every $S' \supseteq S$ and every collection of spherical functions f_v, $v \in S'-S$, every M, $M_0 \subset M \subset L$, and every semi-simple γ in $M(\mathbf{A}_{S'})$ regular in G the equality

$$J_M^L(\gamma, f_S \cdot \prod_{v \in S'-S} f_v) = \sum_{\substack{Q \in \mathcal{F}^L(M) \\ Q \neq L}} J_M(\gamma, F_S^Q \cdot \prod_v f_{Q,v})$$

holds. The functions $f_{Q,v}$ are defined by the equation (6), f_v replacing ϕ_v.

LEMMA 2. <u>Suppose that</u> $f_S = \prod_{v \in S} f_v$ <u>and that for some given</u> v <u>the orbital integral</u> $\Phi(\gamma, f_v) = 0$ <u>for all semi-simple</u> γ <u>in</u> $\mathfrak{m}(F_v)$ <u>regular</u>

in L.

in $L(F_v)$. *Then a collection* F_S^Q *adapted to* f_S *exists*.

For $v = \infty$ the proof of the lemma draws on various facts whose explanation it is convenient to postpone. For v finite it is however an immediate consequence of Lemma 1 and the following lemma of Vignéras (cf. §2 of *Caractérisation des intégrales orbitales sur un groupe réductif p-adique* and App 1.1 of *Représentations des algèbras centrales simples p-adiques*). Gelfand-Kazhdan?

LEMMA 3. *If* v *is finite and* $\Phi(\gamma, f_v) = 0$ *for all regular semi-simple* γ *then* f_v *may be expressed as a sum* $\sum_i f_{i,v}^{h_i} - f_{i,v}$, $h_i \in L(F_v)$.

We set $f_i = f_{i,v} \cdot \prod_{w \neq v} f_v$ and take

$$F_S^Q = \sum_i f_{i,Q,h_i} \quad .$$

LEMMA 4. *Suppose that* f *is a smooth function with compact support on* $M(\mathbf{A}_S)$ *and that for every regular semi-simple* γ *the orbital integral* $\Phi(\gamma, f) = 0$. *Then* f *is a sum* $\sum_{v \in S} f^v$ *where* f^v *is a sum of functions* f_i^v *such that each* f_i^v *is a product* $\prod_{w \in S} F_w$ *and* $\Phi(\gamma, F_v) = 0$ *for all regular semi-simple* γ *in* $M(F_v)$.

Observe that the F_w depend on v and i. It is inconvenient and unnecessary to incorporate this in the notation.

The lemma is proved by induction on the cardinality of S. It is trivial if S_{fin} is empty. So choose $v \in S_{fin}$. If C is a compact subset of M_v and U an open compact subgroup of it, let $\mathcal{H}(C/\!/U)$

be the set of functions on M_v supported by C and bi-invariant under U. We write $m \in M(\mathbf{A}_S)$ as (m_1, m_2), $m_1 \in M(\mathbf{A}_{S_1})$, $m_2 \in M(F_v)$, $S_1 = S - \{v\}$. Choose C and U such that for each m_1 the function $m_2 \longrightarrow f(m_1, m_2)$ lies in $H(C /\!/ U)$.

Since $H(C /\!/ U)$ is finite dimensional we can find, for a suitable ℓ, functions $h_1, \ldots, h_\ell$ in it and ℓ regular semi-simple elements $\gamma_1, \ldots, \gamma_\ell$ in M_v such that $\Phi(\gamma_i, h_j) = \delta_{ij}$ and such that any other function in $H(C /\!/ U)$ is of the form $\sum_{j=1}^{\ell} a_j h_j + h$, where $h \in H(C /\!/ U)$ and all its orbital integrals are 0. In particular

$$f(m_1, m_2) = \sum a_j(m_1) h_j(m_2) + h(m_1, m_2)$$

with

$$a_j(m_1) = \int_{M_\gamma(F_v) \backslash M(F_v)} f(m_1, m_2^{-1} \gamma m_2)\, dm_2 \quad .$$

The lemma follows.

We shall be faced with the following problem. We shall be given a function $\gamma \longrightarrow \Psi(\gamma)$ on regular semi-simple classes in $M(\mathbf{A}_S)$ and we will want to show that there is a smooth, compactly supported function f such that $\Psi(\gamma) = \Phi(\gamma, f)$ for all such γ.

If v is a place in S we say that Ψ is satisfactory at v if for each regular semi-simple γ_1 in $M(\mathbf{A}_{S_1})$ there is a function $f^v_{\gamma_1}$ on $M(F_v)$ smooth and of compact support, such that:

(i) For all regular semi-simple γ_2 in $M(F_v)$ and all regular semi-simple γ_1 in $M(\mathbf{A}_{S_1})$

$$\Psi(\gamma_1, \gamma_2) = \Phi(\gamma_1, f^v_{\gamma_2}) \quad .$$

(ii) If v is finite then there is a C and U such that $f^v_{\gamma_2} \in H(C//U)$ for all γ_2.

LEMMA 5. <u>If</u> Ψ <u>is satisfactory at every</u> v <u>in</u> S <u>then there is a smooth, compactly supported function</u> f <u>on</u> $M(\mathbf{A}_S)$ <u>such that</u> $\Psi(\gamma) = \Phi(\gamma, f)$ <u>for all regular semi-simple</u> γ <u>in</u> $M(\mathbf{A}_S)$.

We argue by induction. If S contains only one element there is nothing to prove. So suppose S_{fin} contains v. Choosing $h_1, \ldots, h_\ell$ and $\gamma_1, \ldots, \gamma_\ell$ as above we may write

$$f^v_\gamma = \sum a_j(\gamma)h_j + h_\gamma \quad ,$$

with

$$a_j(\gamma) = \Phi(\gamma_j, \gamma) \quad .$$

By induction there are functions f'_{γ_j} on $M(\mathbf{A}_{S_1})$, smooth and of compact support, such that

$$\Phi(\gamma_j, \gamma) = \Phi(\gamma, f'_{\gamma_j}) \quad .$$

We may take

$$f(m_1, m_2) = \sum_j f'_{\gamma_j}(m_1)h_j(m_2) \quad .$$

LEMMA 6. *Suppose that* R *is a subset of* S *containing at least two elements and that for every* v *in* R *and every regular semi-simple* γ_2 *in* $M(F_v)$ *there is a smooth compactly supported function* f_{γ_2} *on* $M(\mathbf{A}_{S_1})$ *such that*

$$\Psi(\gamma_1, \gamma_2) = \Phi(\gamma_1, f_{\gamma_2})$$

for regular, semi-simple γ_1 *in* $M(\mathbf{A}_{S_1})$. *Suppose moreover that for each finite* $w \neq v$, *all* γ_2, *and all* $m' \in \prod_{x \neq w, v} M_x$ *the function* $m \longrightarrow f_{\gamma_2}(m', m)$ *lies in* $\mathcal{H}(C//U)$ *for some given* C *and* U. *Then* Ψ *is satisfactory at every place in* S.

This is clear.

A simple problem. The functions $\psi^{\mathfrak{Q}}$ will be chosen inductively and the lack of unicity is somewhat disconcerting. To allay at least some of the unease we consider a Levi factor L of G and the trace formula for a function ψ on $\mathbf{L}$ so chosen that $\psi = \prod_v \psi_v$ and, for some v, all orbital integrals of ψ_v are zero. With this assumption the trace formula should be trivial and we should be able to take all terms from the left to the right without any difficulty.

This means that we should be able to find functions ψ on $\mathbf{M}^Q$, $Q \in \mathcal{F}(M_0)$. $Q \neq G$ such that the analogue of (5) is satisfied,

$$J_M^L(\sigma, \psi) = \sum_{M \subseteq Q \subsetneq L} J_M^L(\sigma, \psi^Q) \quad .$$

Lemma 2 guarantees the existence of ψ^Q satisfying this relation.

Existence.

We come now to the proof of the existence of the functions ψ^Q attached to $\phi = \prod_v \phi_v$. The critical property of the function ϕ is the following (cf. part C of Vignéras's notes).

Suppose $L \neq G$. Then there exist at least two places $v \in S$ such that for all $\gamma \in L(F_v)$ which are regular in G the orbital integrals $\Phi(\gamma, \phi_v)$ are zero.

Let $n_1, \ldots, n_r$ be the partition defining M and let d_v be the denominator of the invariant of D at v. In order that $\Phi(\gamma, \phi_v)$ is not zero for all γ in $L(F_v)$ all $n_1, \ldots, n_r$ must be divisible by d_v. If this were so at all but one v it would be so at all v and then r would be 1 and $L = G$.

For $\rho(M) = 1$ the factors $V_M^G(\gamma, g)$ are linear, and

$$J_M(\gamma, \phi) = |D^G(\gamma)|^{\frac{1}{2}} \sum_v \int_{G_\gamma(\mathbf{A}_{S'})\backslash G(\mathbf{A}_{S'})} \phi(g^{-1}\gamma g) V_M^G(\gamma, g_v) = 0 \quad .$$

Thus we may take $\psi^Q = 0$ if $\rho(Q) = 1$.

We now suppose that for $\rho(Q) < \rho$ we have so defined ψ^Q that

$$J_M(\gamma, \phi) = \sum_{\substack{M \subseteq Q \\ Q \neq G}} J_M(\gamma, \psi^Q) \tag{8}$$

for $\rho(M) < \rho$, and we prove the existence of ψ for $\rho(Q) = \rho$ satisfying the corresponding equation. We apply Lemma 6 to the difference

$$J_M(\gamma, \phi) - \sum_{\substack{M \subseteq Q \\ Q \neq G \\ Q \notin P(M)}} J_M(\gamma, \psi^Q) \quad ,$$

proving that it is equal to ~~ϕ~~ $J_M(\gamma, f)$ for some smooth compactly supported function on $M(\mathbb{A}_{S'})$ and we set

$$\psi^Q = \frac{1}{|\mathcal{P}(M)|} f \quad ,$$

for $Q \in \mathcal{P}(M)$.

Let v be a place in S such that $\Phi(\gamma, \phi_v) = 0$ for all $\gamma \in M_v$ regular in G_v. We need only show that for such a v the condition of Lemma 6 is satisfied.

According to Lemma 6.3 of Arthur's Annals paper we have a decomposition of $v_M^G(g)$ as

$$\sum_{Q \in \mathcal{L}(M)} v_M^Q(g_1) U_Q(g_2)$$

where $g_1 \in G(\mathbb{A}_{S_1})$, $g_2 \in G(F_v)$. This leads to a decomposition

$$(9) \qquad J_M(\gamma, \phi_S) = \sum_{Q \in \mathcal{L}(M)} J_M(\gamma_1, \phi_Q^1) L_Q(\gamma_2, \phi_2)$$

where $\phi(g) = \phi^1(g_1)\phi^2(g_2)$ and

$$\phi_Q^1(m) = \rho_Q(m) \int_{K_1} \int_{N_Q(\mathbb{A}_{S_1'})} \phi^1(k^{-1}mnk)\,dn\,dk \quad .$$

If $Q = G$ then $U_Q(g_2) \equiv 1$ and $L_Q(\gamma_2, \phi^2) = 0$, for it is an ordinary orbital integral and $\gamma_2 \in M_v$. So we drop the term corresponding to $Q = G$ from the sum (9).

For any other Q we apply Lemma 2 to the function ϕ_Q^1 and write

$$J_M(\gamma_1, \phi_Q^1) = \sum_{\substack{Q' \subsetneq Q \\ M \subseteq Q'}} J_M(\gamma_1, F_Q^{Q'}) \quad .$$

Since γ_2 and ϕ_2 play the role of fixed parameters in the discussion we remove them from the notation, setting

$$F^{Q'} = \sum_{Q' \subseteq Q} F_Q^{Q'} L_Q(\gamma_2, \phi_2) \quad .$$

We also have a decomposition

$$J_M(\gamma, \psi^Q) = \sum_{M \subseteq Q' \subseteq Q} J_M(\gamma_1, \psi_{Q'}^{Q,1}) L_{Q'}(\gamma_2, \psi^{Q,2})$$

We set

$$H^{Q'} = \sum_{\substack{Q' \subseteq Q \\ 2 \leq \rho(Q) < \rho}} \psi_{Q'}^{Q,1} L_{Q'}(\gamma_2, \psi^{Q,2}) \quad .$$

Both $H^{Q'}$ and $F^{Q'}$ are smooth, compactly supported functions on $M^{Q'}(\mathbf{A}_{S_1})$.

By assumption

(10) $$\sum_{Q' \in \mathcal{F}(M')} J_{M'}(\gamma_1, F^{Q'}) - \sum_{Q' \in \mathcal{F}(M')} J_{M'}(\gamma_1, H^{Q'}) = 0$$

if $\rho(M') < \rho$.

If $\rho = 2$ then

$$J_M(\gamma_1, \gamma_2, \phi) = \sum_{Q \in \mathcal{P}(M)} J_M(\gamma_1, F^Q) \quad .$$

Thus we may take

$$f_{\gamma_2} = \sum_{Q \in P(M)} F^Q \quad .$$

Now we suppose that $\rho > 2$ and we define inductively on $\rho(M')$, $2 \leq \rho(M') \leq \rho(M)$, functions $E^{M',Q}$ on $M_Q(\mathbf{A}_{S_1'})$, $Q \in \mathcal{P}^{M'}(M)$, $Q \neq M'$, which are smooth and of compact support.

They will be shown inductively to have the property that the orbital integrals of

$$f_{S_1}^{M''} = \sum_{Q'' \in P(M'')} F_{S_1}^{Q''} - \sum_{Q'' \in P(M'')} H_{S_1}^{Q''} + \sum_{\substack{M' \supsetneq M'' \\ \rho(M') \geq 2}} \sum_{Q'' \in P^{M'}(M'')} E_{S_1}^{H',Q''}$$

are zero on regular semi-simple elements if $\rho(M'') < \rho$.

This being so we will take the functions $E_{S_1}^{M'',Q}$ to be those attached to $f_{S_1}^{M''}$ by Lemma 2. The relation (10) assures us that for $\rho(M'') = 2$ the function

$$f_{S_1}^{M''} = \sum_{Q'' \in P(M'')} F_{S_1}^{Q''} - \sum_{Q'' \in P(M'')} H_{S_1}^{Q''}$$

has orbital integrals zero. So we can begin the process.

To verify that it continues we have to evaluate $J_{M''}(\gamma_1, f_{S_1}^{M''})$. Observe first of all that if $M''' \subseteq M''$ then

$$\sum_{Q'' \in P(M'')} J_{M'''}(\gamma_1, F_{S_1}^{Q''}) - \sum_{Q'' \in P(M'')} J_{M'''}(\gamma_1, H_{S_1}^{Q''}) \tag{11}$$

plus

$$\sum_{\substack{M' \supset M'' \\ \rho(\bar{M}') \geq 2}} \sum_{Q'' \in P^{M'}(M'')} J_{M'''}(\gamma_1, E_{S_1}^{M',Q''}) \tag{12}$$

is equal to

(13) $$\sum_{\substack{Q''' \in \mathcal{F}^{M''}(M''') \\ Q''' \neq M''}} J_{M'''}(\gamma_1, E_{S_1}^{M'',Q'''}) \quad .$$

We are interested in calculating $J_{M'''}(\gamma_1, f_{S_1}^{M'''})$. This will involve, for each $M'' \supseteq M'''$, the terms

(14) $$\sum_{Q''' \in \mathcal{P}^{M''}(M''')} J_{M'''}(\gamma_1, E_{S_1}^{M'',Q'''}) \quad .$$

They appear in (11) and thus can be calculated as the sum of (11) and (12) minus the difference of the remaining terms in (13). Moreover there will have to be a sum over $M'' \supsetneq M''$.

This will lead to an expression for $J_{M'''}(\gamma_1, f_{S_1}^{M'''})$ containing first of all

$$\sum_{Q \in \mathcal{F}(M''')} J_{M'''}(\gamma_1, F_{S_1}^{Q}) = J_{M'''}(\gamma, \phi_S)$$

and secondly

$$-\sum_{Q \in \mathcal{F}(M''')} J_{M'''}(\gamma_1, H_{S_1}^{Q}) = \sum_{Q \in \mathcal{F}(M''')} J_{M'''}(\gamma, \psi_S^{Q}) \quad .$$

By assumption these two terms cancel.

The sum of the contributions (12) gives for every triple $M' \supseteq M''$ (with $M'' \supseteq M'''$) and every $Q'' \in \mathcal{P}^{M'}(M'')$ a term $J_{M'''}(\gamma_1, E_{S_1}^{M',Q''})$. On the other hand if in the contributions from (13) we denote M'' by M' and Q''' by Q', denoting $M_{Q'}$ by M'' (which contains M''' but is different from it) we see that they cancel those of (12).

Thus the inductive definition is permitted. We cannot define $f_{S_1}^M$ for the given M with $\rho(M) = \rho$ because $H_{S_1}^Q$ is not defined for $Q \in \mathcal{P}(M)$. However we can introduce the function

$$f_{S_1} = \sum_{Q \in \mathcal{P}(M)} F_{S_1}^Q + \sum_{\substack{M' \supsetneq M \\ \rho(M') \geq 2}} \sum_{Q \in \mathcal{P}^{M'}(M)} E_{S_1}^{H',Q} - \sum_{Q \in \mathcal{P}(M)} \sum_{Q' \supsetneq Q} \psi_Q^{Q',1} L_Q(\gamma_2, \psi^{Q,2}) \quad .$$

The previous calculation now shows that

$$J_M(\gamma_1, f_{S_1}) = J_M(\gamma, \phi_S) - \sum_{\substack{Q \in \mathcal{F}(M) \\ Q \notin \mathcal{P}(M)}} J_M(\gamma, \psi_S^Q) \quad .$$

A similar calculation, using the properties of adapted collections, shows that the relation remains valid if S is replaced by S'. This completes the proof of the existence.

On the global correspondence between GL(n) and division algebras

Marie-France Vignéras

1a. Let D be a division algebra of degree n^2 over a global field F of characteristic zero. We suppose that for each place v of F, D_v is $M(n,F_v)$ or a division algebra. We will use the comparison between the trace formula on GL(n) and D^x and local results to get the global correspondence between automorphic representations of GL(n) and D^x.

The case $n = 2$ is already known (JL or GJ). We suppose $n > 2$. Then at infinity D_∞ is $M(n,F_\infty)$. At a finite place v, Zelevinski (Z) introduced a duality in the Grothendieck group $K(GL(n,F_v))$ of the equivalence classes of the representations of finite length of $GL(n,F_v)$. This duality generalizes the duality introduced by Alvis and Curtis for finite groups, and exchanges the class of the Steinberg representation with the trivial one.

We denote by A the adele ring of F. Recall that an irreducible subrepresentation of $L^2(G_F\backslash G_A,\omega)$ for some central character ω is called a discrete automorphic representation of G_A. We denote by S the set of finite places v of F where D_v is a field. Let π_A be an equivalence class of irreducible representations of $GL(n)_A$, such that for every $v \in S$, π_v is square integrable or the dual of a square integrable representation. By the local correspondence (BDKV), we associate to π_A an equivalence class π'_A of irreducible representation of D^x_A:

$$\pi'_v = \pi_v \qquad \text{if } v \notin S$$

$-\pi'_v$ is such that the characters of π'_v and π_v on the regular elliptic conjugacy classes satisfy

$$\chi_{\pi'_v} = \varepsilon(\pi_v)\chi_{\pi_v} \qquad \text{where } \varepsilon(\pi_v) \in \{\pm 1\} \quad .$$

We will prove the following theorem:

1b. THEOREM: The map $\pi_A \longrightarrow \pi'_A$ induces a bijection from the set of automorphic discrete representations of $GL(n)_A$ such that for every $v \in S$, π_v is square integrable or the dual of a square integrable representation onto the set of automorphic representations of D^x_A.

With the natural definition of duality at infinity for $n = 2$, this theorem includes the theorem of Jacquet-Langlands. We restrict ourselves to the case where D_v is $M(n,F_v)$ or a division algebra because of our ignorance of the residual spectrum for $GL(n)$. In §2, we collect some results on local representations of $GL(n)$. We determine the irreducible representations of $GL(n,F_v)$ whose characters do not vanish on the set G_v^{ell} of regular elliptic conjugacy classes, and we prove that the square-integrable representations and their dual are the only ones which are unitarizable. This last result is a sharpening of a theorem of Casselman (BW). We will use these local results to prove the theorem in §3.

2a. We suppose F local, non-archimedean, of characteristic zero. We let $G = GL(n,F)$ and $E(G)$ be the set of equivalence classes of irreducible representations of G. We denote by $E^2(G)$, $E^2(G)^t$, $E^0(G)$ the subsets given by the quasi-square-integrable, dual of quasi-square-integrable, quasi-cuspidal representations respectively. Recall that a quasi-square-integrable representation is the product of a square-integrable one by a power of ν, where $\nu(g) = |\det g|$, $g \in G$.

Let us recall the classification of $E^2(G)$ given in (Z). Let $X^2(G)$ be the set of (m,ρ) where $m|n$ and $\rho \in E^0(GL(d,F))$ if $md = n$. The unitarily induced representation

$$\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho = i_{P_d}^{G}(\rho \otimes \nu\rho \otimes \cdots \otimes \nu^{m-1}\rho)$$

where P_d is the standard parabolic whose Levi factor is isomorphic to $GL(d,F)^m$, has a unique irreducible quotient. This quotient denoted by $St_m(\rho)$ is quasi-square-integrable. Every quasi-square-integrable irreducible representation of G is equivalent to a unique $St_m(\rho)$. The representation $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$ has a unique submodule. It is the dual $St_m(\rho)^t$ of $St_m(\rho)$. In this classification the Steinberg representation is

$$St_n\left(\nu - \frac{n-1}{2}\right) \quad .$$

2b. <u>THEOREM</u>:

(1) The representations $St_m(\rho)$ and $St_m(\rho)^t$ are unitarizable if and only if their central character is unitary.

(2) No other subquotient of $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$ is unitarizable.

The part (1) is known: it is clear for $St_m(\rho)$ and is proved in (B) for $St_m(\rho)^t$ which is a "segment" in the classification of (Z). The part (2) generalizes a theorem of Casselman, which corresponds to $m = n$. Our proof given in 2d follows closely the proof of this theorem given in (BW,X1,§4, p. 340-343).

2c. Let us recall the description of the Jordan-Hölder composition series J of $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$ given in (Z,§2,p. 176-180), generalizing (BW,X,4.6 and 4.2). We know that it is combinatorial, and depends only on m. We set:

$$\delta = \nu^{\frac{m-1}{2}} \otimes \cdots \otimes \nu^{-\frac{m-1}{2}} \quad .$$

The functor $i = i_{P_d}^G$ of unitary induction is related to the ordinary induction functor $I = I_{P_d}^G$ by the relation

$$i = I\delta \quad .$$

Their left-adjoints r, R verify

$$r = \delta^{-1} R \quad .$$

Let Σ be the standard set of roots of $GL(m)$, Δ the subset of simple positive roots, and W the Weyl group. Given a subset I of Δ, we set

$$W(I) = \{w \in W \ , \ w(\alpha) > 0 \ , \ \forall\alpha \in I \ , \ w(\alpha) < 0 \ , \ \forall\alpha \in \Delta - I\}$$

W acts naturally by permutation on $GL(d)^m$ and by "transport de structure" on the representations of $GL(d)^m$. It is easy to deduce from (Z):

<u>PROPOSITION</u>: J has a composition series whose successive quotients are the irreducible representations π_I such that:

$$R(\pi_I) = \bigoplus_{w \in W(I)} w(\rho \otimes \nu\rho \otimes \cdots \otimes \nu^{m-1}\rho) \cdot \delta$$

each occurring with multiplicity one.

If $I = \phi$, $\pi_\phi = St_m(\rho)$ and if $I = \Delta$, $\pi_\Delta = St_m(\rho)^t$. When $m = n$ and $\rho = \nu^{-(n-1)/2}$, then

$$R(\pi_I) = \bigoplus_{w \in W(I)} w(\delta^{-1}) \cdot \delta \quad .$$

2d. Proof of (2). Suppose $I \neq \phi, \Delta$ and that the central character of π_I is unitary. Then the central character of $w(\rho \otimes \nu\rho \otimes \cdots \otimes \nu^{m-1}\rho) \cdot \delta$ verifies

$$|\chi_w| = w(\delta^{-1}) \cdot \delta \quad .$$

Let W^1 be the longest element of the Weyl group of $\Delta - I$. Then $w^1 \in W(I)$. There is a canonical isomorphism of the center S of $GL(d,F)^m$ to the diagonal group of $GL(m,F)$, then a natural action of Σ on S. The character $|\chi_{w^1}|$ acts trivially on the set of elements

$$C = \{c \in S \ , \ |c^\alpha| \leq 1 \ , \ \text{if } \alpha \in I \ , \ |c^\alpha| = 1 \text{ if } \alpha \in \Delta - I\} \quad .$$

This set is unbounded modulo the center Z of G.

Recall a theorem of Casselman: if $v \in \pi_I$ and $\tilde{v} \in \tilde{\pi}_I$ the contragredient of π_I, and $a \in A^-(\varepsilon)$ where

$$A^-(\varepsilon) = \{a \in S \ , \ |a^\alpha| \leq \varepsilon \ , \ \forall\alpha \in \Delta\} \qquad \varepsilon > 0 \text{ small enough}$$

we have

$$\langle \pi_I(a)v \, , \tilde{v} \rangle = \langle R(\pi_I)(a)u \, , \tilde{u} \rangle$$

if $u, \tilde{u}$ are the canonical images of $v, \tilde{v}$ respectively in $R(\pi_I), R(\tilde{\pi}_I)$.

Let us choose $u, \tilde{u}$ such that $\langle u, \tilde{u}\rangle \neq 0$ in $W^1(\rho \otimes \cdots \otimes \nu^{m-1}\rho)\cdot\delta$ and its contragredient. Let $v, \tilde{v}$ which map onto $u, \tilde{u}$ under the canonical projections. For $a \in A^-(\varepsilon)$, $Ca \subset A^-(\varepsilon)$ and

$$(*) \qquad |\langle \pi_I(ca)v, \tilde{v}\rangle| = |\chi_{w^1}(a)|\,\langle u, \tilde{u}\rangle \quad .$$

There exist a unitary character ν^{ix}, $x \in \mathbb{R}$, of $G = GL(n,F)$ such that $\pi_I \nu^{ix}$ is trivial on a subgroup $Z' \subset Z$, with G/Z', with compact center Z/Z'. We can apply to π_I the Howe theorem: if π_I is unitarizable then the coefficients of π_I vanish at infinity. It follows from (*) since C is unbounded modulo the center that π_I is not unitarizable. Then π_I is not unitarizable.

2e. We determine now the irreducible representations π of G whose characters χ_π do not vanish on the set G^{ell} of elliptic regular conjugacy classes.

We know (Z) that the products (unitary induction) of quasi-square-integrable representations form a **Z**-basis of $K(G)$. Denote by $[\pi]$ the image of π in $K(G)$. For every $\pi \in E(G)$, we have:

$$[\pi] = \sum n(\pi, \pi_1 \times \cdots \times \pi_r)\,[\pi_1 \times \cdots \times \pi_r]$$

where $\pi_i \in E^2(GL(n_i,F))$, $\sum n_i = n$. The sum is finite, contains at most one $St_m(\rho)$, and

$$n(St_m(\rho)^t, St_m(\rho)) = (-1)^{m-1} \quad .$$

We know (BDKV) that the restriction to G^{ell} of the characters of $E^2(G)$ form a complete orthonormal system. Moreover, for every $\pi \in H^2(G)$ there exists $\phi_\pi \in H(G)$ in the Hecke algebra $H(G)$, called a pseudo-coefficient of π such that

$$\langle \pi , \phi_\pi \rangle = 1$$

$$\langle \pi \ , \phi_\pi \rangle = 0$$

if $\pi = St_m(\rho)$ and π is not a subquotient of $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$, $\pi \ \in E(G)$.

We deduce from this the following:

PROPOSITION:

(1) $\chi_\pi = 0$ on G^{ell} if π is not a subquotient of some $\rho \times \nu\rho \times \cdots \times \nu^{m-1}\rho$ and $\chi_\pi = n(\pi , St_m(\rho))\chi_{St_m(\rho)}$ otherwise

(2) The square-integrable-irreducible representations and their duals are the only irreducible unitary representations whose character do not vanish on G^{ell}.

3a. We suppose now F global of characteristic zero. Denote by S a finite set of non-archimedean places of F. We set

$$G_S = \prod_{v \in S} G_v \quad , \quad G_A = G_S G^S \quad .$$

By convention $X_S = (X_v)_{v \in S}$ satisfies (P) if and only if each component X_v satisfies (P). We deduce from 2e the following corollary

COROLLARY: Let $\pi_A = \pi_S \times \pi^S$ be an automorphic representation of $GL(n)_A$. The character of π_S does not vanish on G_S^{ell} if and only if

$$- \pi_S \in E^2(G_S) \qquad \text{if } \pi_A \text{ is cuspidal}$$

$$- \pi_S \in E^2(G_S)^t \qquad \text{if } \pi_A \text{ is not cuspidal} \quad .$$

Proof: From 2e (2) we know that each component π_v of π_S belongs to $E^2(G_v)$ or $E^2(G_v)^t$. If one of them is square integrable, then π_A is cuspidal (this seems to be well known and was indicated to me by Jacquet, it results from the characterization of square integrable representation by the exponents from Jacquet functors, and the computation of the constant terms by Harish-Chandra). It follows that π_v is not degenerated (Sh) at all non-achimedean places v of F. Therefore, for $v \in S$, π_v is square-integrable, because the elements of $E^2(G_v)^t$ are degenerate.

3b. PROPOSITION: A cuspidal automoprhic representation of $GL(n)_A$ and a non-cuspidal one do not have the same G^S-component.

Proof: If they had, their L-function would be equal. This is incompatible with the existence of a pole for an L-function $L(s, \pi_A \times \sigma_A)$ for σ_A cuspidal of $GL(m)_A$, $m < n$ when π_A is automorphic for $GL(n)_A$ is not cuspidal (J.Sh).

3c. We now proceed to the proof of the global correspondence. Let D be as in §1, and S be the set of places v of F where D_v is a division algebra. The comparison of the trace formulas on GL(n) and D^x made by Langlands (L) gives:

(1) trace $\rho(f) = $ trace $\rho_d(f)$

for all $f = \pi f_v$, $f = \pi f_v$ associated to f via orbital integrals:

- $f^s = f^s \in H(GLn)^s)$

- The orbital integrals of f_s on regular elements are zero outside of G_s^{ell}, and equal to the orbital integrals of f_s on $D_s^{\times ell}$ naturally isomorphic to G_s^{ell}.

We use the notations of (BDKV) that we quickly recall: a central characger ω is fixed, ρ is the regular representation of $GL(n)_A$ in $L^2(GL(n,F)\backslash GL(n,A),\omega)$, ρ_d its discrete part, ρ the one for $D_A^\times$.

Using the standard simplification argument (JL) we write (1) in the equivalent form: for all $\pi^s \in E(GL(n)^s)$.

(2) $\Sigma n(\pi_s \otimes \pi^s)$ trace $\pi_s(f_s) = \Sigma n(\pi_s \otimes \pi^s)$ trace $\pi_s(f_s)$ where $n(\pi_A)$ is the multiplicity of π_A in ρ_d, and $n(\pi_A)$ the one for ρ .

The following properties are equivalent:

- (2) does not vanish for all f_s
- π^s is the G^s-component of some $\pi_A \subset \rho$
- π^s is the G^s-component of some $\pi_A \subset \rho_d$ such that the character π_s does not vanish on G_s^{ell} .

We suppose that they are satisfied. We deduce from 3a, 3b, the strong multiplicity one theorem for cuspidal representations of $GL(n)_A$, and the local correspondence (1a), that two disjoint possibilities A, B can occur:

A] π^S is the G^S-component of π_A cuspidal. Then π_A is unique, with multiplicity $n(\pi_A) = 1$, π_s is square-integrable. Let $\pi_s^0 \in E(D_s^x)$ associated to π_s by the local correspondence and $\pi_A^0 = \pi_s^0 \otimes \pi^S$. We have for all $f_s \in H(D_s^x)$:

$$\Sigma\, n\,(\pi_s \otimes \pi^S)\ \text{trace}\ \pi_s(f_s) \;=\; \text{trace}\ \pi_s^0(f_s) \quad .$$

By linear independence we deduce that π^S is the G^S-component of a unique automorphic representation of D_A^x, equal to π_A^0, with multiplicity $n(\pi_A^0) = 1$.

B] π^S is the G^S-component of π_A residual. Then π_s is the dual of a square integrable representation. Let $\pi_s^0 \in E(D_s^x)$ associated to π_s by the dual of the local correspondence and $\pi_A^0 = \pi_s^0 \otimes \pi^S$. We have for all $f_s \in H(D_s^x)$:

$$\Sigma\, n\,(\pi_s \otimes \pi^S)\ \text{trace}\ \pi_s(f_s) \;=\; \Sigma\, n(\pi_s \otimes \pi^S)\ \text{trace}\ \pi_s^0(f_s)\varepsilon(\pi_s)$$

where $\varepsilon(\pi_s) = \pm 1$.

By linear independence we deduce that the set of automorphic representations of D_A^x with G^S-component π^S is equal to the set of the representations π_A^0, where $\pi_A = \pi_s \otimes \pi^S$ is residual for $GL(n)_A$, with multiplicities $n\,(\pi_A^0) = n(\pi_A)$. Moreover $\varepsilon(\pi_s) = 1$ for all such π_A.

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Thursday Morning Seminar

DIVISION ALGEBRAS III

R. Langlands

There were three points left unsettled in the previous lecture that I want to deal with here.

<u>The fine</u> σ-<u>expansion</u>. As already suggested it appears best to write the left side of the trace formula as a sum over $L(M_0)$, the set of Levi subgroups containing M_0. Thus we write

$$\sum_{M \in L(M_0)} \frac{|\Omega^M|}{|\Omega^G|} J_M^T(\phi) = \sum_M \frac{|\Omega^M|}{|\Omega^G|} J_M^T(\phi) = \sum_M \frac{|\Omega^M|}{|\Omega^G|} J_M(\phi)$$

omitting as convenience suggests both the range of summation and the parameter T.

Arthur expects that $J_M(\phi)$ will be a sum over semi-simple conjugacy classes σ in M,

$$J_M(\phi) = \sum_\sigma J_M(\sigma, \phi) \quad ,$$

the $J_M(\sigma, \phi)$ having a form to be described below. I allow myself to anticipate his results, in spite of the element of uncertainty this introduces. It is intended to present them in the Friday morning seminar.

It will in particular have to be proved that

$$J_M(\gamma, \phi) = |D^G(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(\mathbb{A}_{S'})\backslash G(\mathbb{A}_{S'})} \phi_{S'}(g^{-1}\gamma g) V_M^G(\gamma, g)\, dg$$

is defined for those γ in $M(\mathbf{A}_{S'})$ such that $1\text{-Ad}(\gamma)$ is invertible on $\mathfrak{g}/\mathfrak{m}$. It will also have to be proved that it extends by continuity to all of $M(\mathbf{A}_S)$.

Fix a semi-simple γ in $\mathfrak{o}$ and let $U_\gamma(M_{S'})$ be the collection of conjugacy classes of unipotent elements in $\mathbf{M}_{S'}$ which commute with γ, or, better, a set of representatives for such classes. Then for S' sufficiently large

$$J_M(\mathfrak{o}, \phi) = c^{S'}(M) \sum_{\delta \in U_\gamma(M_{S'})} c^M_{S'}(\gamma, \delta) J_M(\gamma\delta, \phi) \quad .$$

The constant $c^{S'}(M)$ outside the summation depends only on S', M, and G and is given by

$$c^{S'}(M) = \text{meas}(K^{S'} \cap \mathbf{M}) \backslash K^{S'} \quad .$$

The constant $c^M_{S'}(\gamma, \delta)$ inside the summation depends on S', γ, δ, and M but not in G.

Lemma 1 of the previous lecture remains valid first for γ such that $1\text{-Ad}\ \gamma$ is invertible in $\mathfrak{g}/\mathfrak{m}$, for which it is proved in the same way, and then, by continuity, for all γ.

This makes it natural to modify the notion of a family F^Q_S adapted to f_S

$$(1) \qquad J_M(\gamma, f_S \prod_{v \in S'-S} f_v) = \sum_{\substack{Q \in \mathcal{F}^L(M) \\ Q \neq L}} J_M(\gamma, F^Q_S \prod_{v \in S'-S} f_{Q,v})$$

for all γ. Lemma 2 remains valid, the proof for finite v being that given in the first lecture. The proof for $v = \infty$ will be given below.

This done we can proceed as in the previous lecture to prove the existence of functions ψ^Q such that

$$J_M(\sigma, \phi) = \sum_{\substack{M \subseteq Q \\ Q \neq G}} J_M(\sigma, \psi^Q)$$

for every σ. Then

$$J_M(\phi) = \sum_{\substack{M \subseteq Q \\ Q \neq G}} J_M(\psi^Q)$$

and the measure-theoretic argument can proceed.

Revision of argument. Although I see no reason to doubt its validity I am unable to prove Lemma 2 of the first lecture for $v = \infty$ in the form there stated. I can only prove it when S consists of ∞ alone. This entails some changes in the proof of the existence of the functions ψ^Q.

Notice first that we had fused all the infinite places together into one so that the statement with which the proof began was not strictly correct. All the places (in the usual sense) at which $\Phi(\gamma, \phi_v) \equiv 0$ may be infinite. Then $n = 2m$, m is odd, and $n_1 = n_2 = m$. In particular $\rho(L) = 1$.

Turning to the construction of the f used to define ψ^Q for $Q \in \mathcal{P}(M)$, $\rho(M) \geq 2$ we distinguish two cases: (a) $\Phi(\gamma, \phi_\infty)$ is not identically zero for all γ in M_∞ regular in G_∞. (b) It is.

In the first case we may continue to use the argument of the first lecture, noticing that if $M \subset Q$ then the two places at which $\Phi(\gamma, \phi_v) \equiv 0$ for all $\gamma \in M_Q(F_v)$ regular in G_v are both finite. In the second case we use Lemma 6 only to show that the function

$$J_M(\gamma, \phi) - \sum_{\substack{M \subseteq Q \\ Q \neq G \\ Q \notin P(M)}} J_M(\gamma, \psi^Q)$$

is satisfactory away from ∞. (Note that in the definition of satisfactory and in Lemmas 5 and 6 the function $\Phi(\gamma, f)$ should be $|D(\gamma)|^{\frac{1}{2}}\Phi(\gamma, f)$ and $\Phi(\gamma_1, f^{\vee}_{\gamma_2})$ should be $|D(\gamma_1)|^{\frac{1}{2}}\Phi(\gamma_1, f^{\vee}_{\gamma_2})$). To show that it is satisfactory at ∞ we use the argument on pp. 14-18 but with $g_1 \in G(\mathbf{R} \times \mathbf{A}^{S'})$, $g_2 \in G(\mathbf{A}_{S_1})$, $S_1 = S - \{\infty\}$.

<u>A lemma of Arthur</u>. To complete the argument of the first lecture we must therefore prove Lemma 2 when $S = \{\infty\}$. The first step is to reduce ourselves to the case that $S' = S$. For this we need a variant of a lemma of Arthur, but I first give the lemma itself, of which we shall in any case have need.

Recall that the Harish-Chandra homomorphism $z \longrightarrow \Gamma_T(z)$ can be factored through Z_M. $z \longrightarrow \Gamma_M(z) \longrightarrow \Gamma_T(\Gamma_M(z))$, where the Γ_T should really be written Γ_T^M for it refers to the pair T, M. For the next lemma G, M_1 and M' can be any connected reductive groups over $\mathbf{R}$.

LEMMA 1. <u>It is possible to attach to any pair</u> $M \subset M'$, <u>any</u> $T \subseteq M$, <u>any</u> γ <u>in</u> $M(\mathbf{R})$ <u>regular in</u> M', <u>and any</u> $z \in Z_M$ <u>an invariant</u>

differential operator $D_M^{M'}(\gamma, z)$ _on_ T _such that_

$$J_M^G(\gamma, zf) = \sum_{M \subset M' \subset G} D_M^{M'}(\gamma, \Gamma_{M'}(z)) J_{M'}^G(\gamma, f)$$

for all $z \in Z_G$.

If $M = G$ then $D_M^G(\gamma, z) = \Gamma_T(z)$ is given by the Harish-Chandra homomorphism and the equality is well known. In general we can proceed by induction and it is easy enough to see that we need only show that

$$f \longrightarrow J_M^G(\gamma, zf) - \sum_{M \subset M' \subsetneq G} D_M^{M'}(\gamma, \Gamma_{M'}) J_{M'}^G(\gamma, f)$$

is an invariant distribution, for it is then an elementary consequence of the theory of distributions that it is given by $f \longrightarrow DJ_G^G(\gamma, f)$, because it is obviously concentrated on the orbit of γ. We define $D_M^G(\gamma, z)$ to be this D.

The proof of invariance relies on two simple, and more or less obvious, identities. The first,

$$(zf)_{Q,h} = \Gamma_{M_Q}(z) f_{Q,h}$$

follows easily from the definition on p. 20 of the Annals paper and the definition of the Harish-Chandra homomorphism. The second,

$$\Gamma_M(\Gamma_{M'}(z)) = \Gamma_M(z) \quad ,$$

is almost formal.

This said, we must verify that

$$J_M(\gamma, zf^h) - J_M(\gamma, zf)$$

is equal to

$$\sum_{M \subset M' \subsetneq G} D_M^{M'}(\gamma, \Gamma_{M'}(z))(J_{M'}^G(\gamma, f^h) - J_{M'}^G(\gamma, f)) \quad .$$

The first difference is equal to

$$\sum_{\substack{Q \in \mathcal{F}(M) \\ Q \neq G}} J_M(\gamma, \Gamma_{M_Q}(z) f_{Q,h}) \quad ,$$

which by induction is equal to

$$\sum_{M \subset M' \subset Q \neq G} D_M^{M'}(\gamma, \Gamma_{M'}(z)) J_{M'}(\gamma, f_{Q,h}) \quad .$$

On the other hand

$$D_M^{M'}(\gamma, \Gamma_{M'}(z))(J_{M'}^G(\gamma, f^h) - J_{M'}^G(\gamma, f))$$

is equal to

$$\sum_{M' \subset Q \neq G} D_M^{M'}(\gamma, \Gamma_{M'}(z)) J_{M'}(\gamma, f_{Q,h}) \quad .$$

The required equality follows.

Observe that in the Friday afternoon seminar we used a special case of the lemma, that for which $\rho(M) - \rho(G) = 1$, obtaining it by a direct argument.

For the second lemma we consider a function $f = f_S f^S$, f_S being $\prod_{v \in S} f_v$ and f^S being $\prod_{v \in S'-S} f_v$ with f_v spherical for $v \in S'-S$. We observe, for it is the key to the proof of the next lemma, that for a spherical $f = f_v$ the function f_Q depends only on M_Q and thus may be written as $\Gamma_M(f_v)$ if $M = M_Q$. Moreover

$$\Gamma_M(\Gamma_{M'}(f_v)) = \Gamma_M(f_v) \quad .$$

If $h \in G(\mathbb{A}_S)$ then

$$f_{Q,h} = f_{S,Q,h}\Gamma_{M_Q}(f^S) \quad .$$

Thus the next lemma can be proved exactly like Lemma 1.

LEMMA 2. <u>For any</u> $\gamma = (\gamma_1, \gamma_2)$ <u>regular in</u> G <u>there are linear forms</u> $\phi \longrightarrow D_M^{M'}(\gamma, \phi)$ <u>on the Hecke algebra of</u> $M'(\mathbb{A}_S^{S'})$ <u>such that</u>

$$J_M(\gamma, f_S f^S) = \sum_{M \subset M' \subset G} D_M^{M'}(\gamma, \Gamma_{M'}(f^S)) J_{M'}(\gamma_1, f) \quad .$$

Suppose then

$$J_M(\gamma_1, f_S) = \sum_{\substack{Q \in \mathcal{F}^L(M) \\ Q \neq L}} J_M(\gamma_1, F_S^Q)$$

for all $M \subseteq L$. Then by the lemma (with G replaced by L)

$$J_M(\gamma, f_S f^S) = \sum_{M \subset M' \subset L} D_M^{M'}(\gamma, \Gamma_{M'}(f^S)) J_{M'}(\gamma_1, f_S)$$

$$= \sum_{M' \subset Q \neq L} D_M^{M'}(\gamma, \Gamma_{M'}(f^S)) J_{M'}(\gamma_1, F_S^Q)$$

$$= \sum_{M' \subset Q \neq L} D_M^{M'}(\gamma, \Gamma_{M'}(f_Q^S)) J_{M'}(\gamma_1, F_S^Q)$$

$$= \sum_Q J_M(\gamma, F_S^Q f_Q^S) \quad .$$

Proof of Lemma 2 for $S' = S = \{\infty\}$. (Notice that in the statement of that lemma γ is to lie in $L(F_v)$ and be regular in it.) The function f_S is now denoted f and we establish the existence of F^Q by induction on $\rho(Q)$. To warm up we begin with $\rho(Q) - \rho(L) = 1$, the weighting factor being then linear. So Arthur's lemma yields differential equations,

$$(2) \qquad J_M(\gamma, zf) = \Gamma_T(z) J_M(\gamma, f) \quad ,$$

the inhomogeneous term falling away because the orbital integrals of f are zero. Notice that at this point we are working entirely within the group L. From the equations (2) and the estimates of the Inventiones paper, to which we shall return, we deduce that $J_M(\gamma, f)$ defines a piecewise smooth function on $T(\mathbb{R})$.

We need to show that there exists a function F on $M_\infty(\mathbb{R})$, smooth and of compact support, such that

$$J_M(\gamma, f) = J_M(\gamma, F) \quad ,$$

at first for γ regular and semi-simple, and then for all γ. Then we set

$$F^Q = \frac{1}{|P(M)|} F \quad .$$

To do this we first use Shelstad's characterization of orbital integrals to obtain an F in the Schwartz space (Shelstad, Characters and inner forms of a quasi-split group over $\mathbb{R}$, Comp. Math. (1979)), observing that for the groups under consideration orbital integrals are necessarily stable, and then we follow a technique of Clozel (App. to Clozel-Delorme, Le théorème de Paley-Wiener invariant pour les groupes de Lie réductifs) which with the help of this Paley-Wiener theorem replaces F by a compactly supported function.

According to Theorem 11 of Shelstad's paper there are several conditions to verify. The first is formal and trivial to verify. The second, invariance under the Weyl group, is clear from the definition. The others refer to the behavior of $J_M(\gamma, f_\infty)$ near semi-regular elements of T. These are defined by a condition $\alpha(\gamma_0) = 1$ with α real or imaginary.

The conditions for α imaginary are the most difficult to state, but the easiest to verify. The point is that near the orbit of such a γ_0 we can set

$$F(m) = \rho_Q(m) \int_{K_\infty} \int_{N_Q(\mathbb{R})} f_\infty(k^{-1}muk) W_M^G(m, u) \, du \, dk$$

where

$$W_M^G(m, u) = V_M^G(m, n)$$

if $m^{-1}nm = mu$. There is a neighborhood X of the orbit of γ_0 such that $(m, n) \longrightarrow (m, u)$ is a homeomorphism of $X \times N_Q(\mathbf{R})$ with itself. Moreover we can so choose X that $V_M^G(m, n)$ is smooth on $X \times N_Q(\mathbf{R})$. Finally

$$J_M(\gamma, f_\infty) = J_M(\gamma, F)$$

and is thus equal to the orbital integral of a smooth compactly supported function near γ_0. So the conditions at imaginary roots are obvious, and need not even be stated. Indeed the conditions at all roots in M are, for the same reason, clearly satisfied.

The condition at a semi-regular element defined by a real root is smoothness. Continuity is already being taken for granted. Just as in the lecture Cancellation of singularities at the real places it is sufficient to verify that $H_\alpha J_M(\gamma, f_\infty)$ does not jump at γ_0, H_α being defined by

$$\left\langle H_\alpha, H' \right\rangle = \frac{2\alpha(H')}{\langle \alpha, \alpha \rangle} \quad ,$$

for all $H' \in \mathfrak{h}$, and α now being a real root not in M.

This is a consequence of the results in Arthur's paper The characters of discrete series as orbital integrals, Inv. (1976). We need only sort out the notation. First of all, writing $f_\infty = f$ and $L = L_\infty$

$$\left\langle R(\zeta, H : Y : 1), f \right\rangle = \varepsilon |D^L(\gamma)|^{\frac{1}{2}} \int_{\gamma(\mathbf{R}) \backslash (\mathbf{R})} f(g^{-1}\gamma g) v_M^L(g)\, dg \quad ,$$

where $\gamma = \zeta \exp H$, the factor ε is locally constant and equal to

$$\prod_{\beta \in R_I^+} \frac{e^{\frac{\beta(H)}{2}} - e^{-\frac{\beta(H)}{2}}}{\left|e^{\frac{\beta(H)}{2}} - e^{-\frac{\beta(H)}{2}}\right|} ,$$

R_I^+ being the set of imaginary roots positive with respect to some order, and Y is the A-orthogonal set obtained by projecting $\{w^{-1}T\}$ on $\mathfrak{a}$, the split component of M.

The weight $v_M^L(g)$ is defined by projecting $w^{-1}(T-H(wg))$ on $\mathfrak{a}$, obtaining thereby for each chamber W in $\mathfrak{a}$ a point x_W. In essence $v_M^L(g)$ is the volume of the convex hull of the x_W. To define $V_M^L(\gamma, g)$ we have to replace x_W by

$$X_W = x_W - \sum_\beta \ln \frac{|1-\beta(\gamma)|}{|\beta(\gamma)|^{\frac{1}{2}}} \bar{H}_\beta ,$$

where $\bar{H}_\beta$ is the projection of H_β on $\mathfrak{a}$ and $\langle H, H_\beta \rangle = \frac{2\beta(H)}{\langle \beta, \beta \rangle}$. The sum is over all positive roots β whose restriction to $\mathfrak{a}$ is not zero and which separate W from W_+.

Digression. It will be observed that we have modified our formulation of Flicker's trick, replacing $\ell n|1-\beta^{-1}(\gamma)|$ by $\ell n \frac{|\beta(\gamma)-1|}{|\beta(\gamma)|^{\frac{1}{2}}}$. This is of no importance if our only concern is to create a continuous function of γ, one modification serving as well as the other. The new modification has however a symmetry which the old lacks and which we have implicitly used. Namely, replacing γ by $w\gamma w^{-1}$ in

$$|D^L(\gamma)|^{\frac{1}{2}} \int f(g^{-1}\gamma g) v_M^L(g)$$

or in

$$|D^L(\gamma)|^{\frac{1}{2}} \int f(g^{-1}\gamma g) V_M^L(\gamma, g)$$

yields the same result as keeping γ but replacing $v_M^L(g)$ by $v_M^L(wg)$ and $V_M^L(\gamma, g)$ by $V_M^L(w\gamma w^{-1}, wg)$. Now $v_M^L(wg)$ is defined by

$$\{s^{-1}T - s^{-1}H(swg)\} = \{w(w^{-1}s^{-1} - w^{-1}s^{-1}H(swg))\} \quad .$$

Thus $v_M^L(wg) = v_M^L(g)$. So the replacement has no effect on the first integral.

Before considering the second we notice that

$$\sum_\beta \ln \frac{|1-\beta(\gamma)|}{|\beta(\gamma)|^{\frac{1}{2}}} H_\beta = \sum_\beta c_\gamma(\beta) H_\beta$$

where $c_\gamma(\beta)$ is defined for all β and $c_\gamma(-\beta) = c_\gamma(\beta)$. The sum is over all roots separating W from W_+, more precisely over those which are positive on W_+ and negative on W. To replace W_+ by another chamber W' is simply to add a common term, independent of W, to all these sums, and that has no effect on the volume.

The factor $V_M^L(w\gamma w^{-1}, wg)$ is defined by

$$w(w^{-1}s^{-1} - w^{-1}s^{-1}H(swg)) - \sum \ell n \frac{|1-\beta(w\gamma w^{-1})|}{|\beta(w\gamma w^{-1})|^{\frac{1}{2}}} H_\beta \quad .$$

According to the preceding remark we may sum over β which are negative on $s^{-1}W_+$ and positive on wW_+. Thus we write the sum as

$$w(\sum c_\gamma(w^{-1}\beta)H_{w^{-1}\beta}) \quad .$$

If $s\beta$ is negative on W_+ then $sw(w^{-1}\beta)$ is also, for they are equal. Thus the sum is in fact equal to

$$w(\sum_{\substack{\beta>0 \\ s\beta<0}} c_\gamma(\beta)H_\beta)$$

and $V_M^L(w\gamma w^{-1}, wg) = V_M^L(\gamma, g)$. So both integrals are invariant under the substitution $\gamma \longrightarrow w\gamma w^{-1}$, w lying in the Weyl group. This ends the digression.

It is convenient to set $X_W = Y_W + Z_W$, where

$$Z_W = - \sum_{\beta\neq\alpha} \ln \frac{|1-\beta(\gamma)|}{|\beta(\gamma)|^{\frac{1}{2}}} \overline{H}_\beta$$

and Y_W equals x_W if α does not separate W from W_+ and is otherwise equal to

$$x_W - \ln \frac{|1-\alpha(\gamma)|}{|\alpha(\gamma)|^{\frac{1}{2}}} \overline{H}_\alpha \quad .$$

The function Z_W is varying smoothly near γ_0. So it is natural to apply Lemma 6.3 of Arthur's Annals paper to write

$$V_M(\gamma, g) = \sum_{Q \in \mathcal{F}^L(M)} V_M^Q(\gamma, g, \{Y_W\})U_Q(\{Z_W\})$$

the notation being I hope obvious. The singularities of $J_M(\gamma, f)$ at γ_0 are therefore determined by those of $J_M'(\gamma, f_Q)$ the prime indicating that we are using the weight factor determined by the family $\{Y_W\}$ rather than that determined by $\{X_W\}$.

There is a simple relation between $V_M^Q(\gamma, g, \{Y_W\})$ and $v_M^Q(g)$, namely

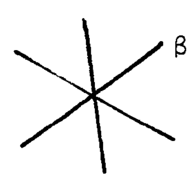
$$V_M^Q(\gamma, g \{Y_W\}) = v_M^Q(g)$$

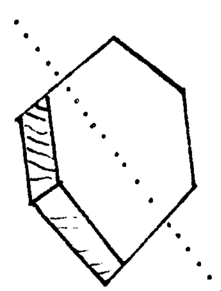
unless α is a root in M_Q. However if α is a root in M_Q and if $M^* \supseteq M$ is the Levi subgroup of M_Q with $\mathfrak{a}^* = \mathfrak{a}_{M^*}$ equal to the null space of α then

$$V_M^Q(\gamma, g, \{Y_W\}) = v_M^Q(g) + \ln \frac{|1-\alpha(\gamma)|}{|\alpha(\gamma)|^{\frac{1}{2}}} v(\bar{H}_\alpha) v_{M^*}^Q(g) \quad ,$$

where $v(\bar{H}_\alpha)$ is the measure of the interval spanned by O and $\bar{H}_\alpha$.

The following diagrams illustrate a typical case of this relation:





The points x_W and Y_W are the same if α is positive on W. The volume of the unshaded region is $v_M^Q(g)$ and that of the shaded region is the difference between $v_M^Q(\gamma, g, \{Y_W\})$ and $v_M^Q(g)$.

To verify the first identity suppose α is not a root M_Q. Then it is either in N_Q or not in Q at all. In the first case $x_W = Y_W$ if $W = W(P)$ and P in $\mathcal{P}(M)$ is contained in Q. In the second case $Y_W = x_W - \ln|1-\alpha^{-1}(\gamma)|\overline{H}_\alpha$. In either case the volumes of their span are equal.

To verify the second we observe that a $P \in \mathcal{P}(M)$ is specified by a $P^* \in \mathcal{P}(M^*)$ and a $P' \in \mathcal{P}^{M^*}(M)$, the second set containing exactly two elements. Moreover if $W = W(P)$ then $x_W = Y_W$ unless α is not a root in P', but then $Y_W = x_W - \ln\frac{|1-\alpha(\gamma)|}{|\alpha(\gamma)|^{\frac{1}{2}}}\overline{H}_\alpha$. At this point one either regards the asserted equality as geometrically obvious or proves it with the algebraic formalism of the Annals paper.

Comparing these relations with the definitions on p. 227 of the Inventiones paper we conclude first of all that

$$J_M'(\gamma, f_Q) = R_{f_Q}(\zeta, H : Y^Q : 1)$$

if α is not a root in M_Q. Observe that in this case an argument used above reduces the study of the weighted orbital integrals near γ_0 to that of ordinary orbital integrals. So we are provided with the required smoothness at no cost. If α is a root in M_Q then

$$J_M'(\gamma, f_Q) = S_{f_Q}^{\alpha}(\zeta, H : Y^Q : 1) \quad .$$

Incidentally, the family Y^Q is defined in §6 of the Annals paper.

This last formula allows us to apply Theorem 6.1 of the Inventiones paper. It asserts, in particular, that the jump in $H_\alpha J'_M(\gamma, f_Q)$ is equal to

$$n_\alpha(A) J_{M^*}(\gamma_0, f_Q) \quad .$$

We deduce that the jump in $J_M(\gamma, f)$ itself at γ_0 is equal to

$$\sum_{M^* \subseteq Q} |D^G(\gamma)|^{\frac{1}{2}} \int f(g^{-1}\gamma_0 g) v^Q_{M^*}(g)\, dg U_Q(\{Z_W\})$$

and another application of Lemma 6.3 shows that this is equal to

$$J_{M^*}(\gamma_0, f) \quad .$$

Notice that the projection of H_α on $\mathfrak{a}_{M^*}$ is zero. At the moment we are dealing with the case that $\rho(M) = \rho(L) + 1$. Thus $M^* = L$ and $J_{M^*}(\gamma_0, f)$ is the limit of ordinary orbital integrals and consequently zero.

This gives us Shelstad's conditions. To convert the function she provides into a compactly supported function we have to assume that f is K-finite and f_Q therefore $K \cap M$ finite. This assumption was overlooked in the statement of Lemma 2. It is a restriction that does not hinder our real purpose. The use of Clozel's technique was suggested by Arthur.

Let F be a function in the Schwartz class on $M(\mathbb{R})$ with

$$J_M(\gamma, f) = J_M(\gamma, F) \quad .$$

Since the orbital integrals of F are well defined, even though F itself is not, the trace $\mathrm{tr}\,\pi(F)$ is well defined for any tempered representation π of $M(\mathbf{R})$ and does not depend on the choice of F. If z is the Casimir operator, $f = (z-\lambda)f'$, $\lambda \in \mathbf{C}$, and

$$J_M(\gamma, f') = J_M(\gamma, F')$$

for all regular semi-simple γ then we may take $F = (\Gamma_M(z) - \lambda)F'$. We conclude that if π is a tempered representation of $M(\mathbf{R})$ and

$$\pi(\Gamma_M(z)) = \lambda I$$

then

$$\mathrm{tr}\,\pi(F) = \mathrm{tr}\,\pi((\Gamma_M(z) - \lambda)F') = 0 \quad .$$

<u>A difficulty</u>. If f is K-finite the equation $f = (z-\lambda)f' = 0$ is solvable for λ positive and large as a consequence of the Plancherel theorem, but it is solvable only in the Schwartz space (Arthur, <u>Harmonic analysis in the Schwartz space of a reductive Lie group</u> (preprint)). Thus in order to make use of the previous observation we must show that F exists not just for compactly supported functions f but also for functions in the Schwartz space, provided of course that their orbital integrals are zero.

To deal with this larger class of functions we need only establish inequalities

$$|DJ_M(\gamma, f)| \leq C(1 + \|H\|)^{-n} \quad ,$$

where D is an arbitrary invariant differential operator on $T(\mathbb{R})$, $C = C(f, D))$ is a constant, and $\gamma = \zeta \exp H$ with ζ in the maximal compact subgroup of $T(\mathbb{R})$ and with $H = H(\gamma)$ in the Lie algebra of its vector part. It must of course be possible to choose the integer n arbitrary.

Corollary 7.4 of Arthur's paper gives us pretty nearly what we want. He works with $v_M^L(g)$ rather than with $V_M^L(\gamma, g)$. However

$$\left| \ln \left(\frac{|\alpha(\gamma)-1|}{|\alpha(\gamma)|^{\frac{1}{2}}} \right) \right| \leq C(1 + \|H\|) \quad .$$

So it is easy to convert the corollary to an estimate for $J_M(\gamma, f)$.

Following Arthur we set $L(\gamma)$ equal to the absolute value of the logarithm of the smallest of the numbers $|1-\alpha(\gamma)^{-1}|$, where α runs over the roots of T which do not vanish on $\mathfrak{a}$. The estimate is

$$(3) \qquad |J_M(\gamma, f)| \leq C(1 + L(\gamma))^{p}(1 + \|H\|)^{-n} \quad ,$$

where $C = C(f, n)$. The question now is whether the technique of the lecture on real groups allows us to rid ourselves of the annoying factor $(1 + L(\gamma))^p$ without losing the factor $(1 + \|H\|)^{-n}$. A glance at that lecture and a moment's reflection convinces us that what we need are inequalities

$$(4) \qquad |DJ_M(\gamma, f)| \leq C\tau(\gamma)^{-p'}(1 + \|H\|)^{-n}$$

valid for an arbitrary invariant differential operator in $T(\mathbb{R})$. The number p' depends on D and $\tau(\gamma)$ is the distance from γ to the set

$$\bigcup_{\alpha} \{t | \alpha(t) = 1\} \quad .$$

This is what the second stage of the argument in §8 of the Inventiones paper gives us. As Arthur remarks it is taken from Harish-Chandra. Notice that (3) and the differential equation (2) provide us with an inequality (4) whenever D is a $\Gamma_T(z)$. This is not the place to recapitulate the argument in all its details. It suffices to note that using the fact that the algebra of invariant differential operators is a finite module over $\Gamma_T(Z_L)$ and the existence of a fundamental solution for powers of the Laplace operator in $T(\mathbf{R})$ one finds $D_1, \ldots, D_r$ in $\Gamma_T(Z_L)$ and functions $E_{1,\varepsilon}, \ldots, E_{r,\varepsilon}, B_\varepsilon$ such that

$$D\varphi(\gamma) = \sum_{j=1}^{r} \int D_j \varphi(\tilde{\gamma}) E_{j,\varepsilon}(\tilde{\gamma}^{-1}\gamma)\, d\tilde{\gamma} - \int \varphi(\tilde{\gamma}) B_\varepsilon(\gamma^{-1}\tilde{\gamma})\, d\tilde{\gamma} \quad .$$

Here φ is any function smooth on the set of regular elements in $T(\mathbf{R})$ and $\varepsilon = \frac{\tau(\gamma)}{5}$, $\tau(\gamma)$ being supposed small. I have allowed myself to work in $T(\mathbf{R})$ rather than in its Lie algebra as Arthur and Harish-Chandra do.

We of course take $\varphi(\gamma) = J_M(\gamma, f)$. The point is that the functions $E_{j,\varepsilon}$ and the function B_ε have support in a ball of radius 3ε. Moreover the $E_{j,\varepsilon}$ are bounded and, as an explicit calculation of the fundamental solution shows,

$$|B_\varepsilon(\tilde{\gamma})| \leq C\varepsilon^{-q} \quad .$$

(See §29 of Harish-Chandra, Invariant eigendistributions on a semisimple Lie group, Trans. AMS (1965).)

So the difficulty can be surmounted. We return to Clozel's technique, taking f once again to have compact support. To show that F has the same orbital integrals as a function with compact support we have to show that the conditions of Theorem A.1 of the appendix to the Clozel-Delorme paper are satisfied. Take a cuspidal parabolic P of M and consider the representations $\pi_{\delta,\lambda}$ induced from a discrete series representation $\delta \otimes \lambda$ on $M_P(\mathbf{R})$. That

$$\lambda \longrightarrow \operatorname{tr} \pi_{\delta,\lambda}(F)$$

is of Paley-Wiener type follows from the explicit formulas for the character of $\pi_{\delta,\lambda}$ and the fact that the orbital integrals of F are compactly supported.

Once this is granted all that is left is to verify that for all but finitely many δ, taken modulo central characters of M, the function $\operatorname{tr} \pi_{\delta,\lambda}(F)$ vanishes on an open set of λ. This however follows from the observation that if z is the Casimir and $A >> 0$ then for all but finitely many δ there is a λ such that $\pi_{\delta,\lambda}(z) = \mu I$ with $\mu > A$.

We have now treated the case $\rho(Q) = 1 + \rho(L)$ and we pass to the general case, proceeding by induction.

All we need do is verify that, for all regular semi-simple γ,

$$\gamma \longrightarrow J_M(\gamma, f) - \sum_{\substack{M \subset Q \neq L \\ M \neq M_Q}} J_M(\gamma, F^Q) = T(\gamma, f)$$

is equal to $J_M(\gamma, F)$ where F is smooth and compactly supported on M. Most of the argument has now been given, but we still have to verify the differential equations

$$T(\gamma, zf) = \Gamma_T(z)T(\gamma, f) \tag{5}$$

and the jump conditions.

To verify the differential equations we have to observe that our inductive assumption allows us to take the family associated to zf to be $\{\Gamma_{M_Q}(z)F^Q\}$. Consequently $T(\gamma, zf)$, which is not well defined unless the family attached to zf is specified for $M \subseteq Q$, $M \neq M_Q$, may be taken to be

$$J_M(\gamma, zf) - \sum_{\substack{Q \in \mathcal{F}(M) \\ Q \notin \mathcal{P}(M)}} J_M(\gamma, \Gamma_{M_Q}(z)F^Q) \quad .$$

Applying Arthur's lemma we see that

$$J_M(\gamma, zf) = \sum_{M \subseteq M'} D_M^{M'}(\gamma, \Gamma_{M'}(z))J_{M'}(\gamma, f)$$

and that

$$J_M(\gamma, \Gamma_{M_Q}(z)F^Q) = \sum_{M \subseteq M' \subseteq Q} D_M^{M'}(\gamma, \Gamma_{M'}(z))J_{M'}(\gamma, F^Q) \quad .$$

Summing over Q and then taking a difference and using the relation

$$J_{M'}(\gamma, f) = \sum_{M' \subseteq Q} J_{M'}(\gamma, F^Q)$$

we obtain

$$\Gamma_T(z)(J_M(\gamma, f) - \sum_{\substack{M \subset Q \\ Q \notin P(M)}} J_M(\gamma, F^Q) \quad .$$

This yields (5).

The jumps only cause difficulty at the semi-regular points defined by roots not in M. Our discussion of the results of the Inventiones paper, especially of Theorem 6.1. show that the jump in the normal derivative of $T(\gamma, f)$ is given by

$$n_\beta(A)\{J_{M^*}(\gamma_0, f) - \sum_{M^* \subseteq Q} J_{M^*}(\gamma_0, F^Q)\} \quad .$$

By the induction assumption this is zero.

Observe that we are implicitly using a result of Harish-Chandra for which no proof has been published. Namely, the induction assumption asserts that

$$J_{M^*}(\gamma, f) - \sum_{M^* \subseteq Q} J_{M^*}(\gamma, F^Q) = 0$$

for γ semi-simple in $M^*(\mathbf{R})$ and regular in $L(\mathbf{R})$. For the inductive argument and, as we have seen, for the fine σ-expansion we need the equality for any γ such that $1 - \mathrm{Ad}\,\gamma$ is invertible on $\mathfrak{l}/\mathfrak{m}$. This follows easily from the fact that for any $\gamma_0 \in M(\mathbf{R})$ there are differential operators $D_1, \ldots, D_r$ on Cartan subgroups $T_1, \ldots, T_r$ of M and sequences γ_i^n of regular elements in $T_i(\mathbb{R})$ such that

$$J_M(\gamma_0, F) = \lim_{n\to\infty} \sum_i D_i J_M(\gamma_i^n, F)$$

for any compactly supported smooth function on $M(\mathbb{R})$.

<u>The measure-theoretic argument sketched</u>. It involves of course the distributions Θ_M, which have never been explicitly defined. Their definition requires an improved form of Theorem 8.2 in Arthur's second Amer. Jour. paper. This involves two things, replacing the hypothetical normalization of §6 of that paper by one deduced from results of Silberger, and proving that the representation on the discrete spectrum is of trace class. These are for the Friday morning seminar.

Observe first of all that we work with functions ϕ on $\mathbf{G}$ and not on $\mathbf{G}^1$. So the integration over $i\mathfrak{a}_L^*/i\mathfrak{a}_G^*$ that appears in Arthur will be an integration over $i\mathfrak{a}_L^*$.

As it is given by Theorem 8.2 the trace formula appears as an absolutely convergent sum over χ of terms $J_\chi(\phi)$. Each $J_\chi(\phi)$ is itself a sum but Arthur does not assert that the double sum is absolutely convergent. Nonetheless it is hoped to present a proof of this in the Friday morning seminar. So we can rearrange at will. The terms appearing in the expression of $J_\chi(\phi)$ as a sum are indexed by two Levi subgroups $M \subseteq L$ and an irreducible unitary representation π of $M(\mathbb{A})$ or, if one prefers, of $M(\mathbb{A})^1$. Only countably many π actually contribute. There is another index s, but it is unimportant. Indeed it is better to use the definition of $M(P, s)$ given on p. 1309 and to express the sum over M, π, and s as a sum over unitary representations of $L(\mathbf{A})$ (induced from $M(\mathbf{A})$. The result will be attributed to $\Theta_L(\phi)$.

Thus $\Theta_L(\phi)$ is a sum

$$\sum_\sigma \int_{i\mathfrak{a}_L^*} \mathrm{tr}(R_\sigma(\lambda) I(\sigma \otimes \lambda, \phi))\, d\lambda \quad ,$$

the iterated operation being, as is to be shown, absolutely convergent. The only σ which actually occur are those which are unramified outside of S. Thus if we choose a place v_0 not in S and replace ϕ by $\phi * \phi_{v_0}$ where ϕ_{v_0} is a spherical function at v_0 the sum is replaced by

(6) $$\sum_\sigma \int_{i\mathfrak{a}_L^*} \mathrm{tr}(R_\sigma(\lambda) I(\sigma \otimes \lambda, \phi))\alpha_{\sigma\otimes\lambda}(\phi_{v_0})\, d\lambda \quad ,$$

$\alpha_{\sigma\otimes\lambda}$ being the homomorphism of the Hecke algebra into $\mathbf{C}$ attached to $\sigma \otimes \lambda$. Now $v = \alpha_{\sigma\otimes\lambda}$ is the homomorphism attached to the unitary representation $\sigma \otimes \lambda$ and thus satisfies $\overline{\alpha(\phi_{v_0})} = \alpha(\phi^*_{v_0})$ with $\phi^*_{v_0}(g) = \bar{\phi}_{v_0}(g^{-1})$.

It is well known that the set of all such homomorphisms may be identified with the quotient C of a compact subset of $\mathfrak{a}^*_v \otimes \mathbf{C}$ by the Weyl group. The Hecke algebra may be identified with an algebra of continuous functions in C. Just as in the study of base change for $GL(2)$ its closure is the algebra of all continuous functions and (6) defines a linear form on this algebra, thus a measure on C. It is clear that the measures associated to non-conjugate L are orthogonal.

What about the terms $\Theta_M(\psi^Q)$. We observe that when ϕ is replaced by $\phi * \phi_{v_0}$ the set S' may increase but the set S does not. Now we have been very careful - because it is of crucial importance - to insist that we could nonetheless take the family of functions ψ^Q attached to $\phi * \phi_{v_0}$ to be $\psi_S^Q \cdot \Gamma_{M_Q}(\phi_{v_0})$, where ψ_S^Q depends on ϕ alone, or, to be more precise, on ϕ <u>and</u> the choice of S alone but not on ϕ_{v_0} or the choice of S'.

Thus Θ_M is a sum of terms like

$$\int_{i\mathfrak{a}_M^*} \operatorname{trace}(R(\lambda)I(\sigma \otimes \lambda, \psi_S^Q)\alpha_{\sigma\otimes\lambda}(\Gamma_{M_Q}(\phi_{v_0}))d\lambda \quad .$$

So it is clear that $\phi_{v_0} \longrightarrow \Theta_M(\psi^Q)$ may be regarded as a measure on C and that measures associated to non-conjugate M are orthogonal.

Putting together the measures associated to G and taking ϕ_{v_0} to be the identity we obtain

$$\Theta_G(\phi) = \Theta_{G'}(\phi') \quad .$$

We have had to assume that ϕ is K_∞-finite but that is of no consequence.

Afternoon Seminar

THE HYPERBOLIC TERMS FOR SL(3) AND SU(3)

R. Langlands

1. Formal properties of J^T. For the ordinary trace formula these are taken from Arthur's paper The trace formula in invariant form. For the twisted trace formula they will have to be verified in the morning seminar, no serious modification of the proof being anticipated.*

The first point to keep in mind is that $J^T(\phi)$ is a polynomial in T. To express this more precisely we introduce for any standard ε-invariant parabolic the function ϕ_Q on $\mathbb{M}$ by

$$\phi_Q(m) = \rho_Q(m) \int_K \int_{\mathbb{N}_Q} \phi(k^{-1}mn\varepsilon(k))dndk \quad .$$

It has the same properties as ϕ but with respect to M rather than G.

In particular the twisted trace formula for M allows us to introduce $J^T(\phi_Q)$. There are polynomials p_Q on $\mathfrak{a}_Q^{\varepsilon}/\mathfrak{a}_G^{\varepsilon}$ of degree equal to $\dim(\mathfrak{a}_Q^{\varepsilon}/\mathfrak{a}_G^{\varepsilon})$ such that

$$J^T(\phi) = \sum_{Q \supseteq P_0} J^{T_1}(\phi_Q)p_Q(T-T_1) \quad . \tag{1}$$

We conclude that $J^T(\phi)$ is a polynomial in T.

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*Observe that, contrary to what has been said more than once in these seminars, the correct domain of integration for obtaining the trace formula from the basic identity is $G\backslash\mathbb{G}_{\varepsilon}^1$ where

$$\mathbb{G}_{\varepsilon}^1 = \{g \in G \,\big|\, |\chi(g)| = 1 \,\forall \chi \in X_{\varepsilon}^*(G)\}$$

the set $X_{\varepsilon}^*(G)$ being the set of ε-invariant rational characters of G defined over $\mathbb{Q}$.

It is inconvenient to be tied to one P_0. So we are going to modify the formula in an essentially trivial way. We do however fix M_0, an ε-invariant Levi factor of P_0 over $\mathbb{Q}$. If $M \supseteq M_0$ is reductive and ε-invariant we let $L_\varepsilon(M)$ be the set of ε-invariant reductive groups over $\mathbb{Q}$ containing M. We let $F_\varepsilon(M)$ be the set of ε-invariant parabolics over $\mathbb{Q}$ containing M and $P_\varepsilon(M) \subseteq F_\varepsilon(M)$ the set of ε-invariant parabolics with M as Levi factor. When $\varepsilon = 1$ it is not included in the notation. Thus $P_1(M) = P(M)$.

Arthur introduces the notion of K admissible relative to M_0. There is no need to rehearse the definition here. The only important point is that the Cartan decomposition $\mathbb{G} = \mathbb{P}_0' K$ is valid for any $P_0' \in P(M_0)$. Thus if $s \in \Omega^\varepsilon(\mathfrak{a}_0, \mathfrak{a}_0) = \Omega(\mathfrak{a}_0^\varepsilon, \mathfrak{a}_0^\varepsilon)$ is represented by $w = w_s$ and $P_0' = w^{-1} P_0 w$ then we can define truncation with respect to P_0'. If T lies in the positive chamber with respect to P_0' and is sufficiently regular then $T' = s^{-1}T + H(w_s^{-1})$ lies in the chamber positive with respect to P_0' and truncation with respect to P_0' allows us to introduce $J^{T'}(\phi)$. We take T to be ε-invariant and to ensure that T' is also ε-invariant we take $H^\varepsilon(w_s^{-1})$ to be the projection of $H(w_s^{-1})$, calculated with respect to P_0, on $\mathfrak{a}_0^\varepsilon$.

We have

$$J^{T'}(\phi) = J^T(\phi) \quad . \tag{2}$$

Notice that $T' = T_s'$ is determined by s alone and is independent of the choice of w_s. An analogue of the identity (1) is valid.

$$J^{T'}(\phi) = \sum_{Q' \supseteq P_0} J^{T'_1}(\phi_{Q'}) p_{Q'}(T'-T'_1) \quad .$$

Let $T^{Q'}$ be the projection of T' on $\mathfrak{a}_0^{Q'}$. It follows readily from (2) that

$$J^{T'}(\phi_{Q'}) = J^{T^{Q'}}(\phi_{Q'})$$

depends only on Q' and not on s, two different choices of s differing by an element in $\Omega^{M_{Q'}}(\mathfrak{a}_0^{\varepsilon}, \mathfrak{a}_0^{\varepsilon})$. Moreover it follows readily from Arthur's definition that

$$p_{Q'}(T'-T'_1) = p_Q(T-T_1) \quad .$$

We denote this polynomial in $T-T_1$ by $p_{M_Q}(T-T_1)$.

The identity (1) may be written

$$(3) \qquad J^T(\phi) = \sum_{Q \in \mathcal{F}_\varepsilon(M_0)} J^{T_1^Q}(\phi_Q) \frac{|\Omega^{M_Q}(\mathfrak{a}_0^{\varepsilon}, \mathfrak{a}_0^{\varepsilon})|}{|\Omega^G(\mathfrak{a}_0^{\varepsilon}, \mathfrak{a}_0^{\varepsilon})|} p_{M_Q}(T-T_1) \quad ,$$

T and T_1 being sufficiently regular and in the Weyl chamber positive with respect to P_0. We abbreviate the quotient appearing here to

$$\frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon^G|}$$

If $\mathfrak{o}$ is a semi-simple conjugacy class in G and $M \in \mathcal{L}_\varepsilon(M_0)$ then $\mathfrak{o} \cap M = \mathfrak{o}_1 \cup \ldots \cup \mathfrak{o}_r$ where the $\mathfrak{o}_i$ are semi-simple ε-conjugacy classes in M. If $P \in \mathcal{P}(M)$ set

$$J_{\mathfrak{o}}^{T^P}(\phi_P) = \sum_i J_{\mathfrak{o}_i}^{J^P}(\phi_P) \quad .$$

There is an analogue of (3),

(4)
$$J_{\mathfrak{o}}^{T}(\phi) = \sum_{Q \in \mathcal{F}_{\varepsilon}(M_0)} J_{\mathfrak{o}}^{T_1^Q}(\phi_Q) \frac{|\Omega_{\varepsilon}^{M_Q}|}{|\Omega_{\varepsilon}^{G}|} p_{M_Q}(T-T_1) \quad .$$

Notice that the value of $J_{\mathfrak{o}}^{T^Q}(\phi_Q)$ is 0 if $\mathfrak{o} \cap M_Q$ is empty. Now associated to γ we have the split center A of the ε-centralizer of γ and $\delta(\mathfrak{o}) = \dim A - \dim \mathfrak{a}_G^{\varepsilon}$ is an invariant of $\mathfrak{o}$. Clearly $\mathfrak{o} \cap M_Q$ is empty unless $\dim \mathfrak{a}_Q^{\varepsilon}/\mathfrak{a}_G^{\varepsilon} \leq \delta(\mathfrak{o})$. We conclude that $J_{\mathfrak{o}}^{T}(\phi)$ is a polynomial of degree at most $\delta(\mathfrak{o})$.

Having recalled how $J^T(\phi)$ and the terms $J_{\mathfrak{o}}^{T}(\phi)$ of the coarse $\mathfrak{o}$-expansion depend upon T, we now see how far J^T and $J_{\mathfrak{o}}^{T}$ depart from ε-invariance. If $h \in \mathbb{G}$ set

$$\phi^h(g) = \phi(hg\varepsilon(h^{-1})) \quad .$$

Observe that we can write $h = ah'$ with $a \in A_G^{\varepsilon}$ and $h' \in G_{\varepsilon}^1$. This is important if the argument in §3 of Arthur's paper is to be imitated to yield

(5)
$$J^T(\phi^h) = \sum_{Q \in \mathcal{F}(M_0)} J^{T^Q}(\phi_{Q,h}) \frac{|\Omega_{\varepsilon}^{M_Q}|}{|\Omega_{\varepsilon}^{G}|}$$

where

$$\phi_{Q,h}(m) = \rho_Q(m) \int_K \int_{\mathbb{N}_Q} \phi(k^{-1}mn\varepsilon(k))u_Q'(k, h)\,dn\,dk \quad .$$

Although we use the same notation as Arthur for the weight factor it does depend on ε.

There is a formula analogous to (5) for $J^T_{\mathfrak{o}}(\phi^h)$.

$$J^T_{\mathfrak{o}}(\phi^h) = \sum_{Q \in F(M_0)} J^{T^Q}_{\mathfrak{o}}(\phi_{Q,h}) \frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon^G|} . \tag{5'}$$

We will be guided by these formulas in our definition of $J^T_M(\phi)$. We shall seek to impose at least two conditions:

$$J^T_M(\phi) = \sum_{Q \in F(H)} \frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon|} J_M^{T_1^Q}(\phi_Q) p_{M_Q}(T-T_1) \tag{a}$$

$$J^T_M(\phi^{h'}) = \sum_{Q \in F(M)} \frac{|\Omega_\varepsilon^{M_Q}|}{|\Omega_\varepsilon|} J_M^{T^Q}(\phi_{Q,h}) . \tag{b}$$

Observe that $F(G)$ consists of G alone, and that p_G is identically 1. Thus (a) and (b) assert in particular that J^T_G is independent of T and ε-invariant.

2. <u>The hyperbolic terms for a Chevalley group</u>. We are concerned with the ordinary trace formula and we want to define $J^T_{M_0}(\phi)$ as a sum

$$\frac{1}{|\Omega^G|} \sum_{\gamma \in M_0} J^T_{M_0}(\gamma, \phi) .$$

If γ_0 lies in M_0 and M_0 is the connected component of the centralizer of γ_0 then the contribution of the semi-simple class

$\mathfrak{o} = \mathfrak{o}(\gamma_0)$ containing γ_0 has been described in the morning seminar. If we divide that contribution among the conjugates of γ_0 in M_0 we see that

$$J^T_{\mathfrak{o}}(\phi) = \frac{1}{|\Omega^G|} \operatorname{meas}(M_0 \backslash \mathbb{M}_0^1) \sum_{\gamma} \int_{\mathbb{M}_0 \backslash \mathbb{G}} \phi(g^{-1}\gamma g) v^G_{M_0}(g) dg \quad .$$

Recall that $v^G_{M_0}(g)$ is the volume of the compact convex set spanned by the points

$$\{s^{-1}T - s^{-1}H(w_s g) | s \in \Omega\} \quad ,$$

when $\Omega = \Omega(\mathfrak{a}_0, \mathfrak{a}_0)$. Notice also that

$$\{s^{-1}T - s^{-1}H(w_s w_r g) | s \in \Omega\} = \{r(s^{-1}T - s^{-1}H(w_s g))\} \quad .$$

Since the action of the Weyl group preserves volume, we conclude that

$$\int_{\mathbb{M}_0 \backslash \mathbb{G}} \phi(g^{-1}\gamma g) v^G_{M_0}(g) dg = J^T_{M_0}(\gamma, \phi)$$

is a symmetric function of γ.

It is for the moment defined only for quasi-regular elements in M_0. To define it more generally we write the integral as

(6) $$\int_K \int_{\mathbb{N}_0} \phi(k^{-1}n^{-1}\gamma nk) v^G_{M_0}(nk) dn dk \quad ,$$

observing that

$$v^G_{M_0}(g) = v^G_{M_0}(nk) - v^G_{M_0}(n)$$

if $g = ank$. For γ in M_0 and quasi-regular the transformation

$$n \longrightarrow u = \gamma^{-1} n^{-1} \gamma n$$

is a measure-preserving bijection from $\mathbb{N}_0$ to itself. So we can change variables in (6) to obtain

$$(7) \qquad \int_K \int_{\mathbb{N}_0} \phi(k^{-1}\gamma u(k) v_{M_0}^G(\gamma, u) du dk \quad .$$

We have set

$$v_{M_0}^G(\gamma, u) = v_{M_0}^G(n) \quad .$$

If (7) were defined on all of $\mathbb{M}_0$ we would simply define $J_{M_0}^T(\gamma, \phi)$ to be its value at γ. It is not. If $W = W(s) = s^{-1}(W_+)$ is the Weyl chamber associated to s we set

$$x_W(\gamma, u) = s^{-1}T - s^{-1}H(w_s n) \quad .$$

The difficulty is that some of the $x_W(\gamma, u)$ may go off to infinity as γ approaches a singular γ_0, u remaining fixed, so that the volume $v_{M_0}^G(\gamma, u)$ becomes infinite. We could tolerate this for some u but if it happens for all u there is no chance of defining the integral (7) at γ_0.

We examine first the group $SL(2)$ for which

$$x_{W_+} = T \qquad\qquad x_{W_-} = -T + H(w_s n) \quad .$$

We have

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad w_s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $H(w_s n)$ is the sum of local contributions $H(w_s n_v)$. Since

$$w_s n = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$$

the standard choice of K yields the following results:

(i) v real,

$$H(w_s n_v) = -\ell n \sqrt{1 + x_v^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(ii) v complex,

$$H(w_s n_v) = -\ell n (1 + |x_v|^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(iii) v finite

$$H(w_s n_v) = -\ell n \max\{1, |x_v|\} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Notice that

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}_0 .$$

If

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and

$$u = \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix}$$

then

$$x' = \left(1 - \frac{b}{a}\right) x \quad .$$

Thus, for a fixed x', both $|x|$ and $H(w_s n_v)$ run off to infinity as $\frac{b}{a} \longrightarrow 1$.

To rectify the situation we notice the following simple identities:

(i) v real,

$$\ln \sqrt{1 + x_v^2} = \ell n \sqrt{\left(1 - \frac{b}{a}\right)^2 + x_v'^2} - \ell n \left|1 - \frac{b}{a}\right|_v$$

(ii) v complex,

$$\ln(1 + |x_v|^2) = \ell n \left|1 - \frac{b}{a}\right|^2 + |x_v'|^2 - \ell n \left|1 - \frac{b}{a}\right|_v$$

(iii) v finite,

$$\ell n \max\{1, |x_v|\} = \ell n \max\{\left|1 - \frac{b}{a}\right|_v, |x_v'|\} - \ell n \left|1 - \frac{b}{a}\right|_v \quad .$$

In all three cases the first term behaves reasonably well as $\left|1 - \frac{b}{a}\right|_v \longrightarrow 0$ provided $|x_v'| \neq 0$. The second term behaves badly, but we begin with regular, retional γ and for these

$$\sum_v \ell n\,|1 - \frac{b}{a}|_v = 0 \quad .$$

Thus we could have begun by defining $v^G_{M_0}(\gamma, u)$ to be the volume of the convex hull of

$$x'_{W_+} = T$$

and

$$x'_{W_-} = -T + H(w_s n) - \sum_v \ell n|1 - \frac{b}{a}|_v H \quad .$$

There is no difference for regular, rational γ.

This leads to difficulties. A little more care is called for. We first observe that for a fixed ϕ there are only finitely many γ in M_0 for which $\phi(k^{-1}\gamma uk)$ does not vanish identically in u and k. So there is a finite set of places $S(\phi)$ such that $|1 - \frac{b}{a}|_v = 1$ if $v \notin S(\phi)$, γ is quasi-regular in M'_0, and $\phi(k^{-1}\gamma uk)$ does not vanish identically as a function of u and k. Thus we begin by defining $V^G_{M_0}(\gamma, u)$ to be the volume of the convex hull of

$$X_{W_+} = T$$

and

$$X_{W_-} = - T + H(w_s n) - \sum_{v \in S(\phi)} \ell n|1 - \frac{b}{a}|_v H \quad .$$

We so choose $S(\phi)$ that it contains S.

With this definition of $V^G_{M_0}(\gamma, u)$. we have, for γ regular in M_0,

$$(8)\quad J^T_{M_0}(\gamma, \phi) = c\chi(\gamma) \int_K \int_{N_0(\mathbb{A}_{S(\phi)})} \phi_{S(\phi)}(k^{-1}\gamma uk) V^G_{M_0}(\gamma, u)\,du\,dk \quad .$$

Here $\chi(\gamma)$ is the characteristic function of the set of elements in $\mathbb{M}_0$ whose projection on $G(\mathbb{A}^{S(\phi)})$ lies in $K^{S(\phi)}$ and

$$c = \int_{N_0(\mathbb{A}^{S(\phi)}) \cap K} du \quad .$$

We are assuming that $K = \prod_v K_v$.

To see this we have to observe first of all that $\chi(\gamma) \neq 0$ and $\phi_{S(\phi)}(k^{-1}\gamma uk) \neq 0$ imply that $\phi(k^{-1}\gamma uk) \neq 0$. We also have to observe that when this is so then

$$\max\{1, |x_v|\} = 1$$

on $N_0(\mathbb{Q}_v) \cap K_v$, which is the support of $u \longrightarrow \phi_v(k^{-1}\gamma uk)$.

The sequence of steps leading to (8) I refer to as Flicker's trick. Before seeing how it works for other Chevalley groups we examine the right side of (8) more carefully for $SL(2)$. If d is the volume of the interval $[0, H]$ the double integral is equal to the product of

$$d \text{ meas } K^{S(\phi)}$$

with the sum of

$$2T \int_{K_{S(\phi)}} \int_{N_0(\mathbb{A}_{S(\phi)})} \phi_{S(\phi)}(k^{-1}\gamma uk)\,du\,dk$$

and

$$\sum_{v \in S(\phi)} \int_{K_{S(\phi)}} \int_{N_0(\mathbb{A}_{S(\phi)})} \phi_{S(\phi)}(k^{-1}\gamma uk) v_{M_0}^G(\gamma, u_v) dudk$$

where

$$v_{M_0}^G(\gamma, u_v)H = -s^{-1}H(w_s n_v) - \ell n |1 - \frac{b}{a}|_v H \quad .$$

The previous formulas make it clear that we can dominate $v_{M_0}^G(\gamma, u_v)$ by a locally integrable function of u on any compact set of γ and thus that (8) defines $J_{M_0}^T(\gamma, \phi)$ as a continuous function of γ on $\mathbb{M}_0$. Thus we can define $J_{M_0}^T(\phi)$ to be

$$\frac{1}{|\Omega^G|} \sum_{\gamma \in M_0} J_{M_0}^T(\gamma, \phi) \quad .$$

One establishes directly for regular γ and by continuity for general γ that

$$(9) \qquad J_{M_0}^T(\gamma, \phi) = \sum_{Q \in F(M_0)} \frac{|\Omega^{M_Q}|}{|\Omega^G|} J_{M_0}^{T_1^Q}(\gamma, \phi_Q) p_{M_Q}(T - T_1)$$

and that

$$(10) \qquad J_{M_0}^T(\gamma, \phi^h) = \sum_{Q \in F(M_0)} \frac{|\Omega^{M_Q}|}{|\Omega^G|} J_{M_0}^{T^Q}(\gamma, \phi_{Q,h}) \quad .$$

Observe that if Q is minimal then

$$J_{M_0}^{T^Q}(\gamma, \phi_Q) = \phi_Q(\gamma)$$

and

$$J_{M_0}^{T^Q}(\gamma,\ \phi_{Q,h}) = \phi_{Q,h}(\gamma) \quad .$$

We now set

$$J_G^T(\mathfrak{o},\ \phi) = J_{\mathfrak{o}}^T(\phi) - \frac{1}{|\Omega^G|}\sum_{\gamma\in\mathfrak{o}\cap M_0} J_{M_0}^T(\gamma,\ \phi) \quad .$$

We readily deduce from (4), (5), (9), (10) and the definition that

(a) $J_G^T(\mathfrak{o},\ \phi) = 0$ if $\mathfrak{o}$ is hyperbolic (that is, not elliptic)

(b) $J_G^T(\mathfrak{o},\ \phi)$ is independent of T

(c) $J_G^T(\mathfrak{o},\ \phi)$ is invariant.

We shall set

$$J_G^T(\phi) = \sum_{\mathfrak{o}} J_G^T(\mathfrak{o},\ \phi) \quad .$$

For an arbitrary Chevalley group the appropriate modification is to replace $x_W = x_W(\gamma,\ u)$ by

(11) $$X_W = x_W - \sum_{v\in S(\phi)}\sum_{\alpha} \ln|1-\alpha^{-1}(\gamma)|_v H_\alpha$$

where H_α is defined by

$$\beta(H_\alpha) = (\beta,\ \alpha)$$

and the inner sum runs over the positive roots separating W from W_+. It is up to the morning seminar to justify this assertion. We shall have to examine it more carefully when we discuss the cancellation of singularities, but for the moment we need it only as a guide for the quasi-split case to which we now turn, contenting ourselves with SU(3), which we realize as the unitary group of the form

$$\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$$

taken on a quadratic extension E of the global field F. (Notice that by working only with groups over $\mathbb{Q}$ we have backed ourselves into a corner, for we have no formalism for dealing easily with an arbitrary number field. It is however easy to imagine what that would be, and even easier to construct it. So we feel free to use it, and in particular to apply the above treatment to Chevalley groups over F.)

<u>The hyperbolic terms for</u> SU(3). The weighting factor is at first defined by

$$x_{W_+} = T$$

and

$$x_{W_-} = s^{-1}T - s^{-1}H(w_s n)$$

where s is the one non-trivial element in the Weyl group. It is represented by

$$w_s = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} .$$

As before we regard the resulting weight factor as a function of γ and u. Thus

$$v_{M_0}^G(n) = v_{M_0}^G(\gamma, u)$$

where

$$u = \gamma^{-1} n^{-1} \gamma n \quad .$$

We want to modify as before to obtain $V_{M_0}^G(\gamma, u)$. It is clear that the choice of $X_{W_-}(\gamma, u)$ is determined by what we have already done, for the group is split at half the places and we shall need to invoke the product formula.

To make this more precise we observe that the global $\mathfrak{a}_0$, which may be identified in the present case with

$$\left\{ \begin{pmatrix} x_0 & \\ & -x \end{pmatrix} \middle| \ x \in \mathbb{R} \right\}$$

is contained in the local $\mathfrak{a}_0$, denoted $\mathfrak{a}_0^v$. If v remains prime in E then $\mathfrak{a}_0^v = \mathfrak{a}_0$. If it splits then $\mathfrak{a}_0^v$ may be identified with

$$\left\{ \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \middle| \; x, y, z \in \mathbb{R}, \; x + y + z = 0 \right\} .$$

In addition

$$s^{-1}H(w_s n) = \sum_v s^{-1}H(w_s n_v)$$

and to calculate $s^{-1}H(w_s n_v)$ at a place split in E we first calculate it on $\mathfrak{a}_0^v$ and then project to $\mathfrak{a}_0$, the projection being

$$\begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{x-z}{2} & & \\ & 0 & \\ & & \frac{z-x}{2} \end{pmatrix} .$$

If

$$\gamma = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$$

the modification to be undertaken in the split group is to subtract

$$\ln\left|1 - \frac{b}{a}\right|_v \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} + \ln\left|1 - \frac{c}{b}\right|_v \begin{pmatrix} a & & \\ & 1 & \\ & & -1 \end{pmatrix} + \ln\left|1 - \frac{c}{a}\right|_v \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} .$$

However we are interested only in the projection, which is

$$(12) \quad \ln\left|1 - \frac{b}{a}\right|\left|1 - \frac{c}{b}\right|_v \begin{pmatrix} \frac{1}{2} & & \\ & 0 & \\ & & -\frac{1}{2} \end{pmatrix} + \ln\left|1 - \frac{c}{a}\right|_v \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} .$$

The split center A_0 of M_0 has two weights in $\mathfrak{n}_0$, say α_1 and α_2 with $\alpha_2 = 2\alpha_1$. Set

$$H_{\alpha_1} = \begin{pmatrix} \frac{1}{2} & & \\ & 0 & \\ & & -\frac{1}{2} \end{pmatrix}, \qquad H_{\alpha_2} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} .$$

Furthermore if α is one of these two weight let $\mathfrak{w}_0(\alpha)$ be the corresponding weight space in $\mathfrak{w}_0$ and consider

$$A_\alpha(\gamma) = \text{ad}\ \gamma \,|_{\mathfrak{w}_0(\alpha)} .$$

The expression (12) is equal to

$$\ell n \left|\det(1 - A_{\alpha_1}^{-1}(\gamma))\right|_v H_{\alpha_1} + \ell n \left|\det(1 - A_{\alpha_2}^{-1}(\gamma))\right| H_{\alpha_2} .$$

We can therefore expect to define $X_{W_-}(\gamma, u)$ by

$$X_{W_-}(\gamma, u) = x_{W_-}(\gamma, u) - \sum_{v \in S(\phi)} \sum_\alpha \ln\left|\det(1 - A_\alpha^{-1}(\gamma))\right|_v H_\alpha ,$$

the existence of the set $S(\phi)$ being proved as before. The inner sum runs over α_1 and α_2, the positive roots separating W_+ from W_-.

The $X_{W_-}(\gamma, u)$ allow us once again to introduce weight factors $V_{M_0}(\gamma, u)$, but we still must verify that the formula (8) serves to define $J_{M_0}^T(\gamma, \phi)$ as a continuous function on all of $\mathbb{M}_0$. This is a local problem and we carry out the calculations only for the case that x does not split in E. The question will have to be taken up afresh in the next lecture anyhow.

Let

$$n_v = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}$$

$$n'_v = \begin{pmatrix} 1 & x' & z' \\ & 1 & y' \\ & & 1 \end{pmatrix}$$

If v does not split in E there are relations to satisfy:

$$b\bar{b} = 1, \ a\bar{c} = 1, \ x = \bar{y}, \ z+\bar{z} = x\bar{x}, \ x' = \bar{y}', \ z'+\bar{z}' = x'\bar{x}' \quad .$$

The bar denotes conjugation in E_v/F_v.

If v is not split in E and is finite we have to choose from amongst those λ in E_v with $\mathrm{tr}\lambda = 1$ one for which $|\lambda|$ is minimal. Then an appropriate choice of K_v is apparently the stabilizer in $G(F_v)$ of the lattice

$$\left\{ \begin{pmatrix} u \\ v \\ \lambda w \end{pmatrix} \middle| \ u, v, w \ \text{integral} \right\} \quad .$$

Thus K_v consists of the matrices (a_{ij}) in $G(F_v)$ for which $a_{11}, a_{12}, a_{21}, a_{22}, a_{33}, \lambda a_{13}, \lambda a_{23}, a_{31}/\lambda, a_{32}/\lambda$ are integral. If v is real then K_v can be taken to be the intersection of $G(F_v)$ with the unitary group attached to the form

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad .$$

Since

$$w_s n_v = \begin{pmatrix} & & 1 \\ & -1 & -y \\ 1 & x & z \end{pmatrix}$$

we see that $-s^{-1}H(w_s n_v)$ is the product of

$$H = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

with the following factor

(i) $-\frac{1}{2}\ln(1 + |x|^2 + |z|^2)$ if v is real.

(ii) $-\ell n(1 + |x|^2 + |z|^2)$ if v is complex.

(iii) $-\ell n \max\{1, |x|, |\lambda z|\} + \ell n|\lambda|$ if v is finite.

The second term in (iii) is harmless, and certainly does not affect the singular behavior at any γ_0.

It is easy to express x', y', z' in terms of x, y, z

$$x' = (1 - \frac{b}{a})x , \quad y' = (1 - \frac{c}{b})x , \quad z' = (1 - \frac{c}{a})z + (\frac{c}{a} - \frac{b}{a})xy \quad .$$

Thus

$$-s^{-1}H(w_s n_v) - \sum_\alpha \ell n|\det(1 - A_\alpha^{-1}(\gamma))|H_\alpha$$

is equal to the product of H with the factor

(i) $-\frac{1}{2}\ell n(|1 - \frac{b}{a}|^2|1 - \frac{c}{a}|^2 + |x'|^2|1 - \frac{c}{a}|^2 + |z'(1 - \frac{b}{a}) + \frac{b}{a}x'y'|^2$ if v is real,

(ii) $-\ln(|1 - \frac{b}{a}|^2|1 - \frac{c}{a}|^2 + |x'|^2|1 - \frac{c}{a}|^2 + |z'(1 - \frac{b}{a}) + \frac{b}{a}x'y'|^2$ if v is complex,

(iii) $-\ell n(\max\{|1 - \frac{b}{a}||1 - \frac{c}{a}|, |x'||1 - \frac{c}{a}|, |\lambda||z'(1 - \frac{b}{a}) + \frac{b}{a}x'y'|\}) + \ell n|\lambda|$

if v is finite.

This factor we denote $V^G_{M_0}(\gamma, u_v)$. We need to verify that

$$\int_{K_{S(\phi)}} \int_{N_0(\mathbb{A}_{S(\phi)})} \phi_{S(\phi)}(k^{-1}\gamma uk) V^G_{\mathbb{M}_0}(\gamma, u_v) du dk$$

defines a continuous function of γ in $M_0(\mathbb{A}_{S(\phi)})$.

To do this we choose a small ε and consider the domains

$$|z'(1 - \frac{b}{a})| \le \varepsilon |x'y'| , \quad |z'(1 - \frac{b}{a})| > \varepsilon |x'y'|$$

separately. On the first domain we can suppose that

$$V^G_{M_0}(\gamma, u_v) \le c_1 + c_2 \ln |x'y'|$$

when $\phi_{S(\phi)}(k^{-1}\gamma uk)$ is not zero. Since $\ln |x'y'|$ is locally integrable the dominated convergence theorem is applicable. Since z' is bounded on the support of $\phi_{S(\phi)}(k^{-1}\gamma uk)$, we can replace the second domain by

$$|x'| \le c_3 |\alpha| , \quad |y'| \le c_3 |\alpha| ,$$

where $\alpha \in E_v$ and $|\alpha^{-2}(1 - \frac{b}{a})|$ is bounded away from infinity and zero. We set $x' = \alpha x''$, $y' = \bar{\alpha} y''$ and change variables to obtain an integral over $|x''| \le c_3$, $|y''| \le c_3$, $|z'| < c_4$ of a function dominated by

$$|1 - \frac{b}{a}|(c_5 + c_6 \ln |1 - \frac{b}{a}| + c_7 \ln |\beta z' + x''y''|) ,$$

where $\beta = \beta(\gamma)$ is bounded away from infinity and zero. The dominated convergence theorem is applicable once again.

We have discussed the quasi-split groups SL(2) and SU(3). We could as easily have discussed a group G whose derived group was isogeneous to one of these two groups, introducing the distribution J_G. The tactics outlined in the previous lecture demand that we show that

$$J_G(\phi) - \sum_{H \neq G} \iota(G, H) SJ_H(\phi^H)$$

is a stably invariant distribution.

For an anisotropic group G obtained from one of the above by an inner twisting there are no hyperbolic terms and we set

$$J_G(\phi) = J_G^T(\phi) = J^T(\phi) \quad .$$

Now we must show that

$$J_G(\phi) = \sum_H \iota(G, H) SJ_H(\phi^H) \quad .$$

These two problems will be dealt with by Rogawski after Christmas. In the next lecture we will turn to a different matter, cancellation of singularities.