Germ expansions for real groups

James Arthur*

Contents

* Supported in part by a Guggenheim Fellowship, the Institute for Advanced Study, and an NSERC Operating Grant.

Introduction

Suppose that G is a connected reductive group over a local field F of characteristic 0. The study of harmonic analysis on $G(F)$ leads directly to interesting functions with complicated singularities. If the field F is p-adic, there is an important qualitative description of the behaviour of these functions near a singular point. It is given by the Shalika germ expansion, and more generally, its noninvariant analogue. The purpose of this paper is to establish similar expansions in the archimedean case $F = \mathbb{R}$.

The functions in question are the invariant orbital integrals, and their weighted generalizations. They are defined by integrating test functions $f \in C_c^\infty\big(G(F)\big)$ over strongly regular conjugacy classes in $G(F)$. We recall that $\gamma \in G(F)$ is strongly regular if its centralizer G_γ in G is a torus, and that the set G_{reg} of strongly regular elements is open and dense in G. If $\gamma \in G_{\mathrm{reg}}(F)$ approaches a singular point c, the corresponding orbital integrals blow up. It is important to study the resulting behaviour in terms of both γ and f.

The invariant orbital integral

$$f_G(\gamma) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\overline{s}G(F)} f(x^{-1}\gamma x)dx, \qquad \gamma \in G_{\mathrm{reg}}(F),$$

is attached to the invariant measure dx on the conjugacy class of γ. Invariant orbital integrals were introduced by Harish-Chandra. They play a critical role in his study of harmonic analysis on $G(F)$. The weighted orbital integral

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\overline{s}G(F)} f(x^{-1}\gamma x)v_M(x)dx, \qquad \gamma \in M(F) \cap G_{\mathrm{reg}}(F),$$

is defined by a noninvariant measure $v_M(x)dx$ on the class of γ. The factor $v_M(x)$ is the volume of a certain convex hull, which depends on both x and a Levi subgroup M of G. Weighted orbital integrals have an indirect bearing on harmonic analysis, but they are most significant in their role as terms in the general trace formula. In the special case that $M = G$, the definitions reduce to $v_G(x) = 1$ and $J_G(\gamma, f) = f_G(\gamma)$. Weighted orbital integrals therefore include invariant orbital integrals.

Suppose that c is an arbitrary semisimple element in $G(F)$. In §1, we shall introduce a vector space $\mathcal{D}_c(G)$ of distributions on $G(F)$. Let $\mathcal{U}_c(G)$ be the union of the set of conjugacy classes $\Gamma_c(G)$ in $G(F)$ whose semisimple part equals the conjugacy class of c. Then $\mathcal{D}_c(G)$ is defined to be the space of distributions that are invariant under conjugation by $G(F)$, and are supported on $\mathcal{U}_c(G)$. If F is p-adic, $\mathcal{D}_c(G)$ is finite dimensional. It has a basis composed of singular invariant orbital integrals

$$f \longrightarrow f_G(\rho), \qquad \rho \in \Gamma_c(G),$$

taken over the classes in $\Gamma_c(G)$. However if $F = \mathbb{R}$, the space $\mathcal{D}_c(G)$ is infinite dimensional. It contains normal derivatives of orbital integrals, as well as more general distributions associated to harmonic differential operators. In §1 (which like the rest of the paper pertains to the case $F = \mathbb{R}$), we shall describe a suitable basis $R_c(G)$ of $\mathcal{D}_c(G)$.

For p-adic F, the invariant orbital integral has a decomposition

$$(1)_p \qquad f_G(\gamma) = \sum_{\rho \in \Gamma_c(G)} \rho^\vee(\gamma) f_G(\rho), \qquad f \in C_c^\infty\big(G(F)\big),$$

into a finite linear combination of functions parametrized by conjugacy classes. This is the original expansion of Shalika. It holds for strongly regular points γ that are close to c, in a sense that depends on f. The terms

$$\rho^\vee(\gamma) = g_G^G(\gamma, \rho), \qquad \rho \in \Gamma_c(G),$$

are known as Shalika germs, since they are often treated as germs of functions of γ around c. One can in fact also treat them as functions, since they have a homogeneity property that allows them to be defined on a fixed neighbourhood of c. The role of the Shalika germ expansion is to free the singularities of $f_G(\gamma)$ from their dependence on f.

In §2, we introduce an analogue of the Shalika germ expansion for the archimedean case $F = \mathbb{R}$. The situation is now slightly more complicated. The sum in $(1)_p$ over the finite set $\Gamma_c(G)$ has instead to be taken over the infinite set $R_c(G)$. Moreover, in place of an actual identity, we obtain only an asymptotic formula

$$(1)_{\mathbb{R}} \qquad f_G(\gamma) \sim \sum_{\rho \in R_c(G)} \rho^\vee(\gamma) f_G(\rho), \qquad f \in C_c^\infty(G).$$

As in the p-adic case, however the terms

$$\rho^\vee(\gamma) = g_G^G(\gamma, \rho), \qquad \rho \in R_c(G),$$

can be treated as functions of γ, by virtue of a natural homogeneity property. The proof of $(1)_{\mathbb{R}}$ is not difficult, and is probably implicit in several sources. We shall derive it from standard results of Harish-Chandra, and the characterization by Bouaziz [B2] of invariant orbital integrals.

Suppose now that M is a Levi subgroup of G, and that c is an arbitrary semisimple element in $M(F)$. It is important to understand something of the behaviour of the general weighted orbital integral $J_M(\gamma, f)$, for points γ near c. For example, in the comparison of trace formulas, one can sometimes establish identities among terms parametrized by strongly regular points γ. One would like to extend such identities to the more general terms parametrized by singular points ρ.

In the p-adic case, there is again a finite expansion*

$$(2)_p \qquad J_M(\gamma, f) = \sum_{L \in \mathcal{L}(M)} \sum_{\rho \in \Gamma_c(L)} g_M^L(\gamma, \rho) J_L(\rho, f), \qquad f \in C_c^\infty\big(G(F)\big).$$

The right hand side is now a double sum, in which L ranges over the finite set $\mathcal{L}(M)$ of Levi subgroups containing M. The terms

$$g_M^L(\gamma, \rho), \qquad L \in \mathcal{L}(M),\ \rho \in \Gamma_c(L),$$

in the expansion are defined as germs of functions of γ in $M(F) \cap G_{\mathrm{reg}}(F)$ near c. The coefficients

$$J_L(\rho, f), \qquad L \in \mathcal{L}(M),\ \rho \in \Gamma_c(L),$$

are singular weighted orbital integrals. These objects were defined in [A3, (6.5)], for F real as well as p-adic, by constructing a suitable measure on the $G(F)$-conjugacy class of the singular element ρ. The role of $(2)_p$ is again to isolate the singularities of $J_M(\gamma, f)$ from their dependence on f.

The goal of this paper is to establish an analogue of $(2)_p$ in the archimedean case $F = \mathbb{R}$. We shall state the results in §5, in the form of two theorems. The main assertion is that there is an infinite asymptotic expansion

$$(2)_{\mathbb{R}} \qquad J_M(\gamma, f) \sim \sum_{L \in \mathcal{L}(M)} \sum_{\rho \in R_c(L)} g_M^L(\gamma, \rho) J_L(\rho, f), \qquad f \in C_c^\infty\big(G(\mathbb{R})\big).$$

* I thank Waldspurger for pointing this version of the expansion out to me. My original formulation [A3, Proposition 9.1] was less elegant.

The double sum here is essentially parallel to $(2)_p$, but its summands are considerably more complicated. The terms

$$g_M^L(\gamma, \rho), \qquad L \in \mathcal{L}(M),\ \rho \in R_c(L), \tag{3}$$

are "formal germs", in that they are defined as formal asymptotic series of germs of functions. The coefficients

$$J_L(\rho, f), \qquad L \in \mathcal{L}(M),\ \rho \in R_c(L), \tag{4}$$

have to be defined for *all* singular invariant distributions ρ, rather than just the singular orbital integrals spanned by $\Gamma_c(L)$. The definitions of [A3] are therefore not good enough. We shall instead construct the distributions $J_L(\rho, f)$ and the formal germs $g_M^L(\gamma, \rho)$ together, in the course of proving the two theorems. We refer the reader to the statement of Theorem 5.1 for a detailed list of properties of these objects.

The proof of Theorems 5.1 and 5.2 will occupy Sections 6 through 9. The argument is by induction. We draw some preliminary inferences from our induction hypothesis in §6. However, our main inspiration is to be taken from the obvious source, the work of Harish-Chandra, specifically his ingenious use of differential equations to estimate invariant orbital integrals. One such technique is the foundation in §3 of some initial estimates for weighted orbital integrals around c. These estimates in turn serve as motivation for the general spaces of formal germs we introduce in §4. A second technique of Harish-Chandra will be the basis of our main estimate. We shall apply the technique in §7 to the differential equations satisfied by the asymptotic series on the right hand side of $(2)_{\mathbb{R}}$, or rather, the difference between $J_M(\gamma, f)$ and that part of the asymptotic series that can be defined by our induction hypothesis. The resulting estimate will be used in §8 to establish two propositions. These propositions are really the heart of the matter. They will allow us to construct the remaining part of the asymptotic series in §9, and to show that it has the the required properties.

In §10, we shall apply our theorems to invariant distributions. We are speaking here of the invariant analogues of weighted orbital integrals, the distributions

$$I_M(\gamma, f), \qquad \gamma \in M(\mathbb{R}) \cap G_{\mathrm{reg}}(\mathbb{R}),$$

that occur in the invariant trace formula. We shall derive an asymptotic expansion

$$I_M(\gamma, f) \sim \sum_{L \in \mathcal{L}(M)} \sum_{\rho \in R_c(L)} g_M^L(\gamma, \rho) I_L(\rho, f), \qquad f \in C_c^\infty\big(G(\mathbb{R})\big), \tag{5}$$

for these objects that is parallel to $(2)_{\mathbb{R}}$.

We shall conclude the paper in §11 with some supplementary comments on the new distributions. In particular, we shall show that the invariant distributions

$$I_L(\rho, f), \qquad L \in \mathcal{L}(M),\ \rho \in R_c(L), \tag{6}$$

in (5), as well as their noninvariant counterparts (4), satisfy a natural descent condition.

The distributions (6) are important objects in their own right. They seem to be essential for dealing with problems of endoscopic transfer. However, their definition is quite indirect. It relies on the construction of noninvariant distributions (4), which as we have noted, is a consequence of our main theorems. Neither set of distributions is entirely determined by the given conditions. We shall frame this lack of uniqueness in terms of a choice of some element in a finite dimensional affine vector space. One can make the choice in either the noninvariant context (Proposition 8.3), or equivalently, the setting of the invariant distributions (as explained at the end of §10). When it comes to comparing invariant distributions (6) on different groups, it will of course be important to make the required choices in a compatible way. As we shall see in another paper, the asymptotic expansion (5) will be an integral part of the process.

§1. Singular invariant distributions

Let G be a connected reductive group over the real field $\mathbb{R}$. If c is a semisimple element in $G(\mathbb{R})$, we write $G_{c,+}$ for the centralizer of c in G, and $G_c = (G_{c,+})^0$ for the connected component of 1 in $G_{c,+}$. Both $G_{c,+}$ and G_c are reductive algebraic groups over $\mathbb{R}$. Recall that c is said to be strongly G-regular if $G_{c,+} = G$ is a maximal torus in G. We shall frequently denote such elements by the symbol γ, reserving c for more general semisimple elements. We write $\Gamma_{ss}(G) = \Gamma_{ss}\big(G(\mathbb{R})\big)$ and $\Gamma_{\mathrm{reg}}(G) = \Gamma_{\mathrm{reg}}\big(G(\mathbb{R})\big)$ for the set of conjugacy classes in $G(\mathbb{R})$ that are, respectively, semisimple and strongly G-regular.

We follow the usual practice of representing the Lie algebra of a group by a corresponding lower case gothic letter. For example, if c belongs to $\Gamma_{ss}(G)$,

$$\mathfrak{g}_c = \{X \in \mathfrak{g} : \ \mathrm{Ad}(c)X = X\}$$

denotes the Lie algebra of G_c. (We frequently do not distinguish between a conjugacy class and some fixed representative of the class.) Suppose that $\gamma \in \Gamma_{\mathrm{reg}}(G)$. Then $T = G_\gamma$ is a maximal torus of G over $\mathbb{R}$, with Lie algebra $\mathfrak{t} = \mathfrak{g}_\gamma$, and we write

$$D(\gamma) = D^G(\gamma) = \det\big(1 - \mathrm{Ad}(\gamma)\big)_{\mathfrak{g}/\mathfrak{t}}$$

for the Weyl discriminant of G. If γ is contained in G_c, we can of course also form the Weyl discriminant

$$D_c(\gamma) = D^{G_c}(\gamma) = \det\big(1 - \mathrm{Ad}(\gamma)\big)_{\mathfrak{g}_c/\mathfrak{t}}$$

of G_c. The function D_c will play an important role in formulating the general germ expansions of this paper.

Suppose that f is a function in the Schwartz space $\mathcal{C}(G) = \mathcal{C}\big(G(\mathbb{R})\big)$ on $G(\mathbb{R})$ [H3], and that γ belongs to $\Gamma_{\mathrm{reg}}(G)$. The *invariant orbital integral* of f at γ is defined by the absolutely convergent integral

$$f_G(\gamma) = J_G(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma(\mathbb{R})\overline{s}G(\mathbb{R})} f(x^{-1}\gamma x)dx.$$

One can regard $f_G(\gamma)$ as a function of f, in which case it is a tempered distribution. One can also regard $f_G(\gamma)$ is a function of γ, in which case it represents a transform from $\mathcal{C}(G)$ to a space of functions on either $\Gamma_{\mathrm{reg}}(G)$ or

$$T_{\mathrm{reg}}(\mathbb{R}) = G_\gamma(\mathbb{R}) \cap G_{\mathrm{reg}}(\mathbb{R}).$$

(Recall that $G_{\mathrm{reg}}(\mathbb{R})$ denotes the open dense subset of strongly G-regular elements in $G(\mathbb{R})$.) We shall generally take the second point of view. In the next section, we shall establish an asymptotic expansion for $f_G(\gamma)$, as γ approaches a fixed singular point.

Let $c \in \Gamma_{ss}(G)$ be a fixed semisimple conjugacy class. Keeping in mind that c also denotes a fixed element within the given class, we write $\mathcal{U}_c(G)$ for the union of those conjugacy classes in $G(\mathbb{R})$ whose semisimple Jordan component equals c. Then $\mathcal{U}_c(G)$ is a closed subset of $G(\mathbb{R})$ on which $G(\mathbb{R})$ acts by conjugation. We define $\mathcal{D}_c(G)$ to be the vector space of $G(\mathbb{R})$-invariant distributions that are supported on $\mathcal{U}_c(G)$. In this section, we shall introduce a suitable basis of $\mathcal{D}_c(G)$.

Elements in $\mathcal{D}_c(G)$ are easy to construct. Let $\mathcal{T}_c(G)$ be a fixed set of representatives of the $G_{c,+}(\mathbb{R})$-orbits of maximal tori on G_c over $\mathbb{R}$, or equivalently, a fixed set of representatives of the $G(\mathbb{R})$-orbits of maximal tori in G over $\mathbb{R}$ that contain c. We shall write $S_c(G)$ for the set of triplets

$$\sigma = (T, \Omega, X),$$

where T belongs to $\mathcal{T}_c(G)$, Ω belongs to the set $\pi_{0,c}\big(T_{\mathrm{reg}}(\mathbb{R})\big)$ of connected components of $T_{\mathrm{reg}}(\mathbb{R})$ whose closure contains c, and X is an invariant differential operator on $T(\mathbb{R})$. (By an invariant differential operator on $T(\mathbb{R})$, we of course mean a linear differential operator that is invariant under translation by $T(\mathbb{R})$.) Let σ be a triplet in $S_c(G)$. A deep theorem of Harish-Chandra [H3] asserts that the orbital integral

$$f_G(\gamma), \qquad\qquad f \in \mathcal{C}(G),\ \gamma \in \Omega,$$

extends to a continuous linear map from $\mathcal{C}(G)$ to the space of smooth functions on the closure of Ω. It follows from this that the limit

$$f_G(\sigma) = \lim_{\gamma \to c} (Xf_G)(\gamma), \qquad\qquad \gamma \in \Omega,\ f \in \mathcal{C}(G),$$

exists, and is continuous in f. If f is compactly supported and vanishes on neighbourhood of $\mathcal{U}_c(G)$, $f_G(\sigma)$ equals 0. The linear form $f \to f_G(\sigma)$ therefore belongs to $\mathcal{D}_c(G)$.

Bouaziz has shown that, conversely, the distributions $f \to f_G(\sigma)$ span $\mathcal{D}_c(G)$. To describe the result in more detail, we need to attach some familiar data to the tori T in $\mathcal{T}_c(G)$. Given T, we write $W_{\mathbb{R}}(G,T)$ for the subgroup of elements in the Weyl group $W(G,T)$ of (G,T) that are defined over $\mathbb{R}$, and $W\big(G(\mathbb{R}), T(\mathbb{R})\big)$ for the subgroup of elements in $W_{\mathbb{R}}(G,T)$ induced from $G(\mathbb{R})$. We also write $W_{\mathbb{R},c}(G,T)$ and $W_c\big(G(\mathbb{R}), T(\mathbb{R})\big)$ for the subgroups of elements in $W_{\mathbb{R}}(G,T)$ and $W\big(G(\mathbb{R}), T(\mathbb{R})\big)$, respectively, that map the element $c \in T(\mathbb{R})$ to itself. We then form the imaginary root sign character

$$\varepsilon_{c,I}(w) = (-1)^b, \qquad\qquad b = |w(\Sigma^+_{c,I}) \cap \Sigma^+_{c,I}|,\ w \in W_{\mathbb{R},c}(G,T),$$

on $W_{\mathbb{R},c}(G,T)$, where $\Sigma^+_{c,I}$ denotes the set of positive imaginary roots on (G_c, T) relative to any chamber. This allows us to define the subspace

$$S\big(\mathfrak{t}(\mathbb{C})\big)^{c,I} = \big\{u \in S\big(\mathfrak{t}(\mathbb{C})\big) :\ wu = \varepsilon_{c,I}(w)u,\ w \in W_c\big(G(\mathbb{R}), T(\mathbb{R})\big\}$$

of elements in the symmetric algebra on $\mathfrak{t}(\mathbb{C})$ that transform under $W_c\big(G(\mathbb{R}), T(\mathbb{R})\big)$ according to the character $\varepsilon_{c,I}$. There is a canonical isomorphism $u \to \partial(u)$ from $S\big(\mathfrak{t}(\mathbb{C})\big)^{c,I}$ onto the space of $\varepsilon_{c,I}$-equivariant differential operators on $T(\mathbb{R})$.

For each $T \in \mathcal{T}_c(G)$, we choose a connected component $\Omega_T \in \pi_{0,c}\big(T_{\mathrm{reg}}(\mathbb{R})\big)$. For any $u \in S\big(A(\mathbb{C})\big)^{c,I}$ and $w \in W_{\mathbb{R},c}(G,T)$, the triplet

$$\sigma_{w,u} = \big(T, w\Omega_T, \partial(u)\big)$$

lies in $S_c(G)$. We obtain a linear transformation

$$\rho : \bigoplus_{T \in \mathcal{T}_c(G)} S\big(\mathfrak{t}(\mathbb{C})\big)^{c,I} \longrightarrow \mathcal{D}_c(G) \tag{1.1}$$

by mapping u to the distribution

$$\rho_u :\ f \longrightarrow \sum_{w \in W_{\mathbb{R},c}(G,T)} \varepsilon_{c,I}(w) f_G(\sigma_{w,u}), \qquad\qquad f \in \mathcal{C}(G),$$

in $\mathcal{D}_c(G)$. For each T, we choose a basis $B\big(\mathfrak{t}(\mathbb{C})\big)^{c,I}$ of $S\big(\mathfrak{t}(\mathbb{C})\big)^{c,I}$, whose elements we take to be homogeneous. We then form the subset

$$R_c(G) = \big\{\rho_u :\ T \in \mathcal{T}_c(G),\ u \in B\big(\mathfrak{t}(\mathbb{C})\big)^{c,I}\big\}$$

of $\mathcal{D}_c(G)$.

Lemma 1.1. *The map (1.1) is an isomorphism, and $R_c(G)$ is a basis of $\mathcal{D}_c(G)$. In particular, $\mathcal{D}_c(G)$ consists of tempered distributions.*

Proof. Since $R_c(G)$ is the image under the linear transformation (1.1) of a basis, it would be enough to establish the assertion that (1.1) is an isomorphism. We could equally well deal with the mapping

$$\rho' : \bigoplus_{T \in \mathcal{T}_c(G)} S\big(\mathfrak{t}(\mathbb{C})\big)^{c,I} \longrightarrow \mathcal{D}_c(G) \tag{1.1$'$}$$

that sends an element $u \in S\big(\mathfrak{t}(\mathbb{C})\big)^{c,I}$ to the distribution

$$\rho'_u : \ f \longrightarrow f_G(\sigma_{1,u}) = f_G\big(T, \Omega_T, \partial(u)\big), \qquad f \in \mathcal{C}(G).$$

For it is an easy consequence of Harish-Chandra's jump conditions for orbital integrals that there is an isomorphism of the domain of (1.1) to itself whose composition with (1.1) equals $(1.1)'$. It would be enough to show that $(1.1)'$ is an isomorphism.

That the mapping $(1.1)'$ is an isomorphism is implicit in the papers [B1] and [B2] of Bouaziz. In the special case that $c = 1$, the corresponding result for the Lie algebra $\mathfrak{g}(\mathbb{R})$ was proved explicitly [B1, Proposition 6.1.1]. The assertion for $G(\mathbb{R})$, again in the special case that $c = 1$, follows immediately from properties of the exponential map. A standard argument of descent then reduces the general assertion for $G(\mathbb{R})$ to the special case, applied to the group $G_c(\mathbb{R})$. □

If $\rho = \big(T, \Omega_T, \partial(u)\big)$ belongs to $R_c(G)$, we set $\deg(\rho)$ equal to the degree of the homogeneous element $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$. Observe that for any nonnegative integer n, the subset

$$R_{c,n}(G) = \{\rho \in R_c(G) : \ \deg(\rho) \le n\}$$

of $R_c(G)$ is finite. This set is in turn a disjoint union of subsets

$$R_{c,(k)}(G) = \{\rho \in R_c(G) : \ \deg(\rho) = k\}, \qquad 0 \le k \le n.$$

The sets $R_{c,(k)}(G)$ will be used in the next section to construct formal germ expansions of invariant orbital integrals.

Let $\mathcal{Z}(G)$ be the center of the universal enveloping algebra of $\mathfrak{g}(\mathbb{C})$. For any torus $T \in \mathcal{T}_c(G)$, we write

$$h_T : \ \mathcal{Z}(G) \longrightarrow S\big(\mathfrak{t}(\mathbb{C})\big)^{W(G,T)}$$

for the Harish-Chandra isomorphism from $\mathcal{Z}(G)$ onto the space of $W(G,T)$-invariant elements in $S\big(\mathfrak{t}(\mathbb{C})\big)$. We then define an action $\sigma \to z\sigma$ of $\mathcal{Z}(G)$ on $\mathcal{D}_c(G)$ by setting

$$z\rho = \big(T, \Omega_T, \partial\big(h_T(z)\big)u\big), \qquad z \in \mathcal{Z}(G),$$

for any $\rho = \big(T, \Omega_T, \partial(u)\big)$ in the basis $R_c(G)$. It follows immediately from Harish-Chandra's differential equations

$$(zf)_G(\gamma) = \partial\big(h_T(z)\big) f_G(\gamma), \qquad f \in \mathcal{C}(G),\ \gamma \in T_{\mathrm{reg}}(\mathbb{R}), \tag{1.2}$$

for invariant orbital integrals that

$$f_G(z\rho) = (zf)_G(\rho). \tag{1.3}$$

There is no special reason to assume that $R_c(G)$ is stable under the action of $\mathcal{Z}(G)$. However, we do agree to identify any function ϕ on $R_c(G)$ with its linear extension to $\mathcal{D}_c(G)$, in order that the values

$$\phi(z\rho), \qquad z \in \mathcal{Z}(G),\ \rho \in R_c(G),$$

be defined. Moreover, for any $z \in \mathcal{Z}(G)$, we write $\widehat{z}$ for the transpose of the linear operator $\sigma \to z\sigma$ on $\mathcal{D}_c(G)$, relative to the basis $R_c(G)$. In other words,

$$\sum_{\rho \in R_c(G)} \phi(\rho)\psi(\widehat{z}\rho) = \sum_{\rho \in R_c(G)} \phi(z\rho)\psi(\rho), \tag{1.4}$$

for any functions ϕ and ψ of finite support on $R_c(G)$.

We note for future reference that as a $\mathcal{Z}(G)$-module, $\mathcal{D}_c(G)$ is free. To exhibit a free basis, we write $\mathcal{D}_{c,\mathrm{harm}}(G)$ for the finite dimensional subspace of $\mathcal{D}_c(G)$ spanned by triplets $\big(T, \Omega, \partial(u)\big)$ in $S_c(G)$ for which u belongs to the subspace $S_{\mathrm{harm}}\big(\mathfrak{t}(\mathbb{C})\big)$ of harmonic elements in $S\big(\mathfrak{t}(\mathbb{C})\big)$. (Recall that u is *harmonic* if as a polynomial on $\mathfrak{t}(\mathbb{C})^*$, $\partial(u^*)u = 0$ for every element $u^* \in S\big(\mathfrak{t}(\mathbb{C})^*\big)^{W(G,T)}$ with zero constant term.) It can be shown that

$$S\big(\mathfrak{t}(\mathbb{C})\big)^{c,I} = S_{\mathrm{harm}}\big(\mathfrak{t}(\mathbb{C})\big)^{c,I} \otimes S\big(\mathfrak{t}(\mathbb{C})\big)^{W(G,T)},$$

where

$$S_{\mathrm{harm}}\big(\mathfrak{t}(\mathbb{C})\big)^{c,I} = S_{\mathrm{harm}}\big(\mathfrak{t}(\mathbb{C})\big) \cap S\big(\mathfrak{t}(\mathbb{C})\big)^{c,I}.$$

Any linear basis of $\mathcal{D}_{c,\mathrm{harm}}(G)$ is therefore a free basis of $\mathcal{D}_c(G)$ as a $\mathcal{Z}(G)$-module.

The remarks above are of course simple consequences of the isomorphism (1.1). Another implication of (1.1) is the existence of a canonical grading on the vector space $\mathcal{D}_c(G)$. The grading is compatible with the natural filtration on $\mathcal{D}_c(G)$ that is inherited from the underlying filtration on the space

$$\mathcal{I}(G) = \big\{f_G(\gamma) :\ f \in \mathcal{C}(G)\big\}.$$

We shall be a bit more precise about this, in order to review how subsets of $R_c(G)$ are related to Levi subgroups.

By a Levi subgroup M of G, we mean an $\mathbb{R}$-rational Levi component of a parabolic subgroup of G over $\mathbb{R}$. For any such M, we write A_M for the $\mathbb{R}$-split component of the center of M. Then $A_M(\mathbb{R})^0$ is a connnected abelian Lie group, whose Lie algebra can be identified with the real vector space

$$\mathfrak{a}_M = \mathrm{Hom}\big(X(M)_{\mathbb{R}}, \mathbb{R}\big).$$

We write

$$W(M) = W^G(M) = \mathrm{Norm}_G(M)/M$$

for the Weyl group of (G, A_M). We shall follow a standard convention of writing $\mathcal{L}(M) = \mathcal{L}^G(M)$ for the finite set of Levi subgroups of G that contain M, and $\mathcal{L}^0(M)$ for the complement of $\{G\}$ in $\mathcal{L}(M)$. Similarly, $\mathcal{F}(M) = \mathcal{F}^G(M)$ stands for the finite set of parabolic subgroups

$$P = M_P N_P, \qquad M_P \in \mathcal{L}(M),$$

of G over $\mathbb{R}$ that contain M, while

$$\mathcal{P}(M) = \mathcal{P}^G(M) = \big\{P \in \mathcal{P}(M) :\ M_P = M\big\}$$

stands for the subset of parabolic subgroups in $\mathcal{F}(M)$ with Levi component M. Again, $\mathcal{F}^0(M)$ denotes the complement of $\{G\}$ in $\mathcal{F}(M)$.

Suppose that M is a Levi subgroup of G. We write $\Gamma_{G\text{-reg}}(M)$ for the set of classes in $\Gamma_{\text{reg}}(M)$ that are strongly G-regular. There is a canonical map from $\Gamma_{G\text{-reg}}(M)$ to $\Gamma_{\text{reg}}(G)$ on whose fibres the group $W(M)$ acts. The dual restriction map of functions is a linear transformation $\phi_G \to \phi_M$ from $\mathcal{I}(G)$ to $\mathcal{I}(M)$. We define $F^M\mathcal{I}(G)$ to be the space of functions ϕ_G in $\mathcal{I}(G)$ such that $\phi_L = 0$ for every Levi subgroup L of G that does not contain a conjugate of M. If $M = G$, $F^M\mathcal{I}(G)$ is the space $\mathcal{I}_{\text{cusp}}(G)$ of cuspidal functions in $\mathcal{I}(G)$. This space is nonzero if and only if G has maximal torus T over $\mathbb{R}$ that is elliptic, in the sense that $T(\mathbb{R})/A_G(\mathbb{R})$ is compact. Letting M vary, we obtain an order reversing filtration on $\mathcal{I}(G)$ over the partially ordered set of G-conjugacy classes of Levi subgroups. The graded vector space attached to the filtration has M-component equal to the quotient

$$G^M\big(\mathcal{I}(G)\big) = F^M\big(\mathcal{I}(G)\big)/\sum_{L\supsetneq M} F^L\big(\mathcal{I}(G)\big).$$

The map $\phi_G \to \phi_M$ is then an isomorphism from $G^M\big(\mathcal{I}(G)\big)$ onto the space $\mathcal{I}_{\text{cusp}}(M)^{W(M)}$ of $W(M)$-invariant cuspidal functions in $\mathcal{I}(M)$. (See [A6]. The definition of $F^M\big(\mathcal{I}(G)\big)$ was unfortunately stated incorrectly on p. 508 of that paper, as was the definition of the corresponding stable space on p. 510.)

Since the distributions in $\mathcal{D}_c(G)$ factor through the projection $f \to f_G$ of $\mathcal{C}(G)$ onto $\mathcal{I}(G)$, they may be identified with linear forms on $\mathcal{I}(G)$. The decreasing filtration on $\mathcal{I}(G)$ therefore provides an increasing filtration on $\mathcal{D}_c(G)$. To be precise, $F^M\big(\mathcal{D}_c(G)\big)$ is defined to be the subspace of distributions in $\mathcal{D}_c(G)$ that annihilate any of the spaces $F^L\big(\mathcal{I}(G)\big)$ with $L \supsetneq M$. The M-component

$$G^M\big(\mathcal{D}_c(G)\big) = F^M\big(\mathcal{D}_c(G)\big)/\sum_{L\subsetneq M} F^L\big(\mathcal{D}_c(G)\big)$$

of the corresponding graded vector space can of course be zero. It is nonzero if and only if $M(\mathbb{R})$ contains some representative of c, and M_c contains a maximal torus T over $\mathbb{R}$ that is elliptic in M. The correspondence $M \to T$ in fact determines a bijection between the set of nonzero graded components of the filtration of $\mathcal{D}_c(G)$ and the set $\mathcal{T}_c(G)$. Moreover, the mapping (1.1) yields an isomorphism between the associated graded component $S\big(\mathfrak{t}(\mathbb{C})\big)^{c,I}$ and $G^M\big(\mathcal{D}_c(G)\big)$. We therefore obtain an isomorphism

$$\mathcal{D}_c(G) \xrightarrow{\sim} \bigoplus_{\{M\}} G^M\big(\mathcal{D}_c(G)\big), \tag{1.5}$$

where $\{M\} = \{M\}/G$ ranges over conjugacy classes of Levi subgroups of G. The construction does depend on the choice of chambers Ω_T that went into the original definition (1.1), but only up to a sign on each summand in (1.5).

The isomorphism (1.5) gives the grading of $\mathcal{D}_c(G)$. We should point out that there is also a natural grading on the original space $\mathcal{I}(G)$. For the elements f_G in $\mathcal{I}(G)$ can be regarded as functions on the set $\Pi_{\text{temp}}(G)$ of irreducible tempered representations of $G(\mathbb{R})$, rather than the set $\Gamma_{\text{reg}}(G)$. The space of functions on $\Pi_{\text{temp}}(G)$ so obtained has been characterized [A5], and has a natural grading that is compatible with the filtration above. (See [A6, §4] for the related p-adic case.) However, this grading on $\mathcal{I}(G)$ is not compatible with (1.5).

We shall say that an element in $\mathcal{D}_c(G)$ is *elliptic* if it corresponds under the isomorphism (1.5) to an element in the space $G^G\big(\mathcal{D}_c(G)\big)$. We write $\mathcal{D}_{c,\text{ell}}(G)$ for the subspace of elliptic elements in $\mathcal{D}_c(G)$, and we write

$$R_{c,\text{ell}}(G) = R_c(G) \cap \mathcal{D}_{c,\text{ell}}(G)$$

for the associated basis of $\mathcal{D}_{c,\mathrm{ell}}(G)$. For any Levi subgroup M of G, we shall also write $\mathcal{D}_{c,\mathrm{ell}}(M,G)$ for the subspace of distributions in $\mathcal{D}_{c,\mathrm{ell}}(M)$ that are invariant under the action of the finite group $W(M)$. (We can assume that $M(\mathbb{R})$ contains a representative c of the given conjugacy class, since the space is otherwise zero.) The set

$$R_{c,\mathrm{ell}}(M,G) = R_c(G) \cap G^M\big(\mathcal{D}_c(G)\big)$$

can then be identified with a basis of $\mathcal{D}_{c,\mathrm{ell}}(M,G)$. The grading (1.5) gives a decomposition

$$R_c(G) = \coprod_{\{M\}} R_{c,\mathrm{ell}}(M,G)$$

of the basis of $\mathcal{D}_c(G)$.

Suppose, finally, that θ is an $\mathbb{R}$-isomorphism from G to another reductive group $G_1 = \theta G$ over $\mathbb{R}$. Then $c_1 = \theta c$ is a class in $\Gamma_{ss}(G_1)$. For any $f \in \mathcal{C}(G)$, the function

$$(\theta f)(x_1) = f(\theta^{-1}x_1), \qquad x_1 \in G_1(\mathbb{R}),$$

belongs to $\mathcal{C}(G_1)$. The map that sends any $\rho \in \mathcal{D}_c(G)$ to the distribution $\theta\rho$ defined by

$$(\theta f)_G(\theta\rho) = f_G(\rho) \tag{1.6}$$

is an isomorphism from $\mathcal{D}_c(G)$ onto $\mathcal{D}_{c_1}(G_1)$. It of course maps the basis $R_c(G)$ of $\mathcal{D}_c(G)$ to the basis $R_{c_1}(G_1) = \theta R_c(G)$ of $\mathcal{D}_{c_1}(G_1)$.

§2. Invariant germ expansions

Let c be a fixed element in $\Gamma_{ss}(G)$ as in §1. We are going to introduce an asymptotic approximation of the invariant orbital integral $f_G(\gamma)$, for elements γ near c. This will be a foundation for the more elaborate asymptotic expansions of weighted orbital integrals that are the main goal of the paper.

Suppose that V is an open, $G(\mathbb{R})$-invariant neighbourhood of c in $G(\mathbb{R})$. We write

$$\mathcal{I}(V) = \big\{ f_G : \ V_{\mathrm{reg}} \longrightarrow \mathbb{C}, \ f \in \mathcal{C}(G) \big\}$$

for the space of functions on

$$V_{\mathrm{reg}} = V \cap G_{\mathrm{reg}}(\mathbb{R})$$

that are restrictions of functions in $\mathcal{I}(G)$. If $\sigma = (T, \Omega, X)$ belongs to the set $S_c(G)$ defined in §1, the intersection

$$V_\Omega = V_{\mathrm{reg}} \cap \Omega$$

is an open neighbourhood of c in the connected component Ω of $T_{\mathrm{reg}}(\mathbb{R})$. The functions ϕ in $\mathcal{I}(V)$ are smooth on V_Ω, and have the property that the seminorms

$$\|\phi\|_\sigma = \sup_{\gamma \in V_\Omega} |(X\phi)(\gamma)| \tag{2.1}$$

are finite. These seminorms make $\mathcal{I}(V)$ into a topological vector space. To deal with neighbourhoods that vary, it will be convenient to work with the algebraic direct limit

$$\mathcal{I}_c(G) = \varinjlim_{V} \mathcal{I}(V)$$

relative to the restriction maps

$$\mathcal{I}(V_1) \longrightarrow \mathcal{I}(V_2), \qquad V_1 \supset V_2.$$

The elements in $\mathcal{I}_c(G)$ are germs of $G(\mathbb{R})$-invariant, smooth functions on invariant neighbourhoods of c in $G_{\mathrm{reg}}(\mathbb{R})$. (We will ignore the topology on $\mathcal{I}_c(G)$ inherited from the spaces $\mathcal{I}(V)$, since it is not Hausdorff.)

As is customary in working with germs of functions, we shall generally not distinguish in the notation between an element in $\mathcal{I}_c(G)$ and a function in $\mathcal{I}(V)$ that represents it. The open neighbourhood V of c is of course not uniquely determined by original germ. The convention is useful only in describing phenomena that do not depend on the choice of V. It does make sense, for example, for the linear forms ρ in $\mathcal{D}_c(G)$. By Lemma 1.1, ρ factors through the map $f \to f_G$. It can be evaluated at a function in any of the spaces $\mathcal{I}(V)$, and the value taken depends only on the image of the functions in $\mathcal{I}_c(G)$. In other words, the notation $\phi(\rho)$ is independent of whether we treat ϕ as a germ in $\mathcal{I}_c(G)$ or a function in $\mathcal{I}(V)$.

For a given V, Bouaziz characterizes the image of the space $C_c^\infty(V)$ under the mapping $f \to f_G$. He proves that the image is the space of $G(\mathbb{R})$-invariant, smooth functions on V_{reg} that satisfy the conditions $\mathrm{I}_1(G)$-$\mathrm{I}_4(G)$ on pp. 579-580 of [B2, §3]. Assume that the open invariant neighbourhood V of c is sufficiently small. The conditions can then be formulated in terms of triplets (T, Ω, X) in $S_c(G)$. Condition $\mathrm{I}_1(G)$ is simply the finiteness of the seminorm (2.1). Condition $\mathrm{I}_2(G)$ asserts that the singularities of ϕ in $T(\mathbb{R}) \cap V$ that do not come from noncompact imaginary roots are removable. Condition $\mathrm{I}_3(G)$ is Harish-Chandra's relation for the jump of $X\phi(\gamma)$ across any wall of V_Ω defined by a noncompact imaginary root. Condition $\mathrm{I}_4(G)$ asserts that the closure in $T(\mathbb{R}) \cap V$ of the support of ϕ is compact. The theorem of Bouaziz leads directly to a characterization of our space $\mathcal{I}_c(G)$.

Lemma 2.1. *$\mathcal{I}_c(G)$ is the space of germs of invariant, smooth functions $\phi \in C^\infty(V_{\mathrm{reg}})$ that for any $(T,\Omega,X) \in S_c(G)$ satisfy the conditions $\mathrm{I}_1(G) - \mathrm{I}_3(G)$ in [B2, §3].*

Proof. Suppose that ϕ belongs to $\mathcal{I}_c(G)$. Then ϕ has a representative in $\mathcal{I}(V)$, for some open invariant neighbourhood V of c. We can therefore identify ϕ with the restriction to V_{reg} of an orbital integral f_G of some function $f \in \mathcal{C}(G)$. It follows from the analytic results of Harish-Chandra that f_G satisfies the three conditions. (See [H3, Lemma 26] and [H4, Theorem 9.1].)

Conversely, suppose that for some small V, ϕ is an invariant function in $C^\infty(V_{\mathrm{reg}})$ that satisfies the three conditions. In order to accommodate the fourth condition, we modify the support of ϕ. Let $\psi_1 \in C^\infty\big(G(\mathbb{R})\big)$ be a smooth, $G(\mathbb{R})$-invariant function whose support is contained in V, and which equals 1 on some open, invariant neighbourhood $V_1 \subset V$ of c. For example, we can chooose a positive, homogeneous, $G_{c,+}(\mathbb{R})$-invariant polynomial q_c on $\mathfrak{g}_c(\mathbb{R})$ whose zero set equals $c\,\mathcal{U}_1(G_c)$, as in the construction on p. 166 of [B1], together with a function $\alpha_1 \in C_c^\infty(\mathbb{R})$ that is supported on a small neighbourhood of 0, and equals 1 on an even smaller neighbourhood of 0. The function

$$\psi_1(x) = \alpha_1\big(q_c(\log\gamma)\big),$$

defined for any

$$x = y^{-1}c\gamma y, \qquad y \in G(\mathbb{R}),\ \gamma \in G_c(\mathbb{R}),$$

has the required property. Given ψ_1, we set

$$\phi_1(x) = \psi_1(x)\phi(x), \qquad x \in G(\mathbb{R}).$$

The function ϕ_1 then satisfies the support condition $\mathrm{I}_4(G)$ of [B2]. It is not hard to see that ϕ_1 inherits the other three conditions $\mathrm{I}_1(G) - \mathrm{I}_3(G)$ of [B2] from the corresponding conditions on ϕ. It follows from the characterization [B2, Théorème 3.2] that $\phi_1 = f_G$, for some function $f \in C_c^\infty(V)$. Since $C_c^\infty(V)$ is contained in $\mathcal{C}(G)$, and since ϕ takes the same values on $V_{1,\mathrm{reg}}$ as the function $\phi_1 = f_G$, the germ of ϕ coincides with the germ of f_G. In other words, the germ of ϕ lies in the image of $\mathcal{C}(G)$. It therefore belongs to $\mathcal{I}_c(G)$. □

In order to describe the asymptotic series of this paper, it will be convenient to fix a "norm" function that is defined on any small $G(\mathbb{R})$-invariant neighbourhood V of c in $G(\mathbb{R})$. We assume that V is small enough that

(i) any element in V is $G(\mathbb{R})$-conjugate to $T(\mathbb{R})$, for some torus $T \in \mathcal{T}(G)$,

(ii) for any $T \in \mathcal{T}_c(G)$ and any w in the complement of $W_c\big(G(\mathbb{R}), T(\mathbb{R})\big)$ in $W\big(G(\mathbb{R}), T(\mathbb{R})\big)$, the intersection

$$w\big(V \cap T(\mathbb{R})\big) \cap \big(V \cap T(\mathbb{R})\big)$$

is empty, and

(iii) for any $T \in \mathcal{T}_c(G)$, the mapping

$$\gamma \longrightarrow \ell_c(\gamma) = \log(\gamma c^{-1}) \tag{2.2}$$

is a diffeomorphism from $\big(V \cap T(\mathbb{R})\big)$ to an open neighbourhood of zero in $\mathfrak{t}(\mathbb{R})$.

We can of course regard the mapping $\gamma \to \ell_c(\gamma)$ as a coordinate system around the point c in $T(\mathbb{R})$. Let us assume that the Cartan subalgebras $\{\mathfrak{t}(\mathbb{R}) :\ T \in \mathcal{T}_c(G)\}$ are all stable under a fixed Cartan involution θ_c of $\mathfrak{g}_c(\mathbb{R})$. We choose a $G_{c,+}(\mathbb{R})$-invariant bilinear form B on $\mathfrak{g}_c$ such that the quadratic form

$$\|X\|^2 = -B\big(X, \theta_c(X)\big), \qquad X \in \mathfrak{g}_c(\mathbb{R}),$$

is positive definite on $\mathfrak{g}_c(\mathbb{R})$. The function

$$\gamma \longrightarrow \|\ell_c(\gamma)\|,$$

defined a priori for γ in any of the sets $V \cap T(\mathbb{R})$, $T \in \mathcal{T}_c(G)$, then extends to a $G(\mathbb{R})$-invariant function on V. It will be used to describe the estimates implicit in our asymptotic series.

We have noted that the elements in $\mathcal{D}_c(G)$ can be identified with linear forms on the space $\mathcal{I}_c(G)$. Let us write $\mathcal{I}_{c,n}(G)$ for the annihilator in $\mathcal{I}_c(G)$ of the finite subset $R_{c,n}(G)$ of our basis $R_c(G)$ of $\mathcal{D}_c(G)$. It is obvious that

$$\mathcal{I}_{c,n}(G) = \varinjlim_{V} \mathcal{I}_{c,n}(V),$$

where $\mathcal{I}_{c,n}(V)$ is the subspace of $\mathcal{I}(V)$ annihilated by $R_{c,n}(G)$. We can think of $\mathcal{I}_{c,n}(G)$ as the subspace of functions in $\mathcal{I}_c(G)$ that vanish of order at least $(n+1)$ at c. For later use, we also set $\mathcal{C}_{c,n}(G)$ equal to the subspace of $\mathcal{C}(G)$ annihilated by $R_{c,n}(G)$. It is clear that the map $f \to f_G$ takes $\mathcal{C}_{c,n}(G)$ surjectively to $\mathcal{I}_{c,n}(G)$.

Suppose that ϕ is an element in $\mathcal{I}_c(G)$. We can take the Taylor series around c, relative to the coordinates $\ell_c(\gamma)$, of each of the functions

$$\phi(\gamma), \qquad\qquad \gamma \subset V_\Omega,\ T \in \mathcal{T}_c(G),\ \Omega \in \pi_{0,c}\big(T_{\mathrm{reg}}(\mathbb{R})\big),$$

that represent ϕ. For any nonnegative integer k, let $\phi^{(k)}$ be the term in the Taylor series of total degree k. Then $\phi^{(k)}$ can be regarded as an invariant, smooth function in $C^\infty(V_{\mathrm{reg}})$. We claim that it belongs to $\mathcal{I}_c(G)$.

Proposition 2.1 asserts that $\phi^{(k)}$ belongs to $\mathcal{I}_c(G)$ if and only if it satisfies the conditions $\mathrm{I}_1(G) - \mathrm{I}_3(G)$ of [B2, §3]. Condition $\mathrm{I}_1(G)$ is trivial. Conditions $\mathrm{I}_2(G)$ and $\mathrm{I}_3(G)$ are similar, since they both concern the jumps of ϕ about walls in V_Ω, for triplets $(T, \Omega, X) \in S_c(G)$. We shall check only $\mathrm{I}_3(G)$. Suppose that β is a noncompact imaginary root of (G_c, T) that defines a wall of $\Omega = \Omega_+$. Let Ω_- be the complementary component in $T_{\mathrm{reg}}(\mathbb{R})$ that shares this wall. By means of the Cayley transform associated to β, one obtains a second triplet $(T_\beta, \Omega_\beta, X_\beta) \in S_c(G)$ for which Ω_β also shares the given wall of Ω. Condition $I_3(G)$ for ϕ asserts that

$$(X\phi_{\Omega_+})(\gamma) - (X\phi_{\Omega_-})(\gamma) = d(\beta)(X_\beta\phi_{\Omega_\beta})(\gamma), \tag{2.3}$$

for γ on the given wall of Ω. Here, ϕ_{Ω_*} represents the restriction of ϕ to V_{Ω_*}, a smooth function that extends to the closure of V_{Ω_*}, while $d(\beta)$ is independent of ϕ. If X is a homogeneous invariant differential operator on $T_*(\mathbb{R})$ of degree d, and ϕ is homogeneous of degree k (in the coordinates $\ell_c(\gamma)$), then $(X\phi_{\Omega_*})(\gamma)$ is homogeneous of degree $k-d$ if $k \geq d$, and vanishes if $k < d$. The relation (2.3) for ϕ then implies the corresponding relation

$$(X\phi^{(k)}_{\Omega_+})(\gamma) - (X\phi^{(k)}_{\Omega_-})(\gamma) = d(\beta)(X_\beta\phi^{(k)}_{\Omega_\beta})(\gamma)$$

for the homogeneous components $\phi^{(k)}$ of ϕ. This is the condition $\mathrm{I}_3(G)$ for $\phi^{(k)}$. The claim follows.

We set

$$\mathcal{I}_c^{(k)}(G) = \big\{\phi \in \mathcal{I}_c(G) :\ \phi^{(k)} = \phi\big\},$$

for any nonnegative integer k. Suppose that n is another nonnegative integer. Then $\mathcal{I}_c^{(k)}(G)$ is contained in $\mathcal{I}_{c,n}(G)$ if $k > n$, and intersects $\mathcal{I}_{c,n}(G)$ only at 0 if $k \leq n$. It follows from what we have just proved that the quotient

$$\mathcal{I}_c^n(G) = \mathcal{I}_c(G)/\mathcal{I}_{c,n}(G)$$

has a natural grading

$$\mathcal{I}_c^n(G) \cong \bigoplus_{0\leq k\leq n} \mathcal{I}_c^{(k)}(G).$$

But $\mathcal{I}_{c,n}(G)$ is the subspace of $\mathcal{I}_c(G)$ annihilated by the finite subset

$$R_{c,n}(G) = \coprod_{0 \leq k \leq n} R_{c,(k)}(G)$$

of $R_c(G)$. It follows that $R_{c,n}(G)$ is a basis of the dual space of $\mathcal{I}_c^n(G)$, and that $R_{c,(k)}(G)$ is a basis of the dual space of $\mathcal{I}_c^{(k)}(G)$.

Let

$$\{\rho^\vee : \rho \in R_{c,(k)}(G)\}$$

be the basis of $\mathcal{I}_c^{(k)}(G)$ that is dual to $R_{c,(k)}(G)$. If $T \in \mathcal{T}_c(G)$ and $\Omega \in \pi_{0,c}\big(T_{\mathrm{reg}}(\mathbb{R})\big)$, the restriction to V_Ω of any function $\rho^\vee$ in this set is a homogeneous polynomial

$$\gamma \longrightarrow \rho^\vee(\gamma), \qquad \gamma \in V_\Omega,$$

of degree k (in the coordinates $\ell_c(\gamma)$). In particular, $\rho^\vee$ has a canonical extension to the set of regular points in any invariant neighbourhood of V of c on which the coordinate functions (2.2) are defined. Thus, unlike a general element in $\mathcal{I}_c(G)$, $\rho^\vee$ really can be treated as a function, as well as a germ of functions.

The union over k of our bases of $\mathcal{I}_c^{(k)}(G)$ is a family of functions

$$\rho^\vee(\gamma), \qquad \gamma \in V_{\mathrm{reg}},\ \rho \in R_c(G),$$

with properties that are dual to those of $R_c(G)$. For example, the dual of the action (1.3) of $\mathcal{Z}(G)$ on $\mathcal{D}_c(G)$ is a differential equation

$$(\widehat{z}\rho)^\vee = h(z)\rho^\vee, \tag{2.4}$$

for any $z \in \mathcal{Z}(G)$ and $\rho \in R_c(G)$. Here $\widehat{z}$ represents the transpose action (1.4) of $\mathcal{Z}(G)$, and $h(z)$ is the $G(\mathbb{R})$-invariant differential operator on V_{reg} obtained from the various Harish-Chandra maps $z \to h_T(z)$. The dual of (1.6) is the symmetry condition

$$\theta\rho^\vee = (\theta\rho)^\vee, \tag{2.5}$$

for any isomorphism θ: $G \to \theta G$ over $\mathbb{R}$, and any $\rho \in R_c(G)$.

The main reason for defining the functions $\{\rho^\vee\}$ is that they represent germs of invariant orbital integrals. It is clear that

$$\phi^{(k)}(\gamma) = \sum_{\rho \in R_{c,(k)}(G)} \rho^\vee(\gamma)\phi(\rho), \qquad k \geq 0,$$

for any function $\phi \in \mathcal{I}_c(G)$. Suppose that f belongs to $\mathcal{C}(G)$. The Taylor polynomial of degree n attached to the function $f_G(\gamma)$ on V_{reg} (taken relative to the coordinates $\ell_c(\gamma)$) is then equal to the function

$$f_G^n(\gamma) = \sum_{0 \leq k \leq n} f_G^{(k)}(\gamma) = \sum_{\rho \in R_{c,n}(G)} \rho^\vee(\gamma) f_G(\rho). \tag{2.6}$$

It follows from Taylor's theorem that there is a constant C_n for each n such that

$$|f_G(\gamma) - f_G^n(\gamma)| \leq C_n \|\ell_c(\gamma)\|^{n+1},$$

for any $\gamma \in V_{\mathrm{reg}}$. Otherwise said, $f_G(\gamma)$ has an asymptotic expansion

$$\sum_{\rho \in R_c(G)} \rho^\vee(\gamma) f_G(\rho),$$

in the sense that $f_G(\gamma)$ differs from the partial sum $f_G^n(\gamma)$ by a function in the class $O(\|\ell_c(\gamma)\|^{n+1})$.

The main points of Sections 1 and 2 may be summarized as follows. There are invariant distributions

$$f \longrightarrow f_G(\rho), \qquad \rho \in R_c(G),$$

supported on $\mathcal{U}_c(G)$, and homogeneous germs

$$\gamma \longrightarrow \rho^\vee(\gamma), \qquad \rho \in R_c(G),$$

in $\mathcal{I}_c(G)$, which transform according to (1.3) and (2.4) under the action of $\mathcal{Z}(G)$, satisfy the symmetry conditions (1.6) and (2.5), and provide an asymptotic expansion

$$f_G(\gamma) \sim \sum_{\rho \in R_c(G)} \rho^\vee(\gamma) f_G(\rho), \qquad \gamma \in V_{\mathrm{reg}}, \tag{2.7}$$

around c for the invariant orbital integral $f_G(\gamma)$.

It is useful to have a formulation of (2.7) that is uniform in f.

Proposition 2.2. *For any $n \geq -1$, the mapping*

$$f \longrightarrow f_G(\gamma) - f_G^n(\gamma), \qquad f \in \mathcal{C}(G),$$

is a continuous linear transformation from $\mathcal{C}(G)$ to the space $\mathcal{I}_{c,n}(V)$.

Proof. We have interpreted $f_G^n(\gamma)$ as the Taylor polynomial of degree n for the function $f_G(\gamma)$. Since $I_{c,n}(V)$ can be regarded as a closed subspace of functions in $\mathcal{I}(V)$ that vanish of order at least $(n+1)$ at c, the difference $f_G(\gamma) - f_G^n(\gamma)$ belongs to $\mathcal{I}_{c,n}(V)$. The continuity assertion of the lemma follows from the integral formula for the remainder in Taylor's theorem [D, (8.14.3)], and the continuity of the mapping $f \to f_G$. □

Remarks. 1. Proposition 2.2 could of course be formulated as a concrete estimate. Given $n \geq -1$, we simplify the notation by writing

$$(n, X) = \big(n + 1 - \deg(x)\big)_+ = \max\{(n + 1 - \deg X), 0\}, \tag{2.8}$$

for any differential operator X. The proposition asserts that for any $\sigma = (T, \Omega, X)$ in $S_c(G)$, there is a continuous seminorm μ_σ^n on $\mathcal{C}(G)$ such that

$$\big|X\big(f_G(\gamma) - f_G^n(\gamma)\big)\big| \leq \mu_\sigma^n(f) \|\ell_c(\gamma)\|^{(n,X)},$$

for any $\gamma \in V_\Omega$ and $f \in \mathcal{C}(G)$.

2. Invariant orbital integrals can be regarded as distributions that are dual to irreducible characters. In this sense, the asymptotic expansion (2.7) is dual to the character expansions introduced by Barbasch and Vogan near the beginning of [BV].

Our goal is to extend these results for invariant orbital integrals to weighted orbital integrals. As background for this, we observe that much of the discussion of Sections 1 and 2 for G applies to the relative setting of a pair (M, G), for a fixed Levi subgroup M of G. In this context, we take c to be a fixed class in $\Gamma_{ss}(M)$. Then c represents a $W(M)$-orbit in $\Gamma_{ss}(M)$ (or equivalently, the intersection of M with a class in $\Gamma_{ss}(G)$), which we also denote by c. With this understanding, we take V to be a small open neighbourhood of c in $M(\mathbb{R})$ that is invariant under the normalizer

$$W(M)M(\mathbb{R}) = \mathrm{Norm}_{G(\mathbb{R})}\big(M(\mathbb{R})\big)$$

of $M(\mathbb{R})$ in $G(\mathbb{R})$.

Given V, we can of course form the invariant Schwartz space $\mathcal{I}(V)$ for M. If f belongs to $\mathcal{C}(G)$, the relative (invariant) orbital integral f_M around c is the restriction of f_G to the subset

$$V_{G\text{-reg}} = V \cap G_{\mathrm{reg}}(\mathbb{R})$$

of $G_{\mathrm{reg}}(\mathbb{R})$. It is easy to see that $f \to f_M$ is a continuous linear mapping from $\mathcal{C}(G)$ into the closed subspace

$$\mathcal{I}(V, G) = \mathcal{I}(V)^{W(M)}$$

of $W(M)$-invariant functions in $\mathcal{I}(V)$. (We identify functions in $\mathcal{I}(V, G)$ with their restrictions to $V_{G\text{-reg}}$.) Other objects defined earlier have obvious relative analogues. For example, $S_c(M, G)$ denotes the set of triplets (T, Ω, X), where T belongs to the set $\mathcal{T}_c(M)$ (defined for M as in §1), Ω is a connected component in $T_{G\text{-reg}}(\mathbb{R})$ (rather than $T_{M\text{-reg}}(\mathbb{R})$) whose closure contains c, and X is an invariant differential operator on $T(\mathbb{R})$ (as before). The elements in $S_c(M, G)$ yield continuous seminorms (2.1) that determine the topology on $\mathcal{I}(V, G)$. We can also define the direct limits

$$\mathcal{I}_c(M, G) = \varinjlim_{V} \mathcal{I}(V, G)$$

and

$$\mathcal{I}_{c,n}(V, G) = \varinjlim_{V} \mathcal{I}_{c,n}(V, G),$$

where $\mathcal{I}_{c,n}(V, G)$ denotes the subspace of $\mathcal{I}(V, G)$ annihilated by the finite subset $R_{c,n}(M)$ of the basis $R_c(M)$. We shall use these relative objects in §4, when we introduce spaces that are relevant to weighted orbital integrals.

We note that there is also a relative analogue of the space of harmonic distributions introduced in §1. We define the subspace $\mathcal{D}_{c,G\text{-harm}}(M)$ of G-*harmonic* distributions in $\mathcal{D}_c(M)$ be the space spanned by those triplets $\big(T, \Omega, \partial(u)\big)$ in $S_c(M, G)$ such that the element $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$ is harmonic relative to G. Any linear basis of $\mathcal{D}_{c,G\text{-harm}}(M)$ is a free basis of $\mathcal{D}_c(M)$, relative to the natural $\mathcal{Z}(G)$-module structure on $\mathcal{D}_c(M)$. In our construction of certain distributions later in the paper, the elements in $\mathcal{D}_{c,G\text{-harm}}(M)$ will be the primitive objects to deal with.

§3. Weighted orbital integrals

We now fix a maximal compact subgroup K of $G(\mathbb{R})$. We also fix a Levi subgroup M of G such that $\mathfrak{a}_M$ is orthogonal to the Lie algebra of K (with respect to the Killing form on $\mathfrak{g}(\mathbb{R})$). There is then a natural smooth function

$$v_M(x) = \lim_{\lambda \to 0} \left(\sum_{P \in \mathcal{P}(M)} \mathrm{e}^{-\lambda(H_P(x))} \theta_P(\lambda)^{-1} \right)$$

on $M(\mathbb{R})\overline{s}G(\mathbb{R})/K$, defined as the volume of a certain convex hull. This function provides a noninvariant measure on the $G(\mathbb{R})$-conjugacy class of any strongly G-regular point in $M(\mathbb{R})$, relative to which any Schwartz function $f \in \mathcal{C}(G)$ is integrable. The resulting integral

$$J_M(\gamma, f) = J_M^G(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma(\mathbb{R})\overline{s}G(\mathbb{R})} f(x^{-1}\gamma x) v_M(x) dx$$

is a smooth, $M(\mathbb{R})$-invariant function of γ in the set

$$M_{G\text{-reg}}(\mathbb{R}) = M(\mathbb{R}) \cap G_{\mathrm{reg}}(\mathbb{R}).$$

(See [A1, Lemma 8.1] and [A2, §6-7].) We recall a few of its basic properties.

For any γ, the linear form $f \to J_M(\gamma, f)$ is a tempered distribution. In contrast to the earlier special case

$$J_G(\gamma, f) = f_G(\gamma)$$

of $M = G$, however, it is not invariant. Let

$$f^y : \ x \longrightarrow f(yxy^{-1}), \qquad x \in G(\mathbb{R}),$$

be the conjugate of f by a fixed element $y \in G(\mathbb{R})$. The weighted orbital integral of f^y can then be expanded as

$$J_M(\gamma, f^y) = \sum_{Q \in \mathcal{F}(M)} J_M^{M_Q}(\gamma, f_{Q,y}), \tag{3.1}$$

in the notation of [A2, Lemma 8.2]. The summand with $Q = G$ is equal to $J_M(\gamma, f)$. The expansion can therefore be written as an identity

$$J_M(\gamma, f^y - f) = \sum_{Q \in \mathcal{F}^0(M)} J_M^{M_Q}(\gamma, f_{Q,y})$$

that represents the obstruction to the distribution being invariant.

Weighted orbital integrals satisfy a generalization of the differential equations (1.2). If z belongs to $\mathcal{Z}(G)$, the weighted orbital integral of zf has an expansion

$$J_M(\gamma, zf) = \sum_{L \in \mathcal{L}(M)} \partial_M^L(\gamma, z_L) J_L(\gamma, f). \tag{3.2}$$

Here $z \to z_L$ denotes the canonical injective homomorphism from $\mathcal{Z}(G)$ to $\mathcal{Z}(L)$, while $\partial_M^L(\gamma, z_L)$ is an $M(\mathbb{R})$-invariant differential operator on $M(\mathbb{R}) \cap L_{\mathrm{reg}}(\mathbb{R})$ that depends only on L. If T is a maximal torus in $\mathcal{T}_c(M)$, $\partial_M^L(\gamma, z_L)$ restricts to an algebraic differential operator on the algebraic variety $T_{L\text{-reg}}$. Moreover, $\partial_M^L(\gamma, z_L)$ is invariant under the finite group $W^L(M)$ of outer automorphisms of M. We can therefore regard $\partial_M^L(\gamma, z_L)$ as a $W^L(M)M(\mathbb{R})$-invariant, algebraic differential operator on the algebraic variety $M_{G\text{-reg}}$. In the case that $L = M$,

$\partial_M^M(\gamma, z_M)$ reduces to the invariant differential operator $\partial\big(h(z)\big)$ on $M(\mathbb{R})$ obtained from the Harish-Chandra isomorphism. The differential equation (3.2) can therefore be written as an identity

$$J_M(\gamma, zf) - \partial\big(h(z)\big)J_M(\gamma, f) = \sum_{L \neq M} \partial_M^L(\gamma, z_L)J_L(\gamma, f)$$

that is easier to compare with the simpler equations (1.2). (See [A1, Lemma 8.5] and [A3, §11-12].)

Suppose that θ: $G \to \theta G$ is an isomorphism over $\mathbb{R}$, as in §1. We can then take weighted orbital integrals on $(\theta G)(\mathbb{R})$ with respect to θK and θM. They satisfy the relation

$$J_{\theta M}(\theta\gamma, \theta f) = J_M(\gamma, f) \tag{3.3}$$

[A7, Lemma 3.3]. In particular, suppose that $\theta = \mathrm{Int}(w)$, for a representative $w \in K$ of some element in the Weyl group $W(M)$. In this case, $J_M(\gamma, \theta f)$ equals $J_M(\gamma, f)$, and $\theta M = M$, from which it follows that

$$J_M(w\gamma w^{-1}, f) = J_M(\gamma, f).$$

Therefore $J_M(\gamma, f)$ is actually a $W(M)M(\mathbb{R})$-invariant function of γ.

At this point, we fix a class $c \in \Gamma_{ss}(M)$ and an open $W(M)M(\mathbb{R})$-invariant neighbourhood V of c in $M(\mathbb{R})$, as at the end of §2. We can assume that V is small. In particular, we assume that the intersection of V with any maximal torus in $M(\mathbb{R})$ is relatively compact.

We propose to study $J_M(\gamma, f)$ as a function of γ in $V_{G\text{-reg}}$. The behaviour of this function near the boundary is more complicated in general than it is in the invariant case $M = G$. In particular, if (T, Ω, X) lies in the set $S_c(M, G)$ introduced at the end of §2, the restriction of $J_M(\gamma, f)$ to the region

$$V_\Omega = V \cap \Omega$$

does not extend smoothly to the boundary of V_Ω. The function satisfies only the weaker estimate of the following lemma.

Lemma 3.1. *For every triplet $\sigma = (T, \Omega, X)$ in $S_c(M, G)$, there is a positive real number a such that the supremum*

$$\mu_\sigma(f) = \sup_{\gamma \in V_\Omega} \big(|X J_M(\gamma, f)||D_c(\gamma)|^a\big), \qquad f \in \mathcal{C}(G),$$

is a continuous seminorm on $\mathcal{C}(G)$. In the case that $X = 1$, we can take a to be any positive number.

Proof. This lemma is essentially the same as Lemma 13.2 of [A3]. The proof is based on an important technique of Harish-Chandra for estimating invariant orbital integrals [H1, Lemma 48]. We shall recall a part of the argument, in order to persuade ourselves that it remains valid under the minor changes here (where, for example, $\mathcal{C}(G)$ replaces $C_c^\infty\big(G(\mathbb{R})\big)$, and $D_c(\gamma)$ takes the place of $D(\gamma)$), referring the reader to [A3] and [H1] for the remaining part.

We fix the first two components $T \in \mathcal{T}_c(M)$ and $\Omega \in \pi_{0,c}\big(T_{G\text{-reg}}(\mathbb{R})\big)$ of a triplet σ. We require an estimate for every invariant differential operator X that can form a third component of σ. As in Harish-Chandra's treatment of invariant orbital integrals, one studies the general problem in three steps.

The first step is to deal with the identity operator $X = 1$. In this case, the required estimate is a consequence of Lemma 7.2 of [A1]. The lemma cited leads to a bound

$$|J_M(\gamma, f)| \leq \mu(f)\big(1 + L(\gamma)\big)^p, \qquad \gamma \in V_\Omega,$$

in which μ is a continuous seminorm on $\mathcal{C}(G)$. The function $L(\gamma)$ is defined at the bottom of p. 245 of [A1] as a supremum of functions

$$\big|\log\big(|1-\alpha(\gamma)|\big)\big|, \qquad \gamma \in V_\Omega,$$

attached to roots α of (G,T). Since V is assumed to be small, the function attached to α is bounded on V_Ω unless α is a root of (G_c,T). It follows that for any $a_1 > 0$, we can choose a constant C_1 such that

$$\big(1+L(\gamma)\big)^p \le C_1|D_c(\gamma)|^{-a_1}, \qquad \gamma \in V_\Omega.$$

Lemma 7.2 of [A1] therefore implies that

$$f \longrightarrow \sup_{\gamma\in V_\Omega}\big(|J_M(\gamma,f)||D_c(\gamma)|^{a_1}\big), \qquad f \in \mathcal{C}(G), \tag{3.4}$$

is a continuous seminorm on $\mathcal{C}(G)$. The required estimate is thus valid in the case $X = 1$, for any positive exponent $a = a_1$.

The next step concerns the case that X is the image under the Harish-Chandra map of a biinvariant differential operator. That is,

$$X = \partial\big(h_T(z)\big), \qquad z \in \mathcal{Z}(G).$$

In this case, the differential equation (3.2) yields an identity

$$\begin{aligned} XJ_M(\gamma,f) &= \partial\big(h_T(z)\big)J_M(\gamma,f) \\ &= J_M(\gamma,zf) - \sum_{L\supsetneq M}\partial_M^L(\gamma,z_L)J_L(\gamma,f) \end{aligned}$$

for the function we are trying to estimate. We have noted that for each L, $\partial_M^L(\gamma,z_L)$ is an algebraic differential operator on $T_{G\text{-reg}}$. In other words, the coefficients of $\partial_M^L(\gamma,z_L)$ are rational functions on T whose poles lie along singular hypersurfaces of T. Since V is small, any singular hypersurface of T that meets the closure of V_Ω is defined by a root of (G_c,T). It follows that for each L, there is a positive integer k_L such that the differential operator

$$D_c(\gamma)^{k_L}\partial_M^L(\gamma,z_L)$$

has coefficients that are bounded on V_Ω. We can assume inductively that Lemma 3.1 is valid if M is replaced by any $L \supsetneq M$. The estimate of the lemma clearly extends to differential operators with bounded coefficients. We can therefore choose $a_L > 0$ for each such L so that

$$f \longrightarrow \sup_{\gamma\in V_\Omega}\big(|D_c(\gamma)^{k_L}\partial_M^L(\gamma,z_L)J_L(\gamma,f)||D_c(\gamma)|^{a_L}\big)$$

is a continuous seminorm on $\mathcal{C}(G)$. We set a equal to the largest of the numbers $k_L + a_L$. The functional

$$f \longrightarrow \sum_{L\supsetneq M}\sup_{\gamma\in V_\Omega}\big(|\partial_M^L(\gamma,z_L)J_L(\gamma,f)||D_c(\gamma)|^{a}\big)$$

is then a continuous seminorm on $\mathcal{C}(G)$. According to the case (3.4) we have already established,

$$f \longrightarrow \sup_{\gamma\in V_\Omega}\big(|J_M(\gamma,zf)||D_c(\gamma)|^{a}\big)$$

is also a continuous seminorm on $\mathcal{C}(G)$. Applying these estimates to the differential equation for $XJ_M(\gamma, f)$ above, we conclude that

$$f \longrightarrow \sup_{\gamma \in V_\Omega} \big(|XJ_M(\gamma, f)||D_c(\gamma)|^a\big), \qquad f \in \mathcal{C}(G), \tag{3.5}$$

is a continuous seminorm on $\mathcal{C}(G)$. We have established the lemma for X of the form $\partial\big(h_T(z)\big)$.

The last step is to treat a general invariant differential operator X on $T(\mathbb{R})$. This is the main step, and the part of the argument that is based on [H1, Lemma 48]. In the proof of [A3, Lemma 13.2], we explained how to apply Harish-Chandra's technique to the weighted orbitals we are dealing with here. Used in this way, the technique reduces the required estimate for X to the case (3.5) obtained above. It thus establishes the assertion of the lemma for any X, and hence for any triplet σ in $S_c(M, G)$. We refer the reader to [A3] and [H1] for the detailed discussion of this step. □

With Lemma 3.1 as motivation, we now introduce some new spaces of functions. We first attach some spaces to any maximal torus T in M over $\mathbb{R}$ that contains c. Given T, let $\Omega \in \pi_{0,c}\big(T_{G\text{-reg}}(\mathbb{R})\big)$ be a connected component whose closure contains c. Then $V_\Omega = V \cap \Omega$ is an open neighbourhood of c in Ω. If a is a nonnegative real number, we write $F_c^a(V_\Omega, G)$ for the Banach space of continuous functions ϕ_Ω on V_Ω such that the norm

$$\|\phi_\Omega\| = \sup_{\gamma \in V_\Omega} \big(|\phi_\Omega(\gamma)||D_c(\gamma)|^a\big)$$

is finite. More generally, if n is an integer with $n \geq -1$, we define $F_{c,n}^a(V_\Omega, G)$ to be the Banach space of continuous functions ϕ_Ω on V_Ω such that the norm

$$\|\phi_\Omega\|_n = \sup_{\gamma \in V_\Omega} \big(|\phi_\Omega(\gamma)||D_c(\gamma)|^a \|\ell_c(\gamma)\|^{-(n+1)}\big)$$

is finite. The first space $F_c^a(V_\Omega, G)$ is of course the special case that $n = -1$. It consists of functions with specified growth near the boundary.

Lemma 3.1 suggests that we introduce a space of smooth functions on the $W(M)M(\mathbb{R})$-invariant set $V_{G\text{-reg}}$ whose derivatives also have specified growth near the boundary. This entails choosing a function to measure the growth. By a *weight function*, we shall mean an assignment

$$\alpha :\ X \longrightarrow \alpha(X)$$

of a nonnegative real number $\alpha(X)$ to each invariant differential operator X on a maximal torus T of M. We assume that

$$\alpha(X) = \overline{\alpha}(\deg X),$$

for an increasing function $\overline{\alpha}$ on the set of nonnegative integers. The weight function is then defined independently of T.

Suppose that α is a weight function, and that V is as above, an open $W(M)M(\mathbb{R})$-invariant neighbourhood of c in $M(\mathbb{R})$. If ϕ is a function on $V_{G\text{-reg}}$, and $\sigma = (T, \Omega, X)$ is a triplet in the set $S_c(M, G)$ introduced in §2, we shall write ϕ_Ω for the restriction of ϕ to V_Ω. We define $\mathcal{F}_c^\alpha(V, G)$ to be the space of smooth, $W(M)M(\mathbb{R})$-invariant functions ϕ on $V_{G\text{-reg}}$ such that for every $\sigma = (T, \Omega, X)$ in $S_c(M, G)$, and every $\varepsilon > 0$, the derivative $X\phi_\Omega$ belongs to the space $F_c^{\alpha(X)+\varepsilon}(V_\Omega, G)$. More generally, suppose $n \geq -1$ is a given integer. We define $\mathcal{F}_{c,n}^\alpha(V, G)$ to be the subspace of functions ϕ in $\mathcal{F}_c^\alpha(V, G)$ such that for any $\sigma = (T, \Omega, X)$ and ε, $X\phi_\Omega$ belongs to the space

$$F_{c,n,X}^{\alpha(X)+\varepsilon}(V_\Omega, G) = F_{c,(n,X)}^{\alpha(X)+\varepsilon}(V_\Omega, G).$$

We recall here that

$$(n, X) = \max\big\{\big(n + 1 - \deg(X)\big), 0\big\}.$$

The seminorms

$$\|\phi\|_{\sigma,\varepsilon,n} = \|X\phi_\Omega\|_n$$

make $\mathcal{F}^\alpha_{c,n}(V, G)$ into a topological vector space. The original space $\mathcal{F}^\alpha_c(V, G)$ is again the special case that $n = -1$. It is the topological vector space of smooth, $W(M)M(\mathbb{R})$-invariant functions ϕ on $V_{G\text{-reg}}$ such that for every $\sigma = (T, \Omega, X)$ and ε, the seminorm

$$\|\phi\|_{\sigma,\varepsilon} = \sup_{x\in V_\Omega} \big(|(X\phi)(\gamma)||D_c(\gamma)|^{\alpha(X)+\varepsilon}\big)$$

is finite.

Lemma 3.1 is an assertion about the mapping that sends $f \in \mathcal{C}(G)$ to the function $J_M(\gamma, f)$ of $\gamma \in V_{G\text{-reg}}$. It can be reformulated as follows.

Corollary 3.2. *There is a weight function α, with $\alpha(1) = 0$, such that the mapping*

$$f \longrightarrow J_M(\gamma, f), \qquad f \in \mathcal{C}(G),$$

is a continuous linear transformation from $\mathcal{C}(G)$ to $\mathcal{F}^\alpha_c(V, G)$. □

There are some obvious operations that can be performed on the spaces $\mathcal{F}^\alpha_{c,n}(V, G)$. Suppose that α_1 is second weight function, and that $n_1 \geq -1$ is a second integer. The multiplication of functions then provides a continuous bilinear map

$$\mathcal{F}^\alpha_{c,n}(V, G) \times \mathcal{F}^{\alpha_1}_{c,n_1}(V, G) \longrightarrow \mathcal{F}^{\alpha+\alpha_1}_{c,n+n_1+1}(V, G),$$

where $\alpha + \alpha_1$ is the weight function defined by

$$\overline{(\alpha + \alpha_1)}(d_+) = \max_{d+d_1=d_+} \big(\overline{\alpha}(d) + \overline{\alpha}_1(d_1)\big), \qquad d_+ \geq 0.$$

In particular, suppose that q is a $W(G, T)$-invariant rational function on a maximal torus T in M that is regular on $T_{G\text{-reg}}$. Then q extends to a $W(M)M(\mathbb{R})$-invariant function on $V_{G\text{-reg}}$ that lies in $\mathcal{F}^{\alpha_q}_c(V, G)$, for some weight function α_q. The multiplication map $\phi \to q\phi$ therefore sends $\mathcal{F}^\alpha_{c,n}(V, G)$ continuously to $\mathcal{F}^{\alpha+\alpha_q}_{c,n}(V, G)$. A similar observation applies to any (translation) invariant differential operator X on T that is also invariant under the action of $W(G, T)$. For X extends to a $W(M)M(\mathbb{R})$-invariant differential operator on $V_{G\text{-reg}}$, and if $X\alpha$ is the weight function $X' \to \alpha(XX')$, the map $\phi \to X\phi$ sends $\mathcal{F}^\alpha_{c,n}(V, G)$ continuously to $\mathcal{F}^{X\alpha}_{c,n,X}(V, G)$. More generally, suppose that $\partial(\gamma)$ is an algebraic differential operator on $T_{G\text{-reg}}$ that is invariant under $W(G, T)$. Then $\partial(\gamma)$ extends to a $W(M)M(\mathbb{R})$-invariant differential operator on $M_{G\text{-reg}}$. One sees easily that there is a weight function $\partial\alpha$ such that $\phi \to \alpha\phi$ is a continuous mapping from $\mathcal{F}^\alpha_{c,n}(V, G)$ to $\mathcal{F}^{\partial\alpha}_{c,n,\partial}(V, G)$.

§4. Spaces of formal germs

We fix a Levi subgroup M of G, and a class $c \in \Gamma_{ss}(M)$, as before. We again take V to be a small, open, $W(M)M(\mathbb{R})$-invariant neighbourhood of c in $M(\mathbb{R})$. In the last section, we introduced some spaces of functions $\mathcal{F}^{\alpha}_{c,n}(V,G)$ on $V_{G\text{-reg}}$. In this section, we shall examine the behaviour of these spaces under operations of localization and completion.

The space $\mathcal{F}^{\alpha}_c(V,G)$ is a generalization of the relative invariant Schwartz space

$$\mathcal{I}(V,G) = \mathcal{I}(V)^{W(M)}.$$

It is an easy consequence of Lemma 2.1 that for each α, there is a continuous injection

$$\mathcal{I}(V,G) \hookrightarrow \mathcal{F}^{\alpha}_c(V,G).$$

As in the special case of $\mathcal{I}(V,G)$, we can localize the spaces $\mathcal{F}^{\alpha}_c(V,G)$ at c. We form the algebraic direct limit

$$\mathcal{G}^{\alpha}_c(M,G) = \varinjlim_{V} \mathcal{F}^{\alpha}_c(V,G), \tag{4.1}$$

relative to the restriction maps

$$\mathcal{F}^{\alpha}_c(V_1,G) \longrightarrow \mathcal{F}^{\alpha}_c(V_2,G), \qquad V_1 \supset V_2.$$

We shall call $\mathcal{G}^{\alpha}_c(M,G)$ the space of *α-germs* for (M,G) at c. The elements of this space are germs of smooth, $W(M)M(\mathbb{R})$-invariant functions on invariant neighbourhoods of c in $M_{G\text{-reg}}(\mathbb{R})$, with α-bounded growth near the boundary. The space has a decreasing filtration by the subspaces

$$\mathcal{G}^{\alpha}_{c,n}(M,G) = \varinjlim_{V} \mathcal{F}^{\alpha}_{c,n}(V,G), \qquad n \geq -1.$$

Asymptotic series are best formulated in terms of the completion of $\mathcal{G}^{\alpha}_c(M,G)$. For any α, and any $n \geq 0$, the quotient

$$\mathcal{G}^{\alpha,n}_c(M,G) = \mathcal{G}^{\alpha}_c(M,G)/\mathcal{G}^{\alpha}_{c,n}(M,G)$$

is a vector space that is generally infinite dimensional. We call it the space of *(α,n)-jets* for (M,G) at c. The completion of $\mathcal{G}^{\alpha}_c(M,G)$ is then defined as the projective limit

$$\widehat{\mathcal{G}}^{\alpha}_c(M,G) = \varprojlim_{n} \mathcal{G}^{\alpha,n}_c(M,G). \tag{4.2}$$

This space is obviously also isomorphic to a projective limit of quotients

$$\widehat{\mathcal{G}}^{\alpha,n}_c(M,G) = \widehat{\mathcal{G}}^{\alpha}_c(M,G)/\widehat{\mathcal{G}}^{\alpha}_{c,n}(M,G),$$

where $\widehat{\mathcal{G}}^{\alpha}_{c,n}(M,G)$ is the kernel of the projection of $\widehat{\mathcal{G}}^{\alpha}_c(M,G)$ onto $\mathcal{G}^{\alpha,n}_c(M,G)$. We call $\widehat{\mathcal{G}}^{\alpha}_c(M,G)$ the space of *formal α-germs* for (M,G) at c. The final step is to remove the dependence on α. We do so by forming the direct limit

$$\widehat{\mathcal{G}}_c(M,G) = \varinjlim_{\alpha} \widehat{\mathcal{G}}^{\alpha}_c(M,G), \tag{4.3}$$

relative to the natural partial order on the set of weight functions. The operations of multiplication and differentiation above clearly extend to this universal space of formal germs. In particular, any $W(M)M(\mathbb{R})$-invariant, algebraic differential operator $\partial(\gamma)$ on $M_{G\text{-reg}}$ has a linear action $g \to \partial g$ on $\widehat{\mathcal{G}}_c(M,G)$.

As an example, consider the case that $G = M = T$ is a torus. The function $D_c(\gamma)$ is then equal to 1, and the various spaces are independent of α. For each α, $\mathcal{G}_c(T) = \mathcal{G}_c^{\alpha}(M,G)$ is the space of germs of smooth functions on $T(\mathbb{R})$ at c, while $\mathcal{G}_{c,n}(T) = \mathcal{G}_{c,n}^{\alpha}(M,G)$ is the subspace of germs of functions that vanish at c of order at least $(n+1)$. The quotient $\mathcal{G}_c^n(T) = \mathcal{G}_c^{\alpha,n}(M,G)$ is the usual space of n-jets on $T(\mathbb{R})$ at c, while $\widehat{\mathcal{G}}_c(T) = \widehat{\mathcal{G}}_c^{\alpha}(M,G)$ is the space of formal Taylor series (in the coordinates $\ell_c(\gamma)$) at c.

If G is arbitrary, but c is G-regular, the group $T = G_c$ is a torus. In this case, the function $\mathcal{D}_c(\gamma)$ is again trivial. The various spaces reduce to the ones above for T, or rather, the subspaces of the ones above consisting of elements invariant under the finite group $M_{c,+}(\mathbb{R})/M_c(\mathbb{R})$. We are of course mainly interested in the case that c is not G-regular. Then $D_c(\gamma)$ has zeros, and the spaces are more complicated. On the other hand, we can make use of the function D_c in this case to simplify the notation slightly. For example, given α and $\sigma = (T, \Omega, X)$, we can choose a positive number a such that for any $n \geq 0$, $\phi \to X\phi_\Omega$ is a continuous linear map from $\mathcal{F}_{c,n}^{\alpha}(V,G)$ to $F_{c,n}^{a}(V_\Omega, G)$ (rather than $F_{c,n,X}^{a}(V_\Omega, G)$). A similar result applies if X is replaced by an algebraic differential operator on $T_{G\text{-reg}}(\mathbb{R})$.

Lemma 4.1. *For any V, α and n, the map*

$$\mathcal{F}_c^{\alpha}(V,G) \longrightarrow \mathcal{G}_c^{\alpha,n}(M,G)$$

is surjective. In other words, any element g^n in $\mathcal{G}_c^{\alpha,n}(M,G)$ has a representative $g^n(\gamma)$ in $\mathcal{F}_c^{\alpha}(V,G)$.

Proof. Suppose that g^n belongs to $\mathcal{G}_c^{\alpha,n}(M,G)$. By definition, g^n has a representative $g_0^n(\gamma)$ in $\mathcal{F}_c^{\alpha}(V_0,G)$, for some $W(M)M(\mathbb{R})$-invariant neighbourhood V_0 of c in $M(\mathbb{R})$ with $V_0 \subset V$. Let ψ_0 be a smooth, compactly supported, $W(M)M(\mathbb{R})$-invariant function on V_0 that equals 1 on some neighbourhood of c. The product

$$g^n(\gamma) = \psi_0(\gamma) g_0^n(\gamma)$$

then extends by 0 to a function on V that lies in $\mathcal{F}_c^{\alpha}(V,G)$. On the other hand, both $g^n(\gamma)$ and $g_0^n(\gamma)$ represent the same germ in $\mathcal{G}_c(M,G)$. They both therefore have the same image g^n in $\mathcal{G}_c^{\alpha,n}(M,G)$. The function $g^n(\gamma)$ is the required representative. □

Lemma 4.2. *For any weight function α, the canonical map from $\widehat{\mathcal{G}}_c^{\alpha}(M,G)$ to $\widehat{\mathcal{G}}_c(M,G)$ is injective.*

Proof. It is enough to show that if α' is a weight function with $\alpha' \geq \alpha$, the map from $\widehat{\mathcal{G}}_c^{\alpha}(M,G)$ to $\widehat{\mathcal{G}}_c^{\alpha'}(M,G)$ is injective. Suppose that g is an element in $\widehat{\mathcal{G}}_c^{\alpha}(M,G)$ that maps to 0 in $\widehat{\mathcal{G}}_c^{\alpha'}(M,G)$. To show that $g = 0$, it would be enough to establish that for any $n \geq 0$, the image g^n of g in $\mathcal{G}_c^{\alpha,n}(M,G)$ equals 0.

Fix n, and let $g^n(\gamma)$ be a representative of g^n in the space of functions $\mathcal{F}_c^{\alpha}(V,G)$ attached to some V. We have to show that $g^n(\gamma)$ lies in $\mathcal{F}_{c,n}^{\alpha}(V,G)$. In other words, we must show that for any $\sigma = (T,\Omega,X)$ in $S_c(M,G)$, and any $\varepsilon > 0$, the derivative $(Xg_\Omega^n)(\gamma)$ lies in $F_{c,n,X}^{\alpha(X)+\varepsilon}(V_\Omega,G)$. This condition is of course independent of the choice of representative g^n. Given σ, we are free to assume that $g^n(\gamma)$ represents the image g^m of g in $\mathcal{G}_c^{\alpha,m}(V,G)$, for some large integer $m > n + \deg X$. Since g^m maps to zero in $\mathcal{G}_c^{\alpha',m}(M,G)$, $g^n(\gamma)$ lies in $\mathcal{F}_{c,m}^{\alpha'}(V,G)$. In other words, $(Xg_\Omega^n)(\gamma)$ lies in $F_{c,n'}^{\alpha'(X)+\varepsilon'}(V_\Omega,G)$, for the large integer $n' = m - \deg(X)$ and for any $\varepsilon' > 0$. But $(Xg_\Omega^n)(\gamma)$ also lies in $F_c^{\alpha(X)+\varepsilon'}(V_\Omega,G)$. We shall apply these two conditions successively to two subsets of V_Ω.

Given $\varepsilon > 0$, we choose $\varepsilon' > 0$ with $\varepsilon' < \varepsilon$. We then write $\delta = \varepsilon - \varepsilon'$, $a = \alpha(X) + \varepsilon$, and $a' = \alpha'(X) + \varepsilon'$. The two conditions amount to two inequalities

$$|(Xg^n_\Omega)(\gamma)| \leq C'|D_c(\gamma)|^{-a'}\|\ell_c(\gamma)\|^{n'}, \qquad \gamma \in V_\Omega,$$

and

$$|(Xg^n_\Omega)(\gamma)| \leq C_\delta |D_c(\gamma)|^{-(\alpha(X)+\varepsilon')} = C_\delta |D_c(\gamma)|^{-a}|D_c(\gamma)|^{\delta}, \qquad \gamma \in V_\Omega,$$

for fixed constants C' and C_δ. We can assume that $a' > a$, since there would otherwise be nothing to prove. (The functions $|D_c(\gamma)|$ and $\|\ell_c(\gamma)\|$ are of course bounded on V_Ω.) We apply the first inequality to the points γ in the subset

$$V\big(\delta, (n, X)\big) = \big\{\gamma \in V_\Omega : \ \|\ell_c(\gamma)\|^{(n,X)} \leq |D_c(\gamma)|^{\delta}\big\}$$

of V_Ω, and the second inequality to each γ in the complementary subset. We thereby deduce that if n' is sufficiently large, there is a constant C such that

$$|(Xg_\Omega)(\gamma)| \leq C|D_c(\gamma)|^{-a}\|\ell_c(\gamma)\|^{(n,X)},$$

for any point γ in V_Ω. In other words, $|(Xg_\Omega)(\gamma)|$ belongs to the space $F^{\alpha(X)+\varepsilon}_{c,n,X}(V_\Omega, G)$. It follows that the vector g^n in $\mathcal{G}^{\alpha,n}_c(M,G)$ vanishes. Since n was arbitrary, the original element g in $\widehat{\mathcal{G}}^{\alpha}_c(M,G)$ vanishes. The map from $\widehat{\mathcal{G}}^{\alpha}_c(M,G)$ to $\widehat{\mathcal{G}}^{\alpha'}_c(M,G)$ is therefore injective. □

The lemma asserts that $\widehat{\mathcal{G}}_c(M,G)$ is the union over all weight functions α of the spaces $\widehat{\mathcal{G}}^{\alpha}_c(M,G)$. Suppose that we are given a formal germ $g \in \mathcal{G}_c(M,G)$, and a positive integer n. We shall write $g^n = g^{\alpha,n}$ for the image of g in the quotient $\mathcal{G}^{\alpha,n}_c(M,G)$, for some fixed α such that $\mathcal{G}^{\alpha}_c(M,G)$ contains g. The choice of α will generally be immaterial to the operations we perform on g^n, so its omission from the notation is quite harmless. If $\phi(\gamma)$ is a function in one of the spaces $\mathcal{F}^{\alpha}_c(V,G)$, we shall sometimes denote the image of $\phi(\gamma)$ in $\widehat{\mathcal{G}}_c(M,G)$ simply by ϕ. This being the case, ϕ^n then stands for an element in $\mathcal{G}^{\alpha,n}_c(M,G)$. This element is of course equal to the projection of the original function $\phi(\gamma)$ onto $\mathcal{G}^{\alpha,n}_c(M,G)$.

We shall need to refer to two different topologies on $\widehat{\mathcal{G}}_c(M,G)$. The first comes from the discrete topology on each of the quotients $\mathcal{G}^{\alpha,n}_c(M,G)$. The corresponding projective limit topology over n, followed by the direct limit topology over α, yields what we call the *adic* topology on $\widehat{\mathcal{G}}_c(M,G)$. This is the usual topology assigned to a completion. A sequence (g_k) converges in the adic topology if there is an α such that each g_k is contained in $\widehat{\mathcal{G}}^{\alpha}_c(M,G)$, and if for any n, the image g^n_k of g_k in $\mathcal{G}^{\alpha,n}_c(M,G)$ is independent of k, for all k sufficiently large.

To describe the second topology, we recall that the quotient spaces $\mathcal{G}^{\alpha,n}_c(M,G)$ are generally infinite dimensional. As an abstract vector space over $\mathbb{C}$, however, each $\mathcal{G}^{\alpha,n}_c(M,G)$ is a direct limit of finite dimensional spaces. The standard topologies on these finite dimensional spaces therefore induce a direct limit topology on each $\mathcal{G}^{\alpha,n}_c(M,G)$. The corresponding projective limit topology over n, followed by the direct limit topology over α, yields what we call the *complex* topology on $\widehat{\mathcal{G}}_c(M,G)$. This is the appropriate topology for describing the continuity properties of maps from some space into $\widehat{\mathcal{G}}_c(M,G)$. A sequence (g_k) converges in the complex topology of $\widehat{\mathcal{G}}_c(M,G)$ if there is an α such that each g_k is contained in $\widehat{\mathcal{G}}^{\alpha}_c(M,G)$, and if for each n, the sequence g^n_k is contained in a finite dimensional subspace of $\mathcal{G}^{\alpha,n}_c(M,G)$, and converges in the standard topology of that space. Unless otherwise stated, any limit in $\widehat{\mathcal{G}}_c(M,G)$ will be understood to be in the adic topology, while any assertion of continuity for a $\widehat{\mathcal{G}}_c(M,G)$-valued function will refer to the complex topology.

Suppose that g lies in $\widehat{\mathcal{G}}_c(M,G)$. We have agreed to write g^n for the image of g in the quotient $\mathcal{G}^{\alpha,n}_c(M,G)$ of $\mathcal{G}^{\alpha}_c(M,G)$. Here n is any nonnegative integer, and α is a fixed weight function such that g lies in $\widehat{\mathcal{G}}^{\alpha}_c(M,G)$. We shall also write $g^n(\gamma)$ for an α-germ of functions in $\mathcal{G}_c^\alpha(M,G)$ that represents g^n, or as in Lemma 4.1, a function in $\mathcal{F}_c^\alpha(V,G)$ that represents the α-germ. The function $g^n(\gamma)$ is of course not uniquely determined by g. To see that this does not really matter, we recall that under the previous convention, g^n also denotes the image of $g^n(\gamma)$ in $\widehat{\mathcal{G}}_c(M,G)$. We are therefore allowing g^n to stand for two objects: an element in $\mathcal{G}_c^{\alpha,n}(M,G)$ that is uniquely determined by g (once α is chosen), and some representative of this element in $\widehat{\mathcal{G}}_c^\alpha(M,G)$ that is not uniquely determined, and that in particular, need not map to g. With this second interpretation, however, the elements g^n do have the property that

$$g = \lim_{n\to\infty}(g^n).$$

These remarks can be phrased in terms of asymptotic series. Suppose that $g_k(\gamma)$ is a sequence of functions in $\mathcal{F}_c^\alpha(V,G)$ such that the corresponding elements $g_k \in \widehat{\mathcal{G}}_c(M,G)$ converge to zero (in the adic topology). In other words, for any n all but finitely many of the functions $g_k(\gamma)$ lie in the space $\mathcal{F}_{c,n}^\alpha(V,G)$. We shall denote the associated asymptotic series by

$$g(\gamma) \sim \sum_k g_k(\gamma), \tag{4.4}$$

where g is the element in $\widehat{\mathcal{G}}_c(M,G)$ such that

$$g = \sum_k g_k \tag{4.5}$$

(in the adic topology). Conversely, any formal germ can be represented in this way. For if g belong to $\widehat{\mathcal{G}}_c(M,G)$, the difference

$$g^{(n)}(\gamma) = g^n(\gamma) - g^{n-1}(\gamma), \qquad n \geq 0,\ g^{-1}(\gamma) = 0,$$

stands for a function in a space $\mathcal{F}_{c,n-1}^\alpha(V,G)$. Therefore

$$g = \sum_{n=0}^{\infty} g^{(n)},$$

so we can represent g by the asymptotic series

$$g(x) \sim \sum_{n=0}^{\infty} g^{(n)}(x).$$

We shall use the notation (4.4) also to denote a convergent sum of asymptotic series. In this more general usage, the terms in (4.4) stand for asymptotic series $g_k(\gamma)$ and $g(\gamma)$, which in turn represent elements g_k and g in $\widehat{\mathcal{G}}_c(M,G)$ that satisfy (4.5).

As a link between the (relative) invariant Schwartz space and the general spaces above, we consider a space $\mathcal{F}_c^{bd}(V,G)$ of bounded functions. For any integer $n \geq -1$, let $\mathcal{F}_{c,n}^{bd}(V,G)$ be the space of smooth, $W(M)M(\mathbb{R})$-invariant functions ϕ on $V_{G\text{-reg}}$ such that for each $\sigma = (T,\Omega,X)$ in $S_c(M,G)$, the derivative $X\phi_\Omega$ belongs to the space $F_{c,n}^0(V,G)$. If α is any weight function, $\mathcal{F}_{c,n}^{bd}(V,G)$ is contained in $\mathcal{F}_{c,n}^\alpha(V,G)$. In fact in the basic case of $n=-1$, the space

$$\mathcal{F}_c^{bd}(V,G) = \mathcal{F}_{c,-1}^{bd}(V,G)$$

is just the subspace of functions in $\mathcal{F}_c^\alpha(V,G)$ whose derivatives are all bounded. As above, we form the localizations

$$\mathcal{G}_{c,n}^{bd}(M,G) = \varinjlim_V \mathcal{F}_{c,n}^{bd}(V,G),$$

the quotients

$$\mathcal{G}_c^{bd,n}(M,G) = \mathcal{G}_c^{bd}(M,G)/\mathcal{G}_{c,n}^{bd}(M,G) = \mathcal{G}_{c,-1}^{bd}(M,G)/\mathcal{G}_{c,n}^{bd}(M,G)$$

and the completion

$$\widehat{\mathcal{G}}_c^{bd}(M,G) = \varprojlim_n \mathcal{G}_c^{bd,n}(M,G).$$

Lemma 4.3. *Suppose that α is a weight function with $\alpha(1) = 0$. Then for any nonnegative integer n, the canonical mapping*

$$\mathcal{G}_c^{bd,n}(M,G) \longrightarrow \mathcal{G}_c^{\alpha,n}(M,G)$$

is injective.

Proof. By Lemma 4.1, there is a canonical isomorphism

$$\mathcal{G}_c^{\alpha,n}(M,G) \cong \mathcal{F}_c^{\alpha}(V,G)/\mathcal{F}_{c,n}^{\alpha}(V,G).$$

On the other hand, any element in $\mathcal{G}_c^{bd,n}(M,G)$ can be identified with a family

$$\big\{\phi_\Omega^n :\ T \in \mathcal{T}_c(M),\ \Omega \in \pi_{0,c}\big(T_{G\text{-reg}}(\mathbb{R})\big)\big\}$$

of Taylor polynomials of degree n (in the coordinates $\ell_c(\gamma)$). This is because the Ω-component of any function in $\mathcal{F}_c^{bd}(V,G)$ extends to a smooth function on the closure of V_Ω. In particular, each element in $\mathcal{G}_c^{bd,n}(M,G)$ has a canonical representative in $\mathcal{F}_c^{bd}(V,G)$, which of course also lies in $\mathcal{F}_c^{\alpha}(V,G)$. With this interpretation, we consider a function ϕ in the intersection

$$\mathcal{G}_c^{bd,n}(M,G) \cap \mathcal{F}_{c,n}^{\alpha}(V,G).$$

We have only to show that ϕ vanishes.

Suppose that $T \in \mathcal{T}_c(M)$ and $\Omega \in \pi_{0,c}\big(T_{G\text{-reg}}(\mathbb{R})\big)$. As an element in $\mathcal{F}_{c,n}^{\alpha}(V,G)$, ϕ_Ω satisfies a bound

$$\sup_{\gamma \in V_\Omega} \big(|\phi_\Omega(\gamma)||D_c(\gamma)|^{\varepsilon}\|\ell_c(\gamma)\|^{-(n+1)}\big) < \infty,$$

for any $\varepsilon > 0$. As an element in $\mathcal{G}_c^{bd,n}(M,G)$, $\phi_\Omega = \phi_\Omega^n$ is a polynomial (in the coordinates $\ell_c(\gamma)$) of degree less than $(n+1)$. Taking ε to be close to zero, we see that no such polynomial can satisfy the bound unless it vanishes. It follows that $\phi_\Omega = 0$. We conclude that that the function ϕ vanishes, and hence, that the original map is injective. □

Corollary 4.4. *For any weight function α, the canonical mapping*

$$\widehat{\mathcal{G}}_c^{bd}(M,G) \longrightarrow \widehat{\mathcal{G}}_c^{\alpha}(M,G)$$

is injective.

Proof. Given α, we choose a weight function $\alpha_0 \leq \alpha$ with $\alpha_0(1) = 0$. The lemma implies that $\widehat{\mathcal{G}}_c^{bd}(M,G)$ maps injectively into $\widehat{\mathcal{G}}_c^{\alpha_0}(M,G)$, while Lemma 4.2 tells us that $\widehat{\mathcal{G}}_c^{\alpha_0}(M,G)$ maps injectively into $\widehat{\mathcal{G}}_c^{\alpha}(M,G)$. The corollary follows. □

Remark. It is not hard to show that if the weight function α is bounded, the injection of Corollary 4.4 is actually an isomorphism. The completion $\widehat{\mathcal{G}}_c^{bd}(M,G)$ is therefore included among the general spaces defined earlier.

The (relative) invariant Schwartz space $\mathcal{I}(V,G)$ is the closed subspace of functions in $\mathcal{F}_c^{bd}(V,G)$ that satisfy the Harish-Chandra jump conditions. Its localization $\mathcal{I}_c(M,G)$ is therefore a subspace of $\mathcal{G}_c^{bd}(M,G)$. Recall that for any n, $\mathcal{I}_{c,n}(M,G)$ is the subspace of $\mathcal{I}_c(M,G)$ annihilated by the finite set of distributions $R_{c,n}(M)$. It follows easily from the discussion of §2 that

$$\mathcal{I}_{c,n}(M,G) = \mathcal{I}_c(M,G) \cap \mathcal{G}_{c,n}^{bd}(M,G).$$

The quotient

$$\mathcal{I}_c^n(M,G) = \mathcal{I}_c(M,G)/\mathcal{I}_{c,n}(M,G)$$

of $\mathcal{I}_c(M,G)$ therefore injects into the quotient $\mathcal{G}_c^{bd,n}(M,G)$ of $\mathcal{G}_c^{bd}(M,G)$. This in turn implies that the completion

$$\widehat{\mathcal{I}}_c(M,G) = \varprojlim_n \mathcal{I}_c^n(M,G) \tag{4.6}$$

injects into $\widehat{\mathcal{G}}_c^{bd}(M,G)$. We thus have embeddings

$$\widehat{\mathcal{I}}_c(M,G) \subset \widehat{\mathcal{G}}_c^{bd}(M,G) \subset \widehat{\mathcal{G}}_c^{\alpha}(M,G) \subset \widehat{\mathcal{G}}_c(M,G),$$

for any weight function α

As a subspace of $\widehat{\mathcal{G}}_c(M,G)$, the completion $\widehat{\mathcal{I}}_c(M,G)$ is particularly suited to the conventions above. If g belongs to $\widehat{\mathcal{I}}_c(M,G)$ and $n \geq 0$, we take $g^n(\gamma)$ to be the *canonical* representative of g^n in $\mathcal{I}_c(M,G)$ that is spanned by the finite set $\{\rho^\vee(\gamma) : \rho \in R_{c,n}(M)\}$. This means that $g^{(n)}(\gamma)$ is the canonical element in $\mathcal{I}_{c,n-1}(M,G)$ that is spanned by the set $\{\rho^\vee(\gamma) : \rho \in R_{c,(n)}(M)\}$. The formal germ g can therefore be represented by a canonical, adically convergent series

$$g = \sum_{\rho \in R_c(M)} g(\rho)\rho^\vee,$$

or if one prefers, a canonical asymptotic expansion

$$g(\gamma) \sim \sum_{\rho \in R_c(M)} g(\rho)\rho^\vee(\gamma),$$

for uniquely determined coefficients $g(\rho)$ in $\mathbb{C}$. In particular, suppose that g equals f_M, for a Schwartz function $f \in \mathcal{C}(G)$. The relative invariant orbital integral $f_M(\gamma)$ then has an asymptotic expansion

$$f_M(\gamma) \sim \sum_{\rho \in R_c(M)} f_M(\rho)\rho^\vee(\gamma).$$

We end this section by remarking that the $W(M)$-invariance we have built into the definitions is not essential. Its purpose is only to reflect the corresponding property for weighted orbital integrals. We shall sometimes encounter formal germs for which the property is absent (notably as individual terms in a finite sum that is $W(M)$-invariant). There is no general need for extra notation. However, one case of special interest arises when M_1 is a Levi subgroup of M, and c is the image of a class c_1 in $\Gamma_{ss}(M_1)$. Under these conditions, we let $\widehat{\mathcal{G}}_{c_1}(M_1 \mid M,G)$ denote the space of formal germs for (M_1,G) at c_1, defined as above, but with $W(M_1)$ replaced by the stabilizer $W(M_1 \mid M)$ of M in $W(M_1)$. There is then a canonical restriction mapping

$$g \longrightarrow g_{M_1}, \qquad g \in \widehat{\mathcal{G}}_c(M,G),$$

from $\widehat{\mathcal{G}}_c(M,G)$ to $\widehat{\mathcal{G}}_{c_1}(M_1 \mid M,G)$.

§5. Statement of the general germ expansions

In §2, we introduced asymptotic expansions for the invariant orbital integrals $J_G(\gamma, f) = f_G(\gamma)$. Our goal is now to establish formal germ expansions for the more general weighted orbital integrals $J_M(\gamma, f)$. We shall state the general expansions in this section. The proof of the expansions will then take up much of the remaining part of the paper.

Recall that the weighted orbital integrals depend on a choice of maximal compact subgroup $K \subset G(\mathbb{R})$, as well as the Levi subgroup M. The formal germ expansions will of course also depend on a fixed element $c \in \Gamma_{ss}(M)$. The theorem we are about to state asserts the existence of two families of objects attached to the 4-tuple (G, K, M, c), which depend also on bases

$$R_c(L) \subset \mathcal{D}_c(L), \qquad L \in \mathcal{L}(M),$$

chosen as in Lemma 1.1.

The first family is a collection of tempered distributions

$$f \longrightarrow J_L(\rho, f), \qquad L \in \mathcal{L}(M),\ \rho \in R_c(L), \tag{5.1}$$

on $G(\mathbb{R})$, which reduce to the invariant distributions

$$J_G(\rho, f) = f_G(\rho), \qquad \rho \in R_c(G),$$

when $L = G$, and in general are supported on the closed, $G(\mathbb{R})$-invariant subset $\mathcal{U}_c(G)$ of $G(\mathbb{R})$. The second family is a collection of formal germs

$$g_M^L(\rho), \qquad L \in \mathcal{L}(M),\ \rho \in R_c(L), \tag{5.2}$$

in $\widehat{\mathcal{G}}_c(M, L)$, which reduce to the homogeneous germs

$$g_M^M(\rho) = \rho^\vee, \qquad \rho \in R_c(M),$$

when $L = M$, and in general have the convergence property

$$\lim_{\rho \to \infty} \big(g_M^L(\rho)\big) = 0. \tag{5.3}$$

This implies that the series

$$g_M^L\big(J_{L,c}(f)\big) = \sum_{\rho \in R_c(L)} g_M^L(\rho) J_L(\rho, f)$$

converges in (the adic topology of) $\widehat{\mathcal{G}}_c(M, L)$, for any $f \in \mathcal{C}(G)$. The continuity of the linear forms (5.1) also implies that the mapping

$$f \longrightarrow g_M^L\big(J_{L,c}(f)\big)$$

from $\mathcal{C}(G)$ to $\widehat{\mathcal{G}}_c(M, L)$ is continuous (in the complex topology of $\widehat{\mathcal{G}}_c(M, L)$.) The objects will also have a functorial property, which can be formulated as an assertion that for any L and f,

$$g_M^L\big(J_{L,c}(f)\big) \text{ is independent of the choice of basis } R_c(L). \tag{5.4}$$

The two families of objects will have other properties, which are parallel to those of weighted orbital integrals. If y lies in $G(\mathbb{R})$, the distributions (5.1) are to satisfy

$$J_L(\rho, f^y) = \sum_{Q\in\mathcal{F}(L)} J_L^{M_Q}(\rho, f_{Q,y}), \qquad f \in \mathcal{C}(G). \tag{5.5}$$

If z belongs to $\mathcal{Z}(G)$, we require that

$$J_L(\rho, zf) = J_L(z_L\rho, f) \tag{5.6}$$

and

$$g_M^L(\widehat{z}_L\rho) = \sum_{S\in\mathcal{L}^L(M)} \partial_M^S(z_S)g_S^L(\rho)_M. \tag{5.7}$$

Finally, suppose that θ: $G \to \theta G$ is an isomorphism over $\mathbb{R}$. The two families of objects are then required to satisfy the symmetry conditions

$$J_{\theta L}(\theta\rho, \theta f) = J_L(\rho, f) \tag{5.8}$$

and

$$g_{\theta M}^{\theta L}(\theta\rho) = \theta g_M^L(\rho), \tag{5.9}$$

relative to the basis $R_{\theta c}(\theta L) = \theta R_c(L)$ of $\mathcal{D}_{\theta c}(\theta L)$.

Given objects (5.1) and (5.2), consider the sum

$$g_{M,c}(f) = \sum_{L\in\mathcal{L}(M)} g_M^L\big(J_{L,c}(f)\big).$$

Then $g_{M,c}$ is, a priori, a continuous map from $\mathcal{C}(G)$ to a space of formal germs that lack the property of symmetry by $W(M)$. However, suppose that $\theta = \mathrm{Int}(w)$, for a representative $w \in K$ of some element in the Weyl group $W(M)$. Then

$$\begin{aligned}\theta g_{M,c}(f) &= \sum_L \sum_\rho \theta g_M^L(\rho) \cdot J_L(\rho, f)\\ &= \sum_L \sum_\rho g_{\theta M}^{\theta L}(\theta\rho) J_{\theta L}(\theta\rho, \theta f),\end{aligned}$$

by (5.8) and (5.9). Since θM equals M and $J_{\theta L}(\theta\rho, \theta f)$ equals $J_{\theta L}(\theta\rho, f)$, we obtain

$$\begin{aligned}\theta g_{M,c}(f) &= \sum_L \sum_\rho g_M^{\theta L}(\theta\rho) J_{\theta L}(\theta\rho, f)\\ &= \sum_L \sum_\rho g_M^L(\rho) J_L(\rho, f) = g_{M,c}(f),\end{aligned}$$

from the condition (5.4). It follows that $g_{M,c}(f)$ is symmetric under $W(M)$, and therefore that $g_{M,c}(f)$ lies in the space $\widehat{\mathcal{G}}_c(M, G)$. In other words, $g_{M,c}$ can be regarded as a continuous linear map from $\mathcal{C}(G)$ to $\widehat{\mathcal{G}}_c(M, G)$.

Theorem 5.1. *There are distributions (5.1) and formal germs (5.2) such that the conditions (5.3)–(5.9) hold, and such that for any $f \in \mathcal{C}(G)$, the weighted orbital integral $J_M(f)$ has a formal germ expansion given by the sum*

$$\sum_{L \in \mathcal{L}(M)} g_M^L\big(J_{L,c}(f)\big) = \sum_{L \in \mathcal{L}(M)} \sum_{\rho \in R_c(L)} g_M^L(\rho) J_L(\rho, f). \tag{5.10}$$

Theorem 5.1 asserts that the sum (5.10) represents the same element in $\widehat{\mathcal{G}}_c(M, G)$ as $J_M(f)$. In other words, the weighted orbital integral has an asymptotic expansion

$$J_M(\gamma, f) \sim \sum_L \sum_\rho g_M^L(\gamma, \rho) J_L(\rho, f).$$

This is the archimedean analogue of the germ expansion for weighted orbital integrals on a p-adic group ([A3], [A8]). We should note that the formal germs $g_M^L(\rho)$ are more complicated in general than in the special case of $L = M = G$ treated in §2. For example, if $M = G$, the formal germs can be identified with homogeneous functions $g_G^G(\gamma, \rho) = \rho^\vee(\gamma)$. In the general case, each $g_M^L(\gamma, \rho)$ does have to be treated as an asymptotic series.

The functorial condition (5.4) seems entirely natural in the light of the main assertion of Theorem 5.1. We observe that (5.4) amounts to a requirement that the individual objects (5.1) and (5.2) be functorial in ρ. More precisely, suppose that for each L, $R'_c(L) = \{\rho'\}$ is a second basis of $\mathcal{D}_c(L)$. The condition (5.4) is equivalent to the transformation formulas

$$J_L(\rho', f) = \sum_\rho a_L(\rho', \rho) J_L(\rho, f) \tag{5.4.1}$$

and

$$g_M^L(\rho') = \sum_\rho a_L^\vee(\rho', \rho) g_M^L(\rho), \tag{5.4.2}$$

where $A_L = \{a_L(\rho', \rho)\}$ is the transformation matrix for the bases $\{\rho'\}$ and $\{\rho\}$, and $A_L^\vee = \{a_L^\vee(\rho', \rho)\} = {}^t A_L^{-1}$ is the transformation matrix for the dual bases $\{(\rho')^\vee\}$ and $\{\rho^\vee\}$. In the special case that $M = G$, these formulas are consequences of the constructions in §1 and §2 (as is (5.4)). In general, they follow inductively from this special case and the condition (5.4) (with L taken to be either M or G). The two formulas tell us that for any L, the two families of objects are functorial in the following sense. The distributions (5.1) are given by a mapping $f \to J_{L,c}(f)$ from $\mathcal{C}(G)$ to $\mathcal{D}_c(L)^*$ such that

$$J_L(\rho, f) = \langle \rho, J_{L,c}(f) \rangle, \qquad \rho \in R_c(L).$$

The formal germs (5.2) are given by an element g_M^L in the (adic) tensor product $\widehat{\mathcal{G}}_c(M, L) \otimes \mathcal{D}_c(L)$ such that

$$\langle g_M^L, \rho \rangle = g_M^L(\rho), \qquad \rho \in R_c(L).$$

The distribution in (5.4) can thus be expressed simply as a pairing

$$g_M^L\big(J_{L,c}(f)\big) = \langle g_M^L, J_{L,c}(f) \rangle.$$

However, we shall retain the basis dependent notation (5.1) and (5.2), in deference to the traditional formulation of p-adic germ expansions.

The formal germ expansion for $J_M(f)$ is the main result of the paper. We shall actually need a quantitative form of the expansion, which applies to partial sums in the asymptotic series, and is slightly stronger than the assertion of Theorem 5.1.

It follows from (5.3) and the definition of the adic topology on $\widehat{\mathcal{G}}_c(M,L)$ that there is a weight function α such that $g_M^L(\rho)$ belongs to $\widehat{\mathcal{G}}_c^{\alpha}(M,L)$, for all L and ρ. Given such an α, and any $n \geq 0$, our conventions dictate that we write $g_M^{L,n}(\rho)$ for the projection of $g_M^L(\rho)$ onto the quotient $\mathcal{G}_c^{\alpha,n}(M,L)$ of $\widehat{\mathcal{G}}_c^{\alpha}(M,L)$, and $g_M^{L,n}(\gamma,\rho)$ for a representative of $g_M^{L,n}(\rho)$ in $\mathcal{F}_c^{\alpha}(V,L)$. We assume that $g_M^{L,n}(\gamma,\rho)=0$, if $g_M^{L,M}(\rho)=0$. The sum

$$J_M^n(\gamma,f) = \sum_L \sum_\rho g^{L,n}(\gamma,\rho) J_L(\rho,f) \tag{5.11}$$

can then be taken over a finite set. Our second theorem will include a slightly sharper form of the symmetry condition (5.9), namely that the functions $g_M^{L,n}(\gamma,\rho)$ can be chosen so that

$$g_{\theta M}^{\theta L,n}(\theta\gamma,\theta\rho) = g_M^{L,n}(\gamma,\rho), \tag{5.9*}$$

for θ as in (5.9). This condition, combined with the remarks prior to the statement of Theorem 5.1, tells us that (5.11) is invariant under the action of $W(M)$ on γ. The function $J_M^n(\gamma,f)$ therefore belongs to $\mathcal{F}_c^{\alpha}(V,G)$. It is uniquely determined up to a finite sum

$$\sum_i \phi_i(\gamma) J_i(f), \tag{5.12}$$

for tempered distributions $J_i(f)$ and functions $\phi_i(\gamma)$ in $\mathcal{F}_{c,n}^{\alpha}(V,G)$.

According to Corollary 3.2, we can choose α so that the weighted orbital integral $J_M(\gamma,f)$ also belongs to $\mathcal{F}_c^{\alpha}(V,G)$.

Theorem 5.1^***.** *We can chose the weight function α above so that $\alpha(1)$ equals 0 and the symmetry condition $(5.9)^*$ is valid, and so that for any n, the mapping*

$$f \longrightarrow J_M(\gamma,f) - J_M^n(\gamma,f), \qquad f \in \mathcal{C}(G),$$

is a continuous linear transformation from $\mathcal{C}(G)$ to the space $\mathcal{F}_{c,n}^{\alpha}(V,G)$.

Remarks. 1. The statement of Theorem 5.1^* is well posed, even though the mapping is determined only up to a finite sum (5.12). For (5.12) represents a continuous linear mapping from $\mathcal{C}(G)$ to $\mathcal{F}_{c,n}^{\alpha}(V,G)$. In other words, the difference

$$K_M^n(\gamma,f) = J_M(\gamma,f) - J_M^n(\gamma,f)$$

is defined up to a function that satisfies the condition of the theorem.

2. In concrete terms, Theorem 5.1^* asserts the existence of a continuous seminorm $\mu_{\sigma,\varepsilon,n}^{\alpha}$ on $\mathcal{C}(G)$, for each $\sigma = (T,\Omega,X)$ in $S_c(M,G)$, each $\varepsilon > 0$, and each $n \geq 0$, such that

$$|XK_M^n(\gamma,f)| \leq \mu_{\sigma,\varepsilon,n}^{\alpha}(f)|D_c(\gamma)|^{-(\alpha(X)+\varepsilon)} \|\ell_c(\gamma)\|^{(n,X)},$$

for every $\gamma \in V_\Omega$ and $f \in \mathcal{C}(G)$. This can be regarded as the analogue of Taylor's formula with remainder. The germ expansion of Theorem 5.1 is of course analogous to the asymptotic series provided by Taylor's theorem. In particular, Theorem 5.1^* implies the germ expansion of Theorem 5.1.

We are going to prove Theorems 5.1 and 5.1* together. The argument will be inductive. We fix the 4-tuple of objects (G, K, M, c), and assume inductively that the two theorems have been established for any other 4-tuple (G_1, K_1, M_1, c_1), with

$$\dim(A_{M_1}/A_{G_1}) < \dim(A_M/A_G).$$

In particular, we assume that the distributions $J_L(\cdot, f)$ have been defined for any $L \supsetneq M$, that the formal germs $g_M^L(\cdot)$ have been defined for any $L \subsetneq G$, and that both sets of objects satisfy conditions of the theorems. Our task will be to construct distributions $J_M(\cdot, f)$ and formal germs $g_M^G(\cdot)$ that also satisfy the required conditions.

We shall begin the proof in the next section. In what remains of this section, let us consider the question of how closely the conditions of Theorem 5.1 come to determining the distributions and formal germs uniquely. Assume that we have been able to complete the induction argument by constructing the remaining distributions $J_M(\cdot, f)$ and formal germs $g_M^G(\cdot)$. To what degree are these objects determined by the distributions and formal germs for lower rank whose existence we have postulated?

Suppose for a moment that $\rho \in R_c(M)$ is fixed. Let ${}^*J_M(\rho, f)$ be an arbitrary distribution on $G(\mathbb{R})$ that is supported on $\mathcal{U}_c(G)$, and satisfies (5.5) (with $L = M$). That is, we suppose that

$${}^*J_M(\rho, f^y) = {}^*J_M(\rho, f) + \sum_{Q \in \mathcal{F}^0(M)} J_M^{M_Q}(\rho, f_{Q,y}),$$

for any $y \in G(\mathbb{R})$. Applying (5.5) to $J_M(\rho, f)$, we deduce that the difference

$$f \longrightarrow {}^*J_M(\rho, f) - J_M(\rho, f)$$

is an invariant tempered distribution that is supported on $\mathcal{U}_c(G)$. It follows that

$$(5.13) \qquad {}^*J_M(\rho, f) = J_M(\rho, f) + \sum_{\rho_G \in R_c(G)} c(\rho, \rho_G) f_G(\rho_G),$$

for complex coefficients $\{c(\rho, \rho_G)\}$ that vanish for almost all ρ_G.

Suppose now that ${}^*J_M(\cdot, f)$ and ${}^*g_M^G(\cdot)$ are arbitrary families of objects that satisfy the relevant conditions of Theorem 5.1. For each $\rho \in R_c(M)$, the distributions ${}^*J_M(\rho, f)$ and $J_M(\rho, f)$ then satisfy an identity (5.13), for complex coefficients

$$(5.14) \qquad c(\rho_M, \rho_G), \qquad \rho_M \in R_c(M),\ \rho_G \in R_c(G),$$

that for any ρ_M, have finite support in ρ_G. The terms with $L \neq M, G$ in the formal germ expansion (5.10) are assumed to have been chosen. It follows that the difference

$$g_M^G\big(J_{G,c}(f)\big) - {}^*g_M^G\big(J_{G,c}(f)\big) = \sum_{\rho_G \in R_c(G)} \big(g_M^G(\rho_G) - {}^*g_M^G(\rho_G)\big) f_G(\rho_G)$$

equals

$$\begin{aligned} g_M^M\big({}^*J_{M,c}(f)\big) - g_M^M\big(J_{M,c}(f)\big) &= \sum_{\rho_M \in R_c(M)} \rho_M^\vee\big({}^*J_M(\rho_M, f) - J_M(\rho_M, f)\big) \\ &= \sum_{\rho_G \in R_c(G)} \left(\sum_{\rho_M \in R_c(M)} \rho_M^\vee c(\rho_M, \rho_G) \right) f_G(\rho_G). \end{aligned}$$

Comparing the coefficients of $f_G(\rho_G)$, we find that

$$ {}^*g^G_M(\rho) = g^G_M(\rho) - \sum_{\rho_M \in R_c(M)} \rho^\vee_M c(\rho_M, \rho), \tag{5.15}$$

for any $\rho \in R_c(G)$. The general objects ${}^*J_M(\cdot, f)$ and ${}^*g^G_M(\cdot)$ could thus differ from the original ones, but only in a way that is quite transparent. Moreover, the coefficients (5.14) are governed by the conditions of Theorem 5.1. If z belongs to $\mathcal{Z}(G)$, they satisfy the equation

$$ c(\rho_M, \, \widehat{z}\rho_G) = c(z_M\rho_M, \rho_G). \tag{5.16}$$

They also satisfy the symmetry condition

$$ c(\theta\rho_M, \theta\rho_G) = c(\rho_M, \rho_G), \tag{5.17}$$

for any isomorphism θ: $G \to \theta G$ over $\mathbb{R}$. Finally, they satisfy the transformation formula

$$ c(\rho'_M, \rho'_G) = \sum_{\rho_M} \sum_{\rho_G} a_M(\rho'_M, \rho_M) c(\rho_M, \rho_G) a^\vee_G(\rho'_G, \rho_G) \tag{5.18}$$

for change of bases, with matrices $\{a_M(\rho'_M, \rho_M)\}$ and $\{a^\vee_G(\rho'_G, \rho_G)\}$ as in (5.4.1) and (5.4.2).

Conversely, suppose that ${}^*J_M(\cdot, f)$ and ${}^*g^G_M(\cdot)$ are defined in terms of $J_M(\cdot, f)$ and $g^G_M(\cdot)$ by (5.13) and (5.15), for coefficients (5.14) that satisfy (5.16)–(5.18). It is then easy to see that ${}^*J_M(\cdot, f)$ and ${}^*g^G_M(\cdot)$ satisfy the conditions of Theorems 5.1 and 5.1^*. We obtain

Proposition 5.2. *Assume that Theorems 5.1 and* 5.1^* *are valid for distributions* $J_L(\cdot, f)$ *and formal germs* $g^L_M(\cdot)$. *Let* ${}^*J_L(\cdot, f)$ *and* ${}^*g^L_M(\cdot)$ *be secondary families of such objects for which* ${}^*J_L(\cdot, f) = J_L(\cdot, f)$ *if* $L \neq M$, *and* ${}^*g^L_M(\cdot) = g^L_M(\cdot)$ *if* $L \neq G$. *Then Theorems 5.1 and* 5.1^* *are valid for* ${}^*J_L(\cdot, f)$ *and* ${}^*g^L_M(\cdot)$ *if and only if the relations (5.13) and (5.15) hold, for coefficients (5.14) that satisfy the conditions (5.16)–(5.18).* □

§6. Some consequences of the induction hypotheses

We shall establish Theorems 5.1 and 5.1^* over the next four sections. In these sections, G, K, M and c will remain fixed. We are assuming inductively that the assertions of the theorems are valid for any (G_1, K_1, M_1, c_1), with

$$\dim(A_{M_1}/A_{G_1}) < \dim(A_M/A_G).$$

In this section, we shall see what can be deduced directly from this induction assumption.

Let L be a Levi subgroup of G in $\mathcal{L}(M)$ that is distinct from both M and G. The terms in the series

$$g_M^L\big(J_{L,c}(f)\big) = \sum_{\rho\in R_c(L)} g_M^L(\rho)J_L(\rho, f), \qquad f \in \mathcal{C}(G),$$

are then defined, according to our induction assumption. The series converges to a formal germ in $\widehat{\mathcal{G}}_c(M, L)$ that is independent of the basis $R_c(L)$, as we see by applying (5.3), (5.4.1) and (5.4.2) inductively to L. Moreover, the mapping

$$f \longrightarrow g_M^L\big(J_{L,c}(f)\big), \qquad f \in \mathcal{C}(G),$$

is a continuous linear transformation from $\mathcal{C}(G)$ to $\widehat{\mathcal{G}}_c(M, L)$. We begin by describing three simple properties of this mapping.

Suppose that $y \in G(\mathbb{R})$. We can then consider the value

$$g_M^L\big(J_{L,c}(f^y)\big) = \sum_{\rho\in R_c(L)} g_M^L(\rho)J_L(\rho, f^y)$$

of the mapping at the y-conjugate of f. Since

$$\dim(A_L/A_G) < \dim(A_M/A_G),$$

we can apply the formula (5.5) inductively to $J_L(\rho, f^y)$. We obtain

$$\begin{aligned}\sum_{\rho\in R_c(L)} g_M^L(\rho)J_L(\rho, f^y) &= \sum_{\rho\in R_c(L)}\sum_{Q\in\mathcal{F}(L)} g_M^L(\rho)J_L^{M_Q}(\rho, f_{Q,y})\\ &= \sum_Q\left(\sum_\rho g_M^L(\rho)J_L^{M_Q}(\rho, f_{Q,y})\right).\end{aligned}$$

It follows that

$$g_M^L\big(J_{L,c}(f^y)\big) = \sum_{Q\in\mathcal{F}(L)} g_M^L\big(J_{L,c}^{M_Q}(f_{Q,y})\big), \qquad f \in \mathcal{C}(G). \tag{6.1}$$

Suppose that $z \in \mathcal{Z}(G)$. Consider the value

$$g_M^L\big(J_{L,c}(zf)\big) = \sum_{\rho\in R_c(L)} g_M^L(\rho)J_L(\rho, zf)$$

of the mapping at the z-transform of f. Since

$$\dim(A_L/A_G) < \dim(A_M/A_G),$$

we can apply the formula (5.6) to $J_L(\rho, zf)$. We obtain

$$\begin{aligned}\sum_{\rho\in R_c(L)} g_M^L(\rho)J_L(\rho, zf) &= \sum_{\rho} g_M^L(\rho)J_L(z_L\rho, f)\\ &= \sum_{\rho\in R_c(L)} g_M^L(\widehat{z}_L\rho)J_L(\rho, f),\end{aligned}$$

by the definition of the transpose $\widehat{z}_L$. Since

$$\dim(A_M/A_L) < \dim(A_M/A_G),$$

we can apply the formula (5.7) inductively to $g_M^L(\widehat{z}_L\rho)$. We obtain

$$\begin{aligned}\sum_{\rho\in R_c(L)} g_M^L(\widehat{z}_L\rho)J_L(\rho, f) &= \sum_{\rho}\sum_{S\in\mathcal{L}^L(M)} \big(\partial_M^S(z_S)g_S^L(\rho)_M\big)J_L(\rho, f)\\ &= \sum_{S}\partial_M^S(z_S)\left(\sum_{\rho} g_S^L(\rho)_M J_L(\rho, f)\right).\end{aligned}$$

It follows that

$$g_M^L\big(J_{L,c}(zf)\big) = \sum_{S\in\mathcal{L}^L(M)} \partial_M^S(z_S)g_S^L\big(J_{L,c}(f)\big)_M, \qquad f\in\mathcal{C}(G). \tag{6.2}$$

Finally, suppose that θ: $G\to\theta G$ is an isomorphism over $\mathbb{R}$. Consider the composition

$$\theta g_M^L\big(J_{L,c}(f)\big) = \sum_{\rho\in R_c(L)} \theta g_M^L(\rho)\cdot J_L(\rho, f)$$

of the mapping with θ. Since

$$\dim(A_L/A_G) < \dim(A_M/A_G),$$

we can apply (5.8) to $J_L(\rho, f)$. Since

$$\dim(A_M/A_L) < \dim(A_M/A_G),$$

we can apply (5.9) to $g_M^L(\rho)$. It follows that

$$g_{\theta M}^{\theta L}\big(J_{\theta L,\theta c}(\theta f)\big) = \theta g_M^L\big(J_{L,c}(f)\big). \tag{6.3}$$

The main assertion of Theorem 5.1 is that the difference

$$K_M(f) = J_M(f) - \sum_{L\in\mathcal{L}(M)} g_M^L\big(J_{L,c}(f)\big),$$

regarded as an element in $\widehat{\mathcal{G}}_c(M,G)$, vanishes. We are not yet in a position to investigate this question, since we have not defined the terms in the series with $L = M$ and $L = G$. We consider instead the partial difference

$$\widetilde{K}_M(f) = J_M(f) - \sum_{\{L\in\mathcal{L}(M):\, L\neq M,G\}} g_M^L\big(J_{L,c}(f)\big), \qquad f\in\mathcal{C}(G), \tag{6.4}$$

regarded again as an element in $\widehat{\mathcal{G}}_c(M,G)$.

Lemma 6.1. *Suppose that $f \in \mathcal{C}(G)$ and $y \in G(\mathbb{R})$. Then*

$$\widetilde{K}_M(f^y) - \widetilde{K}_M(f) = \sum_{Q \in \mathcal{F}^0(M)} g_M^M\big(J_{M,c}^{M_Q}(f_{Q,y})\big). \tag{6.5}$$

Proof. The left hand side of (6.5) equals

$$\big(J_M(f^y) - J_M(f)\big) - \sum_{L \neq M,G} \big(g_M^L\big(J_{L,c}(f^y)\big) - g_M^L\big(J_{L,c}(f)\big)\big).$$

We apply (3.1) to the term on the left, and (6.1) to each of the summands on the right. The expression becomes

$$\sum_{Q \in \mathcal{F}^0(M)} J_M^{M_Q}(f_{Q,y}) - \sum_{L \neq M} \sum_{Q \in \mathcal{F}^0(L)} g_M^L\big(J_{L,c}^{M_Q}(f_{Q,y})\big).$$

We take the second sum over Q outside the sum over L. The new outer sum is then over $Q \in \mathcal{F}^0(M)$, while the new inner sum is over Levi subgroups $L \in \mathcal{L}^{M_Q}(M)$ with $L \neq M$. Since $Q \neq G$, the formal germ $g_M^M\big(J_{M,c}^{M_Q}(f_{Q,y})\big)$ is defined, according to the induction assumption. We can therefore take the new inner sum over all elements $L \in \mathcal{L}^{M_Q}(M)$, provided that we then subtract the term corresponding to $L = M$. The left hand side of (6.5) thus equals the sum of

$$\sum_{Q \in \mathcal{F}^0(M)} \left(J_M^{M_Q}(f_{Q,y}) - \sum_{L \in \mathcal{L}^{M_Q}(M)} g_M^L\big(J_{L,c}^{M_Q}(f_{Q,y})\big) \right)$$

and

$$\sum_{Q \in \mathcal{F}^0(M)} g_M^M\big(J_{M,c}^{M_Q}(f_{Q,y})\big).$$

The first of these expressions reduces to a sum,

$$\sum_{Q \in \mathcal{F}^0(M)} K_M^{M_Q}(f_{Q,y}),$$

whose terms vanish by our induction assumption. The second expression is just the right hand side of (6.5). The formula (6.5) follows. □

In stating the next lemma, we write $f_{G,c}$ for the function

$$J_{G,c}(f) : \ \rho \longrightarrow J_G(\rho, f) = f_G(\rho), \qquad \rho \in R_c(G),$$

to remind ourselves that it is invariant in f.

Lemma 6.2. *Suppose that $f \in \mathcal{C}(G)$ and $z \in \mathcal{Z}(G)$. Then*

$$\widetilde{K}_M(zf) - \partial\big(h(z)\big)\widetilde{K}_M(f) = \sum_{\{L \in \mathcal{L}(M):\, L \neq M\}} \partial_M^L(z_L) g_L^G(f_{G,c})_M. \tag{6.6}$$

Proof. The left hand side of (6.6) equals

$$\big(J_M(zf) - \partial\big(h(z)\big)J_M(f)\big) - \sum_{L \neq M,G} \big(g_M^L\big(J_{L,c}(zf)\big) - \partial\big(h(z)\big)g_M^L\big(J_{L,c}(f)\big)\big).$$

We apply (3.2) to the term on the left, and (6.2) to each of the summands on the right. The expression becomes

$$\sum_S \partial_M^S(z_S)J_S(f) - \sum_{L,S} \partial_M^S(z_S)g_S^L\big(J_{L,c}(f)\big)_M,$$

where the first sum is over Levi subgroups $S \in \mathcal{L}(M)$ with $S \neq M$, and the second sum is over groups L and S in $\mathcal{L}(M)$ with

$$M \subsetneq S \subset L \subsetneq G.$$

This second sum can obviously be represented as an iterated sum over elements $S \in \mathcal{L}(M)$ with $S \neq M$, and elements $L \in \mathcal{L}(S)$ with $L \neq G$. Since $S \neq G$, the formal germ $g_S^L\big(J_{L,c}(f)\big)_M$ is defined, according to the induction assumption. We can therefore sum L over all elements in $\mathcal{L}(S)$, provided that we then subtract the term corresponding to $L = G$. The left hand side of (6.6) thus equals the sum of

$$\sum_{\{S\in\mathcal{L}(M):\,S\neq M\}} \partial_M^S(z_S)\left(J_S(f) - \sum_{L\in\mathcal{L}(S)} g_M^L\big(J_{L,c}(f)\big)_M\right)$$

and

$$\sum_{\{S\in\mathcal{L}(M):\,S\neq M\}} \partial_M^S(z_S)g_S^G\big(J_{G,c}(f)\big)_M.$$

The first of these expressions reduces to a sum,

$$\sum_{S\neq M} \partial_M^S(z_S)K_S(f)_M,$$

whose terms vanish by our induction assumption. The second expression equals

$$\sum_{\{L\in\mathcal{L}(M):\,L\neq M\}} \partial_M^L(z_L)g_L^G(f_{G,c})_M,$$

the right hand side of (6.6). The formula (6.6) follows. □

Lemmas 6.1 and 6.2 can be interpreted as identities

$$\widetilde{K}_M(\gamma, f^y) - \widetilde{K}_M(\gamma, f) = \sum_{Q\in\mathcal{F}^0(M)} g_M^M\big(\gamma, J_{M,c}^{M_Q}(f_{Q,y})\big), \qquad f \in \mathcal{C}(G),$$

and

$$\widetilde{K}_M(\gamma, zf) - \partial\big(h(z)\big)\widetilde{K}_M(\gamma, f) = \sum_{L\neq M} \partial_M^L(\gamma, z_L)g_L^G(\gamma, f_{G,c}), \qquad f \in \mathcal{C}(G),$$

of asymptotic series. What do these identities imply about the partial sums in the series? The question is not difficult, but in the case of Lemma 6.2 at least, it will require a precise answer.

As in the preamble to Theorem 5.1*, we can choose a weight function α such that for each $L \neq G$ and $\rho \in R_c(L)$, $g_M^L(\rho)$ belongs to $\widehat{\mathcal{G}}_c^\alpha(M, L)$. By applying the first assertion of Theorem 5.1* inductively to $(L, K \cap L, M, c)$ (in place of (G, K, M, c)), we see that α may be chosen so that $\alpha(1)$ equals zero. By Corollary 3.2, we can also assume that α is such that $f \to J_M(\gamma, f)$ is a continuous linear transformation from $\mathcal{C}(G)$ to $\mathcal{F}_c^\alpha(V, G)$. Having chosen α, we set

(6.7) $$\widetilde{J}_M^n(\gamma, f) = \sum_{L\neq M,G}\ \sum_{\rho\in R_c(L)} g_M^{L,n}(\gamma, \rho)J_L(\rho, f),$$

for any $n \geq 0$. The sums in this expression can be taken over finite sets, while the functions $g_M^{L,n}(\gamma,\rho)$ can be assumed inductively to satisfy the symmetry condition $(5.9)^*$. The function

$$\widetilde{K}_M^n(\gamma,f) = J_M(\gamma,f) - \widetilde{J}_M^n(\gamma,f), \qquad f \in \mathcal{C}(G), \tag{6.8}$$

is then invariant under the action of $W(M)$ on γ, and is uniquely determined up to a continuous linear mapping from $\mathcal{C}(G)$ to $\mathcal{F}_{c,n}^{\alpha}(V,G)$.

The following analogue of Corollary 3.2 is an immediate consequence of these remarks.

Lemma 6.3. *There is a weight function α with $\alpha(1) = 0$, such that for any n, the mapping*

$$f \longrightarrow \widetilde{K}_M^n(\gamma,f), \qquad f \in \mathcal{C}(G),$$

defines a continuous linear transformation from $\mathcal{C}(G)$ to $\mathcal{F}_c^{\alpha}(V,G)$. □

We shall now state our sharper form of Lemma 6.2. We assume for simplicity that c is not G-regular, or in other words, that the function D_c is nontrivial. The identity in Lemma 6.2 concerns an element $z \in \mathcal{Z}(G)$. In order to estimate the terms in this identity, we fix a triplet $\sigma = (T,\Omega,X)$ in $S_c(M,G)$. For any $n \geq 0$, we then write $k_{z,\sigma}^n(\gamma,f)$ for the function

$$X\big(\widetilde{K}_M^n(\gamma,zf) - \partial\big(h_T(z)\big)\widetilde{K}_M^n(\gamma,f)\big) - \sum_{L\neq M}\partial_M^L(\gamma,z_L)g_L^{G,n}(\gamma,f_{G,c})\big)$$

of $\gamma \in V_\Omega$.

Lemma 6.4. *Given z and σ, we can choose a positive number a with the property that for any $n \geq 0$, the functional*

$$\nu_{z,\sigma}^{a,n}(f) = \sup_{\gamma\in V_\Omega}\big(|k_{z,\sigma}^n(\gamma,f)||D_c(\gamma)|^a\|\ell_c(\gamma)\|^{-(n+1)}\big), \qquad f \in \mathcal{C}(G),$$

is a continuous seminorm on $\mathcal{C}(G)$.

Proof. The assertion is a quantitative reformulation of Lemma 6.2 that takes into account its dependence on f. The proof is in principle the same. However, we do require a few preliminary comments to allow us to interpret the earlier argument.

There is of course some ambiguity in the definition of $k_{z,\sigma}^n(\gamma,f)$. The definition is given in terms of the restrictions to V_Ω of the functions $\widetilde{K}_M^n(\gamma,zf)$, $\widetilde{K}_M^n(\gamma,f)$ and $g_L^{G,n}(\gamma,f_{G,c})$ in $\mathcal{F}_c^{\alpha}(V,G)$. For a given n, these functions are each defined only up to a continuous linear map from $\mathcal{C}(G)$ to $\mathcal{F}_{c,n}^{\alpha}(V,G)$. It is actually the images of the three functions under three linear transformations

$$\phi \longrightarrow X\phi_\Omega,$$

$$\phi \longrightarrow X\partial\big(h_T(z)\big)\phi_\Omega,$$

and

$$\phi \longrightarrow X\partial_M^L(\gamma,z_L)\phi_\Omega, \qquad \phi \in \mathcal{F}_c^{\alpha}(V,G),\ L \neq M,$$

that occur in the definition of $k_{z,\sigma}^n(\gamma,f)$. Each transformation is given by a linear partial differential operator on $T_{G\text{-reg}}(\mathbb{R})$ whose coefficients are at worst algebraic. Since $D_c \neq 1$, the notation of §4 simplifies slightly. Recalling the remark preceding the statement of Lemma 4.1, we see that there is a positive number a_0 with the property that for any $a \geq a_0$, and any n, each of the three linear transformations maps $\mathcal{F}_{c,n}^{\alpha}(V,G)$ continuously to $F_{c,n}^{a}(V,G)$. It follows that $k_{z,\sigma}^n(\gamma,f)$ is determined up to a continuous linear mapping from $\mathcal{C}(G)$ to $F_{c,n}^{a}(V,G)$. In other words, $k^n_{z,\sigma}(\gamma, f)$ is well defined up to a function that satisfies the condition of the lemma. This means that it would suffice to establish the lemma with any particular choice for each of the three functions.

The main ingredient in the proof of Lemma 6.2 was the formula (6.2), which we can regard as an identity

$$g^L_M\big(\gamma, J_{L,c}(zf)\big) = \sum_{S\in\mathcal{L}^L(M)} \partial^S_M(\gamma, z_S) g^L_S\big(\gamma, J_{L,c}(f)\big)$$

of asymptotic series. To prove the lemma at hand, we need a corresponding identity of partial sums. For each S, we choose a weight function β such that the formal germ $g^L_S\big(J_{L,c}(f)\big)$ lies in $\widehat{\mathcal{G}}^\beta_c(S, L)$. For any positive integer m, $g^{L,m}_S\big(\gamma, J_{L,c}(f)\big)$ then denotes a representative in $\mathcal{F}^\beta_c(V, L)$ of the corresponding m-jet. After a moment's thought, it is clear that we can assign an integer $m > n$ to every n such that

$$g^{L,n}_M\big(\gamma, J_{L,c}(zf)\big) = \sum_{S\in\mathcal{L}^L(M)} \partial^S_M(\gamma, z_S) g^{L,m}_S\big(\gamma, J_{L,c}(f)\big).$$

The left hand side here stands for some particular representative of $g^{L,n}_M\big(J_{L,c}(zf)\big)$ in $\mathcal{F}^\alpha_c(V, L)$, rather than the general one. Its sum over Levi subgroups $L \neq M, G$ yields a particular choice for the function $\widetilde{K}^n_M(\gamma, f)$ that occurs in the definition of $k^n_{z,\sigma}(\gamma, f)$. As we have noted, this is good enough for the proof of the lemma.

Armed with the last formula, we have now only to copy the proof of Lemma 6.2. A review of the earlier argument leads us directly to a formula

$$k^n_{z,\sigma}(\gamma, f) = \sum_{S\neq M} X\partial^S_M(\gamma, z_S)K^m_S(\gamma, f), \qquad \gamma \in V_\Omega.$$

We are free to apply Theorem 5.1* inductively to the summand $K^m_S(\gamma, f)$, since $S \neq M$. We thereby observe that $f \to K^m_S(\gamma, f)$ is a continuous linear transformation from $\mathcal{C}(G)$ to $\mathcal{F}^\beta_{c,m}(V, G)$. Since $D_c \neq 1$, we know from the discussion in §4 that there is an $a \geq a_0$ such that for any n, and any $m \geq n$, the linear transformation

$$\phi \longrightarrow X\partial^S_M(\gamma, z_S)\phi_\Omega, \qquad \phi \in \mathcal{F}^\beta_{c,m}(V, G),$$

maps $\mathcal{F}^\beta_{c,m}(V, G)$ continuously to $F^a_{c,n}(V_\Omega, G)$. It follows that for any n, the map

$$f \longrightarrow k^n_{z,\sigma}(\gamma, f), \qquad f \in \mathcal{C}(G),$$

is a continuous linear transformation from $\mathcal{C}(G)$ to $F^a_{c,n}(V_\Omega, G)$. The assertion of the lemma then follows from the definition of $F^a_{c,n}(V_\Omega, G)$. □

Recall that for any nonnegative integer N, $\mathcal{C}_{c,N}(G)$ denotes the subspace of $\mathcal{C}(G)$ annihilated by $R_{c,N}(G)$. This subspace is of finite codimension in $\mathcal{C}(G)$, and is independent of the choice of basis $R_c(G)$.

Lemma 6.5. *For any $n \geq 0$, we can choose an integer N so that if f belongs to $\mathcal{C}_{c,N}(G)$, the function $k^n_{z,\sigma}(\gamma, f)$ simplifies to*

$$k^n_{z,\sigma}(\gamma, f) = X\big(\widetilde{K}^n_M(\gamma, zf) - \partial\big(h_T(z)\big)\widetilde{K}^n_M(\gamma, f)\big), \qquad \gamma \in V_\Omega.$$

Proof. We have to show that the summands

$$g^{G,n}_L(\gamma, f_{G,c}) = \sum_{\rho\in R_c(G)} g^{G,n}_L(\gamma, \rho) f_G(\rho), \qquad L \neq M,$$

in the original definition of $k^n_{z,\sigma}(\gamma, f)$ vanish for the given f. Applying (5.3) inductively (with (G, M) replaced by (G, L)), we see that the (α, n)-jet $g^G_L(\rho)$ vanishes for all but finitely many ρ. We can therefore choose N so that for each L, the function $g_L^{G,n}(\gamma,\rho)$ vanishes for any ρ in the complement of $R_{c,N}(G)$ in $R_c(G)$. The lemma follows. $\square$

Finally, we note that that $\widetilde{K}_M(f)$ transforms in the obvious way under any isomorphism θ: $G \to \theta G$ over $\mathbb{R}$. If we apply (3.3) and (6.3) to the definition (6.4), we see immediately that

$$\widetilde{K}_{\theta M}(\theta f) = \theta \widetilde{K}_M(f), \qquad f \in \mathcal{C}(G). \tag{6.9}$$

Moreover, for any $n \geq 0$, the function (6.8) satisfies the symmetry condition

$$\widetilde{K}^n_{\theta M}(\theta\gamma, \theta f) = \widetilde{K}^n_M(\gamma, f), \qquad \gamma \in V_{G\text{-reg}},\ f \in \mathcal{C}(G). \tag{6.10}$$

This follows from our induction assumption that the relevant terms in (6.7) satisfy $(5.9)^*$.

§7. An estimate

We have been looking at some of the more obvious implications of our induction hypothesis. We are now ready to begin a construction that will eventually yield the remaining objects $g_M^G(\rho)$ and $I_M(\rho, f)$. We shall carry out the process in the next section. The purpose of this section is to establish a key estimate for the mapping $\widetilde{K}_M$, which will be an essential part of the construction. The estimate is based on an important technique [H1] that Harish-Chandra developed from the differential equations (1.2).

Recall that $\widetilde{K}_M$ is a continuous linear transformation from $\mathcal{C}(G)$ to $\widehat{\mathcal{G}}_c(M, G)$. We choose a weight function α as in Lemma 6.3. For any $n \geq 0$, $f \to \widetilde{K}_M^n(f)$ and $f \to \widetilde{K}_M^n(\gamma, f)$ then represent continuous linear mappings from $\mathcal{C}(G)$ onto the respective spaces $\mathcal{G}_c^{\alpha,n}(M, G)$ and $\mathcal{F}_c^\alpha(V, G)$. To focus the discussion, let us write ψ_M^n for the restriction of $\widetilde{K}_M^n$ to some given subspace $\mathcal{C}_{c,N}(G)$ of $\mathcal{C}(G)$. Then

$$f \longrightarrow \psi_M^n(\gamma, f) = \widetilde{K}_M^n(\gamma, f), \qquad f \in \mathcal{C}_{c,N}(G),$$

is a continuous linear mapping from $\mathcal{C}_{c,N}(G)$ to $\mathcal{F}_c^\alpha(V, G)$. For all intents and purposes, we shall take N to be any integer that is large relative to n, and that in particular has the property of Lemma 6.5. This will lead us to an estimate for $\psi_M^n(\gamma, f)$ that is stronger than the bound implied by the definition of $\mathcal{F}_c^\alpha(V, G)$.

To simplify the statement of the estimate, we may as well rule out the trivial case that c is G-regular, as we did in Lemma 6.4. In other words, we assume that $\dim(G_c/T) > 0$, for any maximal torus $T \in \mathcal{T}_c(M)$. We fix T, together with a connected component $\Omega \in \pi_{0,c}\big(T_{G\text{-reg}}(\mathbb{R})\big)$. Consider the open subset

$$V_\Omega(a, n) = \big\{\gamma \in V_\Omega :\ |D_c(\gamma)|^{-a} \|\ell_c(\gamma)\|^n < 1\big\}$$

of $T_{G\text{-reg}}(\mathbb{R})$, defined for any $a > 0$ and any nonnegative integer n. Our interest will be confined to the case that the closure of $V_\Omega(a, n)$ contains c. This condition will obviously be met if n is large relative to a, or more precisely, if n is greater than the integer

$$a^+ = a\ \dim(G_c/T).$$

According to our definitions, any function in the space $\mathcal{F}_{c,n}^\alpha(V, G)$ will be bounded on $V_\Omega(a, n)$, for any $a > \alpha(1)$. (We assume of course that the invariant function $\ell_c(\gamma)$ is bounded on V.) The function $\psi_M^n(\gamma, f)$ above lies a priori only in the larger space $\mathcal{F}_c^\alpha(V, G)$. However, the next lemma asserts that for N large, the restriction of $\psi_M^n(\gamma, f)$ to $V_\Omega(a, n)$ is also bounded.

More generally, we shall consider the derivative $X\psi_M^n(\gamma, f)$, for any (translation) invariant differential operator X on $T(\mathbb{R})$. Given X, we assume that a is greater than the positive number

$$\alpha^+(X) = \alpha(X) + \deg(X) \dim(G_c/T)^{-1}.$$

Then if $n > a^+$, as above, and $\varepsilon > 0$ is small, n will be greater than $\deg(X)$, and

$$\begin{aligned} |D_c(\gamma)|^{-(\alpha(X)+\varepsilon)} \|\ell_c(\gamma)\|^{(n,X)} &= |D_c(\gamma)|^{-(\alpha(X)+\varepsilon)} \|\ell_c(\gamma)\|^{n+1-\deg X} \\ &\leq C |D_c(\gamma)|^{-a} \|\ell_c(\gamma)\|^{n+1}, \qquad \gamma \in V_{\mathrm{reg}}, \end{aligned}$$

for some constant C. It follows that the X-transform of any function in $\mathcal{F}_{c,n}^\alpha(V, G)$ is bounded by a constant multiple of $\|\ell_c(\gamma)\|$ on $V_\Omega(a, n)$, and is therefore absolutely bounded on $V_\Omega(a, n)$. We are going to show that for N large, the function $X\psi_M^n(\gamma, f)$ is also bounded on $V_\Omega(a, n)$.

Lemma 7.1. *Given the triplet $\sigma = (T, \Omega, X)$, we can choose a positive integer $a > \alpha^+(X)$ with the property that for any $n > a^+$, and for N large relative to n, the function*

$$f \longrightarrow \sup_{\gamma \in V_\Omega(a,n)} |X\psi_M^n(\gamma, f)|, \qquad f \in \mathcal{C}_{c,N}(G),$$

is a continuous seminorm on $\mathcal{C}_{c,N}(G)$.

Proof. We should first check that the statement of the lemma is well posed, even though the function $\psi_M^n(\gamma, f)$ is not uniquely determined. As in the remark preceding the statement of Theorem 5.1^*, we observe that $\psi_M^n(\gamma, f)$ is defined only up to a finite sum

$$\sum_i \phi_i(\gamma) J_i(f), \qquad f \in \mathcal{C}_{c,N}(G),$$

for tempered distributions $J_i(f)$ and functions $\phi_i(\gamma)$ in $\mathcal{F}_{c,n}^\alpha(V, G)$. From the discussion above, we see that the function

$$\sum_i \big(Xg_i(\gamma)\big) J_i(f)$$

is bounded on $V_\Omega(a, n)$, and in fact, can be bounded by a continuous seminorm in f. In other words, $X\psi_M^n(\gamma, f)$ is well defined up to a function that satisfies the condition of the lemma. The condition therefore makes sense for $X\psi_M^n(\gamma, f)$.

Let $u_1 = 1, u_2, \ldots, u_q$ be a basis of the G-harmonic elements in $S\big(\mathfrak{t}(\mathbb{C})\big)$. Any element in $S\big(\mathfrak{t}(\mathbb{C})\big)$ can then be written uniquely in the form

$$\sum_j u_j h_T(z_j), \qquad z_j \in \mathcal{Z}(G).$$

For any $n \geq 0$, $f \in \mathcal{C}_{c,N}(G)$, and $\gamma \in V_\Omega$, we write

$$\psi_i^n(\gamma, f) = \psi_{M,i}^n(\gamma, f) = \partial(u_i)\psi_M^n(\gamma, f), \qquad 1 \leq i \leq q.$$

Our aim is to estimate the functions

$$\partial(u)\psi_i^n(\gamma, f), \qquad u \in S\big(\mathfrak{t}(\mathbb{C})\big),\ 1 \leq i \leq q. \tag{7.1}$$

The assertion of the lemma will then follow from the case $i = 1$ and $X = \partial(u)$.

Consider a fixed element $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$. For any i, we can write

$$uu_i = \sum_{j=1}^q h_T(z_{ij})u_j,$$

for operators $z_{ij} = z_{u,ij}$ in $\mathcal{Z}(G)$. This allows us to write (7.1) as the sum of

$$\sum_j \psi_j^n(\gamma, z_{ij}f)$$

with

$$\sum_j \partial(u_j)\big(\partial\big(h_T(z_{ij})\big)\psi_M^n(\gamma, f) - \psi_M^n(\gamma, z_{ij}f)\big). \tag{7.2}$$

We shall estimate the two expressions separately.

The first step is to apply Lemma 6.4 to the summands in (7.2). For any given n, we choose N to be large enough that the summands have the property of Lemma 6.5. In other words, the expression (7.2) is equal to a sum of functions

$$-\sum_j k^n_{z_{ij},\sigma_j}(\gamma, f), \qquad \sigma_j = \big(T, \Omega, \partial(u_j)\big),\ f \in \mathcal{C}_{c,N}(G),$$

defined as in the preamble to Lemma 6.4. Applying Lemma 6.4 to each summand, we obtain a positive number a with the property that for any n, and for each i and j, the functional

$$\nu^{a,n}_{z_{ij},\sigma_j}(f) = \sup_{\gamma \in V_\Omega} \big(|k^n_{z_{ij},\sigma_j}(\gamma, f)||D_c(\gamma)|^a \|\ell_c(\gamma)\|^{-(n+1)}\big)$$

is a continuous seminorm on $\mathcal{C}_{c,N}(G)$. Given a, we write $\nu^{a,n}_u(f)$ for the supremum over $1 \leq i \leq q$ and γ in $V_\Omega(a,n)$ of the absolute value of (7.2). It then follows from the definition of $V_\Omega(a,n)$ that

$$\nu^{a,n}_u(f) \leq C_0 \sup_i \Big(\sum_j \nu^n_{z_{ij},\sigma_j}(f)\Big), \qquad f \in \mathcal{C}_{c,N}(G),$$

where

$$C_0 = \sup_{\gamma \in V_\Omega(a,n)} \|\ell_c(\gamma)\|.$$

We conclude that $\nu^{a,n}_u$ is a continuous seminorm on $\mathcal{C}_{c,N}(G)$. The exponent a depends on the elements $z_{ij} \in \mathcal{Z}(G)$, and these depend in turn on the original elements u. It will be best to express this dependence in terms of an arbitrary positive integer d. For any such d, we can choose an exponent $a = a_d$ so that for any $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$ with $\deg(u) \leq d$, the functional $\nu^{a,n}_u(f)$ is a continuous seminorm on $\mathcal{C}_{c,N}(G)$.

The next step is to combine the estimate we have obtained for (7.2) with the estimate for the functions

$$\psi^n_j(\gamma, z_{ij} f) = \partial(u_j)\widetilde{K}^n_M(\gamma, z_{ij} f)$$

provided by Lemma 6.3. It is a consequence of this lemma that there is an integer b such that for any n, i, and j, the mapping

$$f \longrightarrow \psi^n_j(\gamma, z_{ij} f), \qquad f \in \mathcal{C}_{c,N}(G),$$

is a continuous linear transformation from $\mathcal{C}_{c,N}(G)$ to $F^b_c(V_\Omega, G)$. (Here, N can be any nonnegative integer.) In other words, each functional

$$\sup_{\gamma \in V_\Omega} \big(|D_c(\gamma)|^b |\psi^n_j(\gamma, z_{ij} f)|\big), \qquad f \in \mathcal{C}_{c,N}(G),$$

is a continuous seminorm on $\mathcal{C}_{c,N}(G)$. We can now handle both expressions in the original decomposition of (7.1). Our conclusion is that there is a continuous seminorm μ^n_u on $\mathcal{C}_{c,N}(G)$ such that

$$|\partial(u)\psi^n_i(\gamma, f)| \leq |D_c(\gamma)|^{-b}\mu^n_u(f) + \nu^{a,n}_u(f), \qquad f \in \mathcal{C}_{c,N}(G),$$

for every γ in $V_\Omega(a,n)$. In particular,

$$|\partial(u)\psi^n_i(\gamma, f)| \leq \mu^{a,n}_u(f)|D(\gamma)|^{-b}, \qquad \gamma \in V_\Omega(a,n),\ f \in \mathcal{C}_{c,N}(G),$$

where

$$\mu^{a,n}_u(f) = \mu^n_u(f) + \Big(\sup_{\gamma \in V_\Omega(a,n)} |D_c(\gamma)|^b\Big)\nu^{a,n}_u(f)$$

is a continuous seminorm on $\mathcal{C}_{c,N}(G)$. We need be concerned only with the index $i = 1$. We shall write

$$\psi_u^n(\gamma, f) = \partial(u)\psi_M^n(\gamma, f) = \partial(u)\psi_1^n(\gamma, f),$$

in this case. The last estimate is then

$$|\psi_u^n(\gamma, f)| \leq \mu_u^{a,n}(f)|D_c(\gamma)|^{-b}, \qquad \gamma \in V_\Omega(a,n),\ f \in \mathcal{C}_{c,N}(G),$$

for any element $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$ with $\deg(u) \leq d$. Our task is to establish a stronger estimate, in which $b = 0$. We emphasize that in the estimate we have already obtained, b is independent of u, and therefore also of d and $a = a_d$. It is this circumstance that allows an application of the technique of Harish-Chandra from [H1].

For any $\delta > 0$, set

$$V_{\Omega,\delta}(a,n) = \big\{\gamma \in V_\Omega(a,n) :\ \|\ell_c(\gamma)\| < \delta\big\}.$$

If γ belongs to the complement of $V_{\Omega,\delta}(a,n)$ in $V_\Omega(a,n)$, we have

$$|D_c(\gamma)|^a > \|\ell_c(\gamma)\|^n \geq \delta^n.$$

It follows that the function $|D_c(\gamma)|^b$ is bounded away from 0 on the complement of $V_{\Omega,\delta}(a,n)$ in $V_\Omega(a,n)$. We have therefore only to show that for some δ, the function

$$\sup_{\gamma \in V_{\Omega,\delta}(a,n)} \big(|\psi_u^n(\gamma, f)|\big), \qquad f \in \mathcal{C}_{c,N}(G), \tag{7.3}$$

is a continuous seminorm on $\mathcal{C}_{c,N}(G)$.

Given a and n, we simply choose any $\delta > 0$ that is sufficiently small. We then assign a vector $H \in \mathfrak{t}(\mathbb{R})$ to each point γ in $V_{\Omega,\delta}(a,n)$ in such a way that the line segments

$$\psi_t = \gamma \exp tH, \qquad \gamma \in V_{\Omega,\delta}(a,n),\ 0 \leq t \leq 1, \tag{7.4}$$

are all contained in $V_\Omega(a,n)$, and the end points $\gamma_1 = \gamma \exp H$ all lie in the complement of $V_{\Omega,\delta}(a,n)$ in $V_\Omega(a,n)$. We can in fact arrange that the correspondence $\gamma \to H$ has finite image in $\mathfrak{t}(\mathbb{R})$. We can also assume that the points (7.4) satisfy an inequality

$$|D_c(\gamma_t)|^{-1} \leq C_1 t^{-\dim(G_c/T)}, \qquad 0 < t \leq 1,$$

where C_1 is a constant that is independent of the starting point γ in $V_{\Omega,\delta}(a,n)$. Setting p equal to the product of $\dim(G_c/T)$ with the integer b, and absorbing the constant C_1^b in the seminorm $\mu_u^{a,n}(f)$ above, we obtain an estimate

$$|\psi_u^n(\gamma_t, f)| \leq \mu_u^{a,n}(f)t^{-p}, \qquad f \in \mathcal{C}_{c,N}(G),$$

for each of the points γ_t in (7.4), and any $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$ with $\deg(u) \leq d$.

The last step is to apply the argument from [H1, Lemma 49]. Observe that

$$\frac{d}{dt}\psi_u^n(\gamma_t, f) = \partial(H)\psi_u^n(\gamma_t, f) = \psi_{Hu}^n(\gamma_t, f).$$

Therefore

$$\left|\frac{d}{dt}\psi_u^n(\gamma_t, f)\right| \leq \mu_{Hu}^{a,n}(f)t^{-p},$$

for any γ_t as in (7.4), and any $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$ with $\det(u) \leq d-1$. Combining this estimate with the fundamental theorem of calculus, we obtain

$$\begin{aligned} |\psi_u^n(\gamma_t, f)| &\leq \left| \int_t^1 \Big(\frac{d}{ds} \psi_u^n(\gamma_s, f) \Big) ds \right| + |\psi_u^n(\gamma_1, f)| \\ &\leq \int_t^1 \mu_{Hu}^{a,n}(f) s^{-p} ds + \mu_u^{a,n}(f) \\ &\leq \Big(\frac{1}{p-1} \Big) \mu_{Hu}^{a,n}(f)(t^{-p+1} - 1) + \mu_u^{a,n}(f). \end{aligned}$$

It follows that there is a continuous seminorm $\mu_{u,1}^{a,n}$ on $\mathcal{C}_{c,N}(G)$ such that

$$|\psi_u^n(\gamma_t, f)| \leq \mu_{u,1}^{a,n}(f) t^{-p+1},$$

for any γ_t as in (7.4), and any $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$ with $\deg(u) \leq d-1$. Following the proof of [H1, Lemma 49], we repeat this operation p times. We obtain a continuous seminorm $\mu_{\mu,p}^{a,n}$ on $\mathcal{C}_{c,N}(G)$ such that

$$|\psi_u^n(\gamma_t, f)| \leq \mu_{u,p}^{a,n}(f) |\log t|,$$

for any γ_t as in (7.4), and any $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$ with $\deg(u) \leq d-p$. Repeating the operation one last time, and using the fact that $\log t$ is integrable over $[0,1]$, we conclude that there is a continuous seminorm $\lambda_u^{a,n} = \mu_{u,p+1}^{a,n}$ on $\mathcal{C}_{c,N}(G)$ such that

$$|\psi_u^n(\gamma_t, f)| \leq \lambda_u^{a,n}(f), \qquad f \in \mathcal{C}_{c,N}(G),$$

for all γ_t as in (7.4), and any $u \in S\big(\mathfrak{t}(\mathbb{C})\big)$ with $\deg(u) \leq d-(p+1)$. Setting $t=0$, we see that the supremum (7.3) is bounded by $\lambda_u^{a,n}(f)$, and is therefore a continuous seminorm on $\mathcal{C}_{c,N}(G)$.

We have now finished. Indeed, for the given differential operator $X = \partial(u)$, we set

$$d = \deg(u) + b \dim(G_c/T) + 1 = \deg(u) + p + 1,$$

where b is the absolute exponent above. We then take a to be the associated number a_d. Given a, together with a positive integer n, we choose $\delta > 0$ as above. The functional (7.3) is then a continuous seminorm on $\mathcal{C}_{c,N}(G)$. As we have seen, this yields a proof of the lemma. □

Corollary 7.2. *Given the triplet $\sigma = (T, \Omega, X)$, we can choose a positive number $a > \alpha^+(X)$ with the property that for any $n > a^+$, and for N large relative to n, the limit*

$$\chi_M(\sigma, f) = \lim_{\gamma \to c} X\psi_M^n(\gamma, f), \qquad \gamma \in V_\Omega(a,n),\ f \in \mathcal{C}_{c,N}(G), \tag{7.5}$$

exists, and is continuous in f.

Proof. Once again, the statement is well posed, even though $\psi_M^n(\gamma, f)$ is defined only up to a function

$$\sum_i \phi_i(\gamma) J_i(f)$$

in $\mathcal{F}_{c,n}^\alpha(V,G)$. For it follows from the preamble to Lemma 7.1 that the X-transform of any function in $\mathcal{F}_{c,n}^\alpha(V,G)$ can be written as a product of $\ell_c(\gamma)$ with a function that is bounded on $V_\Omega(a,n)$. In particular, $X\psi_M^n(\gamma, f)$ is well defined up to a function on $V_\Omega(a,n)$ whose limit at c vanishes.

Given σ, and thus X, we choose a so that the assertion of the lemma holds for all the differential operators

$$\partial(H)X, \qquad H \in \mathfrak{t}(\mathbb{C}).$$

The first derivatives

$$\partial(H)X\psi_M^n(\gamma, f), \qquad \gamma \in V_\Omega(a,n),\ f \in \mathcal{C}_{c,N}(G),$$

of the function $X\psi_M^n(\gamma, f)$ are then all bounded on $V_\Omega(a,n)$ by a fixed, continuous seminorm in f. It follows that the function $\gamma \to X\psi_M^n(\gamma, f)$ extends continuously to the closure of $V_\Omega(a,n)$ in a way that is also continuous in f. The limit $\chi_M(\sigma, f)$ therefore exists, and is continuous in f. $\square$

Remarks. 1. As the notation suggests, the limit $\chi_M(\sigma, f)$ is independent of n. For if $m > n$, $\psi_M^m(\gamma, f)$ differs from $\psi_M^n(\gamma, f)$ by a function of γ that lies in $\mathcal{F}_{c,n}^\alpha(V, G)$. As we noted at the beginning of the proof of the corollary, the X-transform of any such function converges to 0 as γ approaches c in $V_\Omega(a,n)$. Of course n must be large relative to $\deg(X)$, and N has in turn to be large relative to n. The point is that for any $\sigma = (T, \Omega, X)$, and for N sufficiently large relative to $\deg(X)$, the limit

$$\chi_M(\sigma, f), \qquad f \in \mathcal{C}_{c,N}(G),$$

can be defined in terms of any appropriately chosen n.

2. Lemma 7.1 and Corollary 7.2 were stated under the assumption that $\dim(G_c/T) > 0$. The excluded case that $\dim(G_c/T) = 0$ is trivial. For in this case, the function $\psi_M^n(\gamma, f)$ on V_Ω extends to a smooth function in an open neighbourhood of c. The lemma and corollary then hold for any n.

§8. The mapping $\widetilde{\chi}_M$

We fix a weight function α satisfying the conditions of Lemma 6.3, as we did in the last section. Then $\alpha(1)$ equals zero, and the continuous mapping

$$\widetilde{K}_M : \mathcal{C}(G) \longrightarrow \widehat{\mathcal{G}}_c(M,G)$$

takes values in the subspace $\widehat{\mathcal{G}}_c^{\alpha}(M,G)$ of $\widehat{\mathcal{G}}_c(M,G)$. Recall that the space $\widehat{\mathcal{I}}_c(M,G)$ introduced in §4 is contained in $\widehat{\mathcal{G}}_c^{\alpha}(M,G)$. The goal of this section is to construct a continuous linear mapping

$$\widetilde{\chi}_M : \mathcal{C}(G) \longrightarrow \widehat{\mathcal{I}}_c(M,G)$$

that approximates $\widetilde{K}_M$.

The main step will be the next proposition, which applies to the restrictions

$$\psi_M^n = \widetilde{K}_M^n : \mathcal{C}_{c,N}(G) \longrightarrow \mathcal{G}_c^{\alpha,n}(M,G)$$

treated in the last section.

Proposition 8.1. *Suppose that $n \geq 0$, and that N is large relative to n. Then there is a uniquely determined continuous linear transformation*

$$\chi_M^n : \mathcal{C}_{c,N}(G) \longrightarrow \mathcal{I}_c^n(M,G),$$

such that for any $f \in \mathcal{C}_{c,N}(G)$, the image of $\chi_M^n(f)$ in $\mathcal{G}_c^{\alpha,n}(M,G)$ equals $\psi_M^n(f)$. More precisely, the mapping

$$f \longrightarrow \psi_M^n(\gamma,f) - \chi_M^n(\gamma,f), \qquad f \in \mathcal{C}_{c,N}(G), \tag{8.1}$$

is a continuous linear transformation from $\mathcal{C}_{c,N}(G)$ to the space $\mathcal{F}_{c,n}^{\alpha}(V,G)$.

Proof. Recall that $\mathcal{I}_c(M,G)$ is contained in the space $\mathcal{G}_c^{bd}(M,G)$ of bounded germs. The first step is to construct χ_M^n as a mapping from $\mathcal{C}_{c,N}(G)$ to the quotient $\mathcal{G}_c^{bd,n}(M,G)$ of $\mathcal{G}_c^{bd}(M,G)$. As we observed in the proof of Lemma 4.3, any element in $\mathcal{G}_c^{bd,n}(M,G)$ can be identified with a family

$$\phi^n = \big\{\phi_\Omega^n : T \in \mathcal{T}_c(M),\ \Omega \in \pi_{0,c}\big(T_{G\text{-reg}}(\mathbb{R})\big)\big\}$$

of Taylor polynomials of degree n (in the coordinates $\ell_c(\gamma)$) on the neighbourhoods V_Ω. In particular, $\mathcal{G}_c^{bd,n}(M,G)$ is finite dimensional. The subspace $\mathcal{I}_c^n(M,G)$ consists of those families that satisfy Harish-Chandra's jump conditions (2.3).

Suppose that f belongs to $\mathcal{C}_{c,N}(G)$, for some N that is large relative to n. We define $\chi_M^n(f)$ as a family of Taylor polynomials of degree n by means of the limits $\chi_M(\sigma,f)$ provided by Corollary 7.2. More precisely, we define

$$\chi_{M,\Omega}^n(\gamma,f), \qquad T \in \mathcal{T}_c(M),\ \Omega \in \pi_{0,c}\big(T_{G\text{-reg}}(\mathbb{R})\big),\ \gamma \in V_\Omega,$$

to be the polynomial of degree n such that

$$\lim_{\gamma \to c} \big(X\chi_{M,\Omega}^n(\gamma,f)\big) = \chi_M(\sigma,f), \qquad \gamma \in V_\Omega, \tag{8.2}$$

where X ranges over the invariant differential operators on $T(\mathbb{R})$ of degree less than or equal to n, and where $\sigma = (T,\Omega,X)$. If (T,Ω) is replaced by a pair (T',Ω') that is $W(M)M(\mathbb{R})$-conjugate to (T,Ω), the corresponding polynomial $\chi_{M,\Omega'}^n(\gamma',f)$ is $W(M)M(\mathbb{R})$-conjugate to $\chi_{M,\Omega}^n(\gamma,f)$. This follows from Corollary 7.2 and the analogous property for the (α,n)-jet $\psi_M^n(\gamma,f)$. Therefore $\chi_M^n(f)$ is a well defined element in $\mathcal{G}_c^{bd,n}(M,G)$.

Moreover, Corollary 7.2 tells us that each limit (8.2) is continuous in f. It follows that $f \to \chi_M^n(f)$ is a continuous linear map from $\mathcal{C}_{c,N}(G)$ to the finite dimensional space $\mathcal{G}_c^{bd,n}(M, G)$.

The main step will be to establish the continuity of the mapping (8.1), defined for some weight α that satisfies the conditions of Lemma 6.3. This amounts to showing that for any triplet $\sigma = (T, \Omega, X)$ in $S_c(M, G)$, and any $\varepsilon > 0$, there is a continuous seminorm $\mu(f)$ on $\mathcal{C}_{c,N}(G)$ such that

$$\big|X\big(\psi_M^n(\gamma, f) - \chi_M^n(\gamma, f)\big)\big| \le \mu(f)|D_c(\gamma)|^{-(\alpha(X)+\varepsilon)}\|\ell_c(\gamma)\|^{(n,X)}, \tag{8.3}$$

for $\gamma \in V_\Omega$. Observe that if $\deg|X| > n$, (8.3) reduces to an inequality

$$|X\widetilde{K}_M^n(\gamma, f)| \le \mu(f)|D_c(\gamma)|^{-(\alpha(X)+\varepsilon)},$$

since $X\chi_M^n(\gamma, f) = 0$ and $(n, X) = 0$. We know from Lemma 6.3 that such an inequality actually holds for any f in the full Schwartz space $\mathcal{C}(G)$. We may therefore assume that $\deg(X) \le n$. We shall derive (8.3) in this case from four other inequalities, in which $\mu_1(f)$, $\mu_2(f)$, $\mu_3(f)$, and $\mu_4(f)$ denote four continuous seminorms on $\mathcal{C}_{c,N}(G)$.

We have first to combine Taylor's formula with Lemma 7.1. This lemma actually applies only to the case that $\dim(G_c/T) > 0$. However, if $\dim(G_c/T) = 0$, $D_c(\gamma)$ equals 1, and the weight function α plays no role. In this case, the estimate (8.3) is a direct application of Taylor's formula, which we can leave to the reader. We shall therefore assume that $\dim(G_c/T) > 0$.

We have fixed data n, $\sigma = (T, \Omega, X)$ and ε, with $\deg(X) \le n$, for which we are trying to establish (8.3). For later use, we also fix a positive number ε', with $\varepsilon' < \varepsilon$. At this point, we have removed from circulation the symbols X and n in terms of which Lemma 7.1 was stated, so our application of the lemma will be to a pair of objects denoted instead by Y and m. We allow Y to range over the invariant differential operators on $T(\mathbb{R})$ with $\deg(Y) \le n + 1$. If

$$a = a_n > \sup_Y \big(\alpha^+(Y)\big) = \overline{\alpha}(n+1) + (n+1)\dim(G_c/T)^{-1},$$

as in Lemma 7.1, we choose m with

$$m > a_n^+ = a_n \dim(G_c/T).$$

The lemma applies to functions $f \in \mathcal{C}_{c,N}(G)$, for N large relative to m (which is the same as being large relative to n, if m is fixed in terms of n). In combination with the fundamental theorem of calculus, it tells us that $\psi_M^m(\gamma, f)$ extends to a function on an open neighbourhood of the closure of $V_\Omega(a_n, m)$ that is continuously differentiable of order $(n + 1)$. The derivatives of this function at $\gamma = c$ are the limits treated in Corollary 7.2. They are independent of m, and can be identified with the coefficients of the polynomial $\chi_M^n(\gamma, f)$ on V_Ω. We can therefore regard $\chi_M^n(\gamma, f)$ as the Taylor polynomial of degree n at $\gamma = c$ (relative to the coordinates $\ell_c(\gamma)$) of the function $\psi_M^m(\gamma, f)$ on $V_\Omega(a_n, m)$. Now if γ belongs to $V_\Omega(a_n, m)$, the set

$$\lambda_t(\gamma) = c\,\exp\big(t\ell_c(\gamma)\big), \qquad 0 < t < 1,$$

is contained in $V_\Omega(a_n, m)$, and may be regarded as the line segment joining c with γ. Applying the bound of Lemma 7.1 to the remainder term (of order $(n + 1)$) in Taylor's theorem, we obtain an estimate

$$\big|X(\psi_M^m(\gamma, f) - \chi_M^n(\gamma, f))\big| \le \mu_1(f)\|\ell_c(\gamma)\|^{(n,X)},$$

for any γ in $V_\Omega(a_n, m)$. We are assuming that $m > n$ and that α satisfies the conditions of Lemma 6.3. The projection of $\psi_M^m(f)$ onto $\mathcal{G}_c^{\alpha,n}(M,G)$ therefore exists, and is equal to $\psi_M^n(f)$. The definitions then yield a second estimate

$$\big|X(\psi_M^m(\gamma,f) - \psi_M^n(\gamma,f)\big)\| \leq \mu_2(f)|D_c(\gamma)|^{-(\alpha(X)+\varepsilon)}\|\ell_c(\gamma)\|^{(n,X)},$$

that is valid for any γ in V_Ω. Combining the two estimates, we see that

$$\big|X\big(\psi_M^n(\gamma,f) - \chi_M^n(\gamma,f)\big)\big| \leq \mu_3(f)|D_c(\gamma)|^{-(\alpha(X)+\varepsilon)}\|\ell_c(\gamma)\|^{(n,X)}, \tag{8.4}$$

for any γ in $V_\Omega(a_n, m)$.

The functions $\psi_M^n(\gamma,f)$ and $\chi_M^n(\gamma,f)$ in (8.4) both belong to the space $\mathcal{F}_c^\alpha(V,G)$. Applying the estimate that defines this space to each of the functions, we obtain a bound

$$\big|X\big(\psi_M^n(\gamma,f) - \chi_M^n(\gamma,f)\big)\big| \leq \mu_4(f)|D_c(\gamma)|^{-(\alpha(X)+\varepsilon')},$$

that holds for every γ in V_Ω. Suppose that γ lies in the complement of $V_\Omega(a_n,m)$ in V_Ω. Then

$$|D_c(\gamma)| \leq \|\ell_c(\gamma)\|^{m'},$$

for the exponent $m' = ma_n^{-1}$. Setting $\delta = \varepsilon - \varepsilon' > 0$, we write

$$|D_c(\gamma)|^{-(\alpha(X)+\varepsilon')} = |D_c(\gamma)|^{-(\alpha(X)+\varepsilon)}|D_c(\gamma)|^{\delta} \leq |D_c(\gamma)|^{-(\alpha(X)+\varepsilon)}\|\ell_c(\gamma)\|^{\delta m'}.$$

We are free to choose m to be as large as we like. In particular, we can assume that

$$\delta m' \geq (n,X),$$

and therefore that

$$|D_c(\gamma)|^{-(\alpha(X)+\varepsilon')} \leq C'|D_c(\gamma)|^{-(\alpha(X)+\varepsilon)}\|\ell_c(\gamma)\|^{(n,X)},$$

for some constant C'. Absorbing C' in the seminorm $\mu_4(f)$, we conclude that

$$\big|X\big(\psi_M^n(\gamma,f) - \chi_M^n(\gamma,f)\big)\big| \leq \mu_4(f)|D_c(\gamma)|^{-(\alpha(X)+\varepsilon)}\|\ell_c(\gamma)\|^{(n,X)}, \tag{8.5}$$

for any γ in the complement of $V_\Omega(a_n,m)$ in V_Ω.

The estimates (8.4) and (8.5) account for all the points γ in V_Ω. Together, they yield an estimate of the required form (8.3), in which we can take

$$\mu(f) = \mu_3(f) + \mu_4(f).$$

We have established the required assertion that for N large relative to n,

$$f \longrightarrow \psi_M^n(\gamma,f) - \chi_M^n(\gamma,f), \qquad f \in \mathcal{C}_{c,N}(G),$$

is a continuous linear transformation from $\mathcal{C}_{c,N}(G)$ to $\mathcal{F}_{c,n}^\alpha(V,G)$. From this, it follows from the definitions that the image of $\chi_M^n(f)$ in $\mathcal{G}_c^{\alpha,n}(M,G)$ equals $\psi_M^n(f)$. In particular, $\psi_M^n(f)$ lies in the subspace $\mathcal{G}_c^{bd,n}(M,G)$ of $\mathcal{G}_c^n(M,G)$.

The space $\mathcal{I}_c^n(M,G)$ is in general a proper subspace of $\mathcal{G}_c^{bd,n}(M,G)$, by virtue of the extra constraints imposed by the jump conditions (2.3). The last step is to show that for suitable N, χ_M^n takes $\mathcal{C}_{c,N}(G)$ to the smaller space $\mathcal{I}_c^n(M,G)$. This will be an application of Lemma 6.1.

Let ξ be a linear form on the finite dimensional space $\mathcal{G}_c^{bd,n}(M,G)$ that vanishes on the subspace $\mathcal{I}_c^n(M,G)$. The mapping

$$J_\xi :\ f \longrightarrow \xi\big(\chi_M^n(f)\big) = \xi\big(\psi_M^n(f)\big), \qquad f \in \mathcal{C}_{c,N}(G),$$

is continuous, and is therefore the restriction to $\mathcal{C}_{c,N}(G)$ of a tempered distribution. Suppose that

$$f = h^y - h, \qquad h \in \mathcal{C}(G),\ y \in G(\mathbb{R}).$$

Then

$$\begin{aligned} J_\xi(f) &= \xi\big(\psi_M^n(h^y - h)\big) \\ &= \xi\big(\widetilde{K}_M^n(h^y) - \widetilde{K}_M^n(h)\big) \\ &= \sum_{Q\in\mathcal{F}^0(M)} \xi\big(g_M^{M,n}\big(J_{M,c}^{M_Q}(h_{Q,y})\big)\big), \end{aligned}$$

by Lemma 6.1. The sum of the n-jets

$$g_M^{M,n}\big(J_{M,c}^{M_Q}(h_{Q,y})\big), \qquad Q \in \mathcal{F}^0(M),$$

lies in the subspace $\mathcal{I}_c^n(M,G)$ on which ξ vanishes. The distribution J_ξ thus annihilates any function of the form $h^y - h$, and is therefore invariant. On the other hand, if $f_0 \in \mathcal{C}(G)$ is compactly supported, and vanishes on a neighbourhood of the closed invariant subset $\mathcal{U}_c(G)$ of $G(\mathbb{R})$, one sees easily from the definitions that $\widetilde{K}_M^n(f_0)$ equals 0. It follows that the distribution J_ξ is supported on $\mathcal{U}_c(G)$. We have established that J_ξ belongs to the space $\mathcal{D}_c(G)$, and is therefore a finite linear combination of distributions in the basis $R_c(G)$. Increasing N if ncessary, we can consequently assume that for each ξ, J_ξ annihilates the space $\mathcal{C}_{c,N}(G)$. In other words, ψ_M^n takes any function $f \in \mathcal{C}_{c,N}(G)$ to the subspace $\mathcal{I}_c^n(M,G)$ of $\mathcal{G}_c^{bd,n}(M,G)$. Since $\psi_M^n(f)$ equals $\chi_M^n(f)$, the image of χ_M^n is also contained in $\mathcal{I}_c^n(M,G)$.

We have now proved that for N large relative to n, $f \to \chi_M^n(f)$ is a continuous linear mapping from $\mathcal{C}_{c,N}(G)$ to $\mathcal{I}_c^n(M,G)$. We have also shown that the image of $\chi_M^n(f)$ in $\mathcal{G}_c^{\alpha,n}(M,G)$ equals $\psi_M^n(f)$. But Lemma 4.3 implies that the mapping of $\mathcal{I}_c^n(M,G)$ into $\mathcal{G}_c^{\alpha,n}(M,G)$ is injective. We conclude that $\chi_M^n(f)$ is uniquely determined. With this last observation, the proof of the proposition is complete. □

The germs $\chi_M^n(f)$ share some properties with the (α,n)-jets $\widetilde{K}_M^n(f)$ from which they were constructed. For example, suppose that $m \geq n$, and that N is large relative to m. Then if f belongs to $\mathcal{C}_{c,N}(G)$, both $\chi_M^n(f)$ and $\chi_M^m(f)$ are defined. But $\psi_M^n(f) = \widetilde{K}_M^n(f)$ is the projection of $\psi_M^m(f) = \widetilde{K}_M^m(f)$ onto $\mathcal{G}_c^{\alpha,n}(M,G)$. It follows that $\chi_M^n(f)$ is the projection of $\chi_M^m(f)$ onto $\mathcal{I}_M^n(M,G)$.

We can reformulate this property in terms of the dual pairing between $\mathcal{D}_c(M)$ and $\widehat{\mathcal{I}}_c(M)$. Recall that $\widehat{\mathcal{I}}_c(M,G)$ is the subspace of $W(M)$-invariant elements in $\widehat{\mathcal{I}}_c(M)$. The pairing

$$\langle\sigma,\phi\rangle, \qquad \sigma \in \mathcal{D}_c(M),\ \phi \in \widehat{\mathcal{I}}_c(M,G), \tag{8.6}$$

therefore identifies $\widehat{\mathcal{I}}_c(M,G)$ with the dual $\mathcal{D}_c(M)^*_{W(M)}$ of the space $\mathcal{D}_c(M)_{W(M)}$ of $W(M)$-covariants of $\mathcal{D}_c(M)$. If σ belongs to the finite dimensional subspace

$$\mathcal{D}_{c,n}(M) = \{\sigma \in \mathcal{D}_c(M) :\ \deg(\sigma) \leq n\}$$

of $\mathcal{D}_c(M)$ spanned by $R_{c,n}(M)$, the value

$$\langle\sigma,\phi^n\rangle = \langle\sigma,\phi\rangle$$

depends only on the image ϕ^n of ϕ in $\mathcal{I}_c^n(M,G)$. With this notation, we set

$$\langle\sigma,\chi_M(f)\rangle = \langle\sigma,\chi_M^n(f)\rangle, \qquad \sigma\in\mathcal{D}_c(M),\ f\in\mathcal{C}_{c,N}(G), \tag{8.7}$$

for any $n\geq \deg(\sigma)$ and for N large relative to n. In view of the projection property above, the pairing (8.7) is independent of the choice of n. It is defined for any N that is large relative to $\deg(\sigma)$.

Another property that $\chi_M^n(f)$ inherits is the differential equation (6.6) satisfied by $\widetilde{K}_M(f)$. We shall state it in terms of the pairing (8.7).

Lemma 8.2. *Suppose that $z\in\mathcal{Z}(G)$ and $\sigma\in\mathcal{D}_c(M)$, and that N is large relative to $\deg(\sigma)+\deg(z)$. Then*

$$\langle\sigma,\chi_M(zf)\rangle = \langle z_M\sigma,\chi_M(f)\rangle, \tag{8.8}$$

for any $f\in\mathcal{C}_{c,N}(G)$.

Proof. The assertion is an reformulation of Lemma 6.2 in terms of the objects $\chi_M^n(f)$. Its proof, like that of Lemmas 6.4 and 6.5, is quite straightforward. We can afford to be brief.

We choose positive integers $n_1\geq\deg(\sigma)$ and $n\geq n_1+\deg(z)$, and we assume that N is large relative to n. If f belongs to $\mathcal{C}_{c,N}(G)$, zf belongs to the space $\mathcal{C}_{c,N_1}(G)$, where $N_1=N-\deg(z)$ is large relative to n_1. We can therefore take the pairing

$$\langle\sigma,\chi_M(zf)\rangle = \langle\sigma,\chi_M^{n_1}(zf)\rangle.$$

We can also form the pairing

$$\langle z_M\sigma,\chi_M(f)\rangle = \langle z_M\sigma,\chi_M^n(f)\rangle,$$

which can be written as

$$\big\langle\sigma,\partial\big(h(z)\big)\chi_M^n(f)\big\rangle = \big\langle\sigma,\big(\partial\big(h(z)\big)\chi_M^n(f)\big)^{n_1}\big\rangle,$$

since the action of z_M on $\mathcal{D}_c(M)$ is dual to the action of $\partial\big(h(z)\big)$ on $\widehat{\mathcal{I}}_c(M,G)$. We have to show that the difference

$$\langle\sigma,\chi_M(zf)\rangle-\langle z_M\sigma,\chi_M(f)\rangle = \big\langle\sigma,\chi_M^{n_1}(zf)-\big(\partial\big(h(z)\big)\chi_M^n(f)\big)^{n_1}\big\rangle$$

vanishes.

Combining Lemma 6.2 with the various definitions, we see that

$$\begin{aligned}
&\chi_M^{n_1}(zf)-\big(\partial\big(h(z)\big)\chi_M^n(f)\big)^{n_1}\\
=&\psi_M^{n_1}(zf)-\big(\partial\big(h(z)\big)\psi_M^n(f)\big)^{n_1}\\
=&\big(\widetilde{K}_M^{n_1}(zf)-\partial\big(h(z)\big)\widetilde{K}_M^n(f)\big)^{n_1}\\
=&\big(\widetilde{K}_M(zf)-\partial\big(h(z)\big)\widetilde{K}_M(f)\big)^{n_1}\\
=&\sum_{L\neq M}\big(\partial_M^L(z_L)g_L^G(f_{G,c})_M\big)^{n_1}\\
=&\sum_{L\neq M}\big(\partial_M^L(z_L)g_L^{G,n}(f_{G,c})_M\big)^{n_1}.
\end{aligned}$$

We apply (5.3) inductively to the formal germs

$$g_L^{G,n}(f_{G,c})_M=\sum_{\rho\in R_c(G)}g_L^{G,n}(\rho)_M f_G(\rho), \qquad L\neq M,$$

as in the proof of Lemma 6.5. Since N is large relative to n, we conclude that these objects all vanish. The equation (8.8) follows. $\square$

Finally, it is clear that $\chi_M^n(f)$ inherits the symmetry property (6.9), relative to an isomorphism θ: $G \to \theta G$ over $\mathbb{R}$. If N is large relative to a given $\sigma \in \mathcal{D}_c(M)$, we obtain

$$\langle \theta\sigma, \chi_{\theta M}(\theta f)\rangle = \langle \sigma, \chi_M(f)\rangle, \tag{8.9}$$

for any $f \in \mathcal{C}_{c,N}(G)$. This property will be of special interest in the case that θ belongs to the group $\mathrm{Aut}(G, K, M, c)$ of automorphisms of the 4-tuple (G, K, M, c).

We shall now construct $\widetilde{\chi}_M$ as an extension of the family of mappings $\{\chi_M^n\}$.

Proposition 8.3. *There is a continuous linear mapping*

$$\widetilde{\chi}_M : \mathcal{C}(G) \to \widehat{\mathcal{I}}_c(M, G) \tag{8.10}$$

that satisfies the restriction condition

$$\langle \sigma, \widetilde{\chi}_M(f)\rangle = \langle \sigma, \chi_M(f)\rangle, \qquad f \in \mathcal{C}_{c,N}(G), \tag{8.11}$$

for any $\sigma \in \mathcal{D}_c(M)$ and N large relative to $\deg(\sigma)$, *the differential equation*

$$\langle \sigma, \widetilde{\chi}_M(zf)\rangle = \langle z_M\sigma, \widetilde{\chi}_M(f)\rangle, \qquad z \in \mathcal{Z}(G),\ f \in \mathcal{C}(G), \tag{8.12}$$

for any $\sigma \in \mathcal{D}_c(M)$, and the symmetry condition

$$\langle \theta\sigma, \widetilde{\chi}_M(\theta f)\rangle = \langle \sigma, \widetilde{\chi}_M(f)\rangle, \qquad f \in \mathcal{C}(G), \tag{8.13}$$

for any $\sigma \in \mathcal{D}_c(M)$ and $\theta \in \mathrm{Aut}(G, K, M, c)$.

Proof. Let

$$\mathcal{D}_{c,1}(M) = \mathcal{D}_{c,G\text{-harm}}(M)$$

be the space of G-harmonic elements in $\mathcal{D}_c(M)$. This is a finite dimensional subspace of $\mathcal{D}_c(M)$, which is invariant under the action of $W(M)$ and, more generally, the group $\mathrm{Aut}(G, K, M, c)$. (Here we are regarding c as a $W(M)$-orbit in $\Gamma_{ss}(M)$.) Choose a positive integer N_1 that is large enough that the pairing (8.7) is defined for every σ in $\mathcal{D}_{c,1}(M)$ and f in $\mathcal{C}_{c,N_1}(G)$. We thereby obtain a continuous linear map

$$\chi_{M,1} : \mathcal{C}_{c,N_1}(G) \longrightarrow \mathcal{D}_{c,1}(M)^*,$$

which by (8.9) is fixed under the action of the group $\mathrm{Aut}(G, K, M, c)$. Let

$$\widetilde{\chi}_{M,1} : \mathcal{C}(G) \longrightarrow \mathcal{D}_{c,1}(M)^* \tag{8.14}$$

be any linear extension of this mapping to $\mathcal{C}(G)$ that remains fixed under the action of $\mathrm{Aut}(G, K, M, c)$. Since $\mathcal{C}_{c,N_1}(G)$ is of finite codimension in $\mathcal{C}(G)$, $\widetilde{\chi}_{M,1}$ is automatically continuous. With this mapping, we obtain a pairing $\langle \sigma, \widetilde{\chi}_{M,1}(f)\rangle$, for elements $\sigma \in \mathcal{D}_{c,1}(M)$ and functions $f \in \mathcal{C}(G)$, that satisfies (8.11) and (8.13).

The extension of the pairing to all elements $\sigma \in \mathcal{D}_c(M)$ is completely determined by the differential equations (8.12). According to standard properties of harmonic polynomials, the map

$$z \otimes \sigma \longrightarrow z_M\sigma, \qquad z \in \mathcal{Z}(G),\ \sigma \in \mathcal{D}_{c,1}(M),$$

is a linear isomorphism from $\mathcal{Z}(G) \otimes \mathcal{D}_{c,1}(M)$ onto $\mathcal{D}_c(M)$. Any element in $\mathcal{D}_c(M)$ is therefore a finite linear combination of elements

$$\sigma = z_{1,M}\sigma_1, \qquad z_1 \in \mathcal{Z}(G),\ \sigma_1 \in \mathcal{D}_{c,1}(M).$$

For any σ of this form, we define

$$\langle \sigma, \widetilde{\chi}_M(f)\rangle = \langle \sigma_1, \widetilde{\chi}_{M,1}(z_1 f)\rangle, \qquad f \in \mathcal{C}(G).$$

Since $z_{1,M}$ is $W(M)$-invariant, the values taken by this pairing at a given f determine a linear form on the quotient $\mathcal{D}_c(M)_{W(M)}$ of $\mathcal{D}_c(M)$. The pairing therefore defines a continuous mapping (8.10) that satisfies the differential equation (8.12). The restriction condition (8.11) follows from (8.12), the associated differential equation (8.8) for $\langle \sigma, \chi_M(f)\rangle$, and the special case of $\sigma \in \mathcal{D}_{c,1}(M)$ that was built into the definition. The symmetry condition (8.13) follows from the compatibility of $\mathrm{Aut}(G, K, M, c)$ with the action of $\mathcal{Z}(G)$, and again the special case of $\sigma \in \mathcal{D}_{c,1}(M)$. Our construction is complete. □

Remarks. 1. The three properties of $\widetilde{\chi}_M$ can of course be stated without recourse to the pairing (8.6). The restriction condition (8.11) can be formulated as the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{C}_{c,N}(G) & \xrightarrow{\chi_M^n} & \mathcal{I}_c^n(M,G) \\ \big\downarrow & & \big\uparrow \\ \mathcal{C}(G) & \xrightarrow{\widetilde{\chi}_M} & \widehat{\mathcal{I}}_c(M,G), \end{array}$$

for any $n \geq 0$ and N large relative to n. The differential equation (8.12) has a dual version

$$\widetilde{\chi}_M(zf) = \partial\big(h(z)\big)\widetilde{\chi}_M(f), \qquad z \in \mathcal{Z}(G),\ f \in \mathcal{C}(G),$$

that is similar to (1.2). The symmetry condition (8.13) is essentially just the equation

$$\theta(\widetilde{\chi}_M(f) = \widetilde{\chi}_M(\theta f), \qquad \theta \in \mathrm{Aut}(G, K, M, c),\ f \in \mathcal{C}(G).$$

2. The mapping $\widetilde{\chi}_M$ is completely determined up to translation by an $\mathrm{Aut}(G, K, M, c)$-fixed linear transformation

$$C:\ \mathcal{C}(G)/\mathcal{C}_{c,N_1}(G) \longrightarrow \mathcal{D}_{c,G\text{-harm}}(M)^*. \tag{8.15}$$

The space of such linear transformations is of course finite dimensional.

§9. Completion of the proof

We shall now complete the proof of Theorems 5.1 and 5.1^*. We have to construct distributions (5.1) with $L = M$, and formal germs (5.2) with $L = G$, that satisfy the conditions (5.3)–(5.9). The key to the construction is the mapping

$$\widetilde{\chi}_M : \mathcal{C}(G) \longrightarrow \widehat{\mathcal{I}}_c(M, G)$$

of Proposition 8.3.

The distributions (5.1) are in fact built into $\widetilde{\chi}_M$. If ρ belongs to $R_c(M)$, we simply set

$$J_M(\rho, f) = \langle \rho, \widetilde{\chi}_M(f) \rangle, \qquad f \in \mathcal{C}(G). \tag{9.1}$$

Since $\widetilde{\chi}_M$ is continuous, the linear forms

$$f \longrightarrow J_M(\rho, f), \qquad \rho \in R_c(M),$$

are tempered distributions. We must check that they are supported on $\mathcal{U}_c(G)$.

Suppose that f_0 is a function in $\mathcal{C}(G)$ that is compactly supported, and vanishes on a neighbourhood of $\mathcal{U}_c(G)$. As we noted near the end of the proof of Proposition 8.1, the definitions imply that the (α, n)-jet

$$\widetilde{K}^n_M(f_0) = J^n_M(f_0) - \sum_{L \neq M, G} \sum_{\rho_L \in R_c(L)} g^{L,n}_M(\rho_L) J_L(\rho_L, f_0)$$

vanishes for $n \geq 0$. Indeed, the weighted orbital integral $J_M(\gamma, f_0)$ vanishes for γ near $\mathcal{U}_c(G)$, while our induction hypothesis includes the assumption that the distributions $J_L(\rho_L, f)$ are supported on $\mathcal{U}_c(G)$. Given $\rho \in R_c(M)$, we choose any $n \geq \deg(\rho)$. Then

$$\begin{aligned} J_M(\rho, f_0) &= \langle \rho, \widetilde{\chi}_M(f_0) \rangle = \langle \rho, \chi_M(f_0) \rangle \\ &= \langle \rho, \chi^n_M(f_0) \rangle = \langle \rho, \psi^n_M(f_0) \rangle \\ &= \langle \rho, \widetilde{K}^n_M(f_0) \rangle = 0, \end{aligned}$$

by (8.11), (8.7) and Proposition 8.1. The distribution $J_M(\rho, f)$ is therefore supported on $\mathcal{U}_c(G)$.

We have now constructed the distributions (5.1), in the remaining case that $L = M$. The required conditions (5.4)–(5.6) (with $L = M$) amount to properties of $\widetilde{\chi}_M$ we have already established. The functorial condition (5.4) concerns the (adically) convergent series

$$g^M_M\big(J_{M,c}(f)\big) = \sum_{\rho \in R_c(M)} \rho^\vee J_M(\rho, f).$$

Observe that

$$\rho\big(g^M_M\big(J_{M,c}(f)\big)\big) = J_M(\rho, f) = \langle \rho, \widetilde{\chi}_M(f) \rangle,$$

for any $\rho \in R_c(M)$. It follows that

$$g^M_M\big(J_{M,c}(f)\big) = \widetilde{\chi}_M(f), \qquad f \in \mathcal{C}(G). \tag{9.2}$$

Since $\widetilde{\chi}_M(f)$ was constructed without recourse to the basis $R_c(M)$, the same is true of $g^M_M\big(J_{M,c}(f)\big)$.

To check the variance condition (5.5), we note that for $f \in \mathcal{C}(G)$ and $y \in G(\mathbb{R})$, the function $f^y - f$ belongs to each of the spaces $\mathcal{C}_{c,N}(G)$. Given $\rho \in R_c(M)$, we may therefore write

$$\begin{aligned} J_M(\rho, f^y - f) &= \langle \rho, \widetilde{\chi}_M(f^y - f)\rangle = \langle \rho, \chi_M(f^y - f)\rangle \\ &= \langle \rho, \psi_M(f^y - f)\rangle = \langle \rho, \widetilde{K}_M(f^y - f)\rangle \\ &= \sum_{Q \in \mathcal{F}^0(M)} \rho\big(g_M^M\big(J_{M,c}^{M_Q}(f_{Q,y})\big)\big) \\ &= \sum_{Q \in \mathcal{F}^0(M)} J_M^{M_Q}(\rho, f_{Q,y}), \end{aligned}$$

by Lemma 6.1. The formula (5.5) follows.

The differential equation (5.6) is even simpler. If $z \in \mathcal{Z}(G)$ and $\rho \in R_c(M)$, we use (8.12) to write

$$\begin{aligned} J_M(\rho, zf) &= \langle \rho, \widetilde{\chi}_M(zf)\rangle \\ &= \langle z_M\rho, \widetilde{\chi}_M(f)\rangle = J_M(z_M\rho, f). \end{aligned}$$

This is the required equation.

To deal with the other assertions of the two theorems, we set

$$K'_M(f) = \widetilde{K}_M(f) - \widetilde{\chi}_M(f) = \widetilde{K}_M(f) - g_M^M\big(J_{M,c}(f)\big).$$

Then $f \to K'_M(f)$ is a continuous linear transformation from $\mathcal{C}(G)$ to $\widehat{\mathcal{G}}_c(M, G)$, which can be expanded in the form

$$K'_M(f) = J_M(f) - \sum_{L \in \mathcal{L}^0(M)} \sum_{\rho \in R_c(L)} g_M^L(\rho) J_L(\rho, f).$$

Suppose that α is as before, a weight function that satisfies the conditions of Lemma 6.3. Then $K'_M(f)$ lies in the subspace $\widehat{\mathcal{G}}_c^\alpha(M, G)$ of $\widehat{\mathcal{G}}_c(M, G)$. For any n, $K'_M(f)$ projects to the (α, n)-jet

$$K_M^{\prime,n}(f) = \widetilde{K}_M^n(f) - \widetilde{\chi}_M^n(f)$$

in $\mathcal{G}_c^{\alpha,n}(M, G)$, which in turn comes with a representative

$$\begin{aligned} K_M^{\prime,n}(\gamma, f) &= \widetilde{K}_M^n(\gamma, f) - \widetilde{\chi}_M^n(\gamma, f) \\ &= J_M(\gamma, f) - \sum_{L \in \mathcal{L}^0(M)} \sum_{\rho \in R_c(L)} g_M^{L,n}(\gamma, \rho) J_L(\rho, f) \end{aligned}$$

in $\mathcal{F}_c^\alpha(V, G)$. We shall use the mappings $K_M^{\prime,n}$ to construct the remaining germs (5.2). The argument at this point is quite similar to that of the p-adic case [A3, §9].

Since we are considering the case $L = G$ of (5.2), we take ρ to be an element in the basis $R_c(G)$. If N is a large positive integer, let f_ρ^N denote a function in $\mathcal{C}(G)$ with the property that for any ρ_1 in the subset $R_{c,N}(G)$ of $R_c(G)$, the condition

(9.3) $$f_{\rho,G}^N(\rho_1) = \begin{cases} 1, & \text{if } \rho_1 = \rho, \\ 0, & \text{if } \rho_1 \neq \rho, \end{cases}$$

holds. Suppose that $n \geq 0$. Taking N to be large relative to n, we define

(9.4) $$g_M^{G,n}(\rho) = K_M^{\prime,n}(f_\rho^N).$$

Then $g_M^{G,n}(\rho)$ is an element in $\mathcal{G}_c^{\alpha,n}(M,G)$. Suppose that N' is another integer, with $N' \geq N$, and that $f_\rho^{N'}$ is a corresponding function (9.3). The difference

$$f_\rho^{N,N'} = f_\rho^N - f_\rho^{N'}$$

then lies in $\mathcal{C}_{c,N}(G)$. From Propositions 8.1 and 8.3, we see that

$$\begin{aligned} K_M'^{,n}(f_\rho^N) - K_M'^{,n}(f_\rho^{N'}) &= K_M'^{,n}(f_\rho^{N,N'}) \\ &= \widetilde{K}_M^n(f_\rho^{N,N'}) - \widetilde{\chi}_M^n(f_\rho^{N,N'}) \\ &= \psi_M^n(f_\rho^{N,N'}) - \chi_M^n(f_\rho^{N,N'}) = 0. \end{aligned}$$

It follows that the (α,n)-jet $g_M^{G,n}(\rho)$ depends only on ρ and n. It is independent of both N and the choice of function f_ρ^N.

Suppose that $m \geq n$, and that N is large relative to m. Then

$$g_M^{G,m}(\rho) = K_M'^{,m}(f_\rho^N)$$

is an element in $\mathcal{G}_c^{\alpha,m}(M,G)$. Since the image of $K_M'^{,m}(f_\rho^N)$ in $\mathcal{G}_c^{\alpha,n}(M,G)$ equals $K_M'^{,n}(f_\rho^N)$, by definition, the image of $g_M^{G,m}(\rho)$ in $\mathcal{G}_c^{\alpha,n}(M,G)$ equals $g_M^{G,n}(\rho)$. We conclude that the inverse limit

$$g_M^G(\rho) = \varprojlim_n g_M^{G,n}(\rho)$$

exists, and defines an element in $\widehat{\mathcal{G}}_c^\alpha(M,G)$. This completes our construction of the formal germs (5.2), in the remaining case that $L = G$. As we agreed in §4, we can represent them by asymptotic series

$$g_M^G(\gamma,\rho) = \sum_{n=0}^{\infty} g_M^{G,(n)}(\gamma,\rho),$$

where

$$g_M^{G,(n)}(\gamma,\rho) = g_M^{G,n}(\gamma,\rho) - g_M^{G,n-1}(\gamma,\rho),$$

and

$$g_M^{G,n}(\gamma,\rho) = K_M'^{,n}(\gamma,f_\rho^N) = \widetilde{K}_M^n(\gamma,f_\rho^N) - \widetilde{\chi}_M^n(\gamma,f_\rho^N). \tag{9.5}$$

Suppose that N is large relative to n, and that ρ lies in the complement of $R_{c,N}(G)$ in $R_c(G)$. Taking $f_\rho^N = 0$ in this case, we deduce that

$$g_M^{G,n}(\rho) = K_M'^{,n}(f_\rho^N) = 0.$$

In other words, $g_M^{G,n}(\rho)$ vanishes whenever $\deg(\rho)$ is large relative to n. This is the property (5.3). It implies that for any $f \in \mathcal{C}(G)$, the series

$$g_M^G(f_{G,c}) = g_M^G\big(J_{G,c}(f)\big) = \sum_{\rho \in R_c(G)} g_M^G(\rho) f_{G,c}(\rho)$$

converges in (the adic topology of) $\widehat{\mathcal{G}}_c(M,G)$.

Before we establish other properties of the formal germs $g_M^G(\rho)$, let us first prove the main assertion of Theorem 5.1*. Having defined the series $g_M^G(f_{G,c})$, we set

$$\begin{aligned} K_M(f) &= K'_M(f) - g_M^G(f_{G,c}) \\ &= J_M(f) - \sum_{L\in\mathcal{L}(M)}\sum_{\rho\in R_c(L)} g_M^L(\rho)J_L(\rho,f). \end{aligned}$$

Then $f \to K_M(f)$ is a continuous linear mapping from $\mathcal{C}(G)$ to $\widehat{\mathcal{G}}_c(M,G)$. For any n, $K_M(f)$ projects to the element

$$K_M^n(f) = K_M'^{,n}(f) - g_M^{G,n}(f_{G,c})$$

in $\mathcal{G}_c^{\alpha,n}(M,G)$, which in turn comes with a representative

$$\begin{aligned} K_M^n(\gamma,f) &= K_M'^{,n}(\gamma,f) - g_M^{G,n}(\gamma,f_{G,c}) \\ &= K_M'^{,n}(\gamma,f) - \sum_{\rho\in R_c(G)} g_M^{G,n}(\gamma,\rho)f_G(\rho) \end{aligned}$$

in $\mathcal{F}_c^\alpha(V,G)$. We can also write

$$\begin{aligned} K_M^n(\gamma,f) &= J_M(\gamma,f) - \sum_{L\in\mathcal{L}(M)}\sum_{\rho\in R_c(L)} g_M^{L,n}(\gamma,\rho)J_L(\rho,f) \\ &= J_M(\gamma,f) - J_M^n(\gamma,f), \end{aligned}$$

in the notation (5.11). By construction, $f \to K_M^n(\gamma,f)$ is a continuous linear mapping from $\mathcal{C}(G)$ to $\mathcal{F}_c^\alpha(V,G)$. Theorem 5.1^* asserts that the mapping actually sends $\mathcal{C}(G)$ continuously to the space $\mathcal{F}_{c,n}^\alpha(V,G)$.

Given n, we once again choose N to be large. It is a consequence of (5.3) that the sum in the first formula for $K_M^n(\gamma,f)$ may be taken over the finite subset $R_{c,N}(G)$ of $R_c(G)$. It follows that

$$\begin{aligned} K_M^n(\gamma,f) &= K_M'^{,n}(\gamma,f) - \sum_{\rho\in R_{c,N}(G)} g_M^{G,n}(\gamma,\rho)f_G(\rho) \\ &= K_M'^{,n}(\gamma,f) - \sum_{\rho\in R_{c,N}(G)} K_M'^{,n}(\gamma,f_\rho^N)f_G(\rho). \end{aligned}$$

The mapping

$$f \longrightarrow K_M'^{,n}(\gamma,f) = \widetilde{K}_M^n(\gamma,f) - \widetilde{\chi}_M^n(\gamma,f)$$

from $\mathcal{C}(G)$ to $\mathcal{F}_c^\alpha(V,G)$ is of course linear. Consequently,

$$K_M^n(\gamma,f) = K_M'^{,n}(\gamma,f^{c,N}),$$

where

$$f^{c,N} = f - \sum_{\rho\in R_{c,N}(G)} f_G(\rho)f_\rho^N.$$

Observe that the mapping

$$f \longrightarrow f^{c,N}, \qquad f\in\mathcal{C}(G), \tag{9.6}$$

is a continuous linear operator on $\mathcal{C}(G)$ that takes values in the subspace $\mathcal{C}_{c,N}(G)$ of $\mathcal{C}(G)$. In particular, the function

$$K_M^n(\gamma,f) = \widetilde{K}_M^n(\gamma,f^{c,N}) - \widetilde{\chi}_M^n(\gamma,f^{c,N})$$

equals

$$\psi_M^n(\gamma, f^{c,N}) - \chi_M^n(\gamma, f^{c,N}),$$

by the restriction condition (8.11). Composing (9.6) with the mapping (8.1) of Proposition 8.1, we conclude that

$$f \longrightarrow K_M^n(\gamma, f), \qquad f \in \mathcal{C}(G),$$

is a continuous linear mapping from $\mathcal{C}(G)$ to $\mathcal{F}_{c,n}^{\alpha}(V, G)$. This is the main assertion of Theorem 5.1^*.

We have finished our inductive construction of the objects (5.1) and (5.2). We have also established the continuity assertion of Theorem 5.1^*. As we noted in §5, this implies the assertion of Theorem 5.1 that the weighted orbital integral $J_M(f)$ represents the same element in $\widehat{\mathcal{G}}_c(M, G)$ as the formal germ (5.10). We have therefore an identity

$$g_M^G\big(J_{G,c}(f)\big) = J_M(f) - \sum_{L \in \mathcal{L}^0(M)} g_M^L\big(J_{L,c}(f)\big) \tag{9.7}$$

of formal germs, which holds for any function $f \in \mathcal{C}(G)$. According to our induction assumption, the summands in (9.7) with $L \neq M$ are independent of the choice of bases $R_c(L)$. The same is true of the summand with $L = M$, as we observed earlier in this section. Since the other term on the right hand side of (9.7) is just the weighted orbital integral $J_M(f)$, the left hand side of (9.7) is also independent of any choice of bases. We have thus established the functorial condition (5.4) in the remaining case that $L = G$.

We can also use (9.7) to prove the differential equation (5.7). Suppose that $z \in \mathcal{Z}(G)$. In §6, we established the identity (6.2) as a consequence of the two sets of equations (5.6) and (5.7). Since we now have these equations for any $L \neq G$, we can assume that (6.2) also holds for any $L \neq G$. It follows that

$$\begin{aligned}
&\sum_{L \in \mathcal{L}^0(M)} g_M^L\big(J_{L,c}(zf)\big) \\
&= \sum_{L \in \mathcal{L}^0(M)} \sum_{S \in \mathcal{L}^L(M)} \partial_M^S(z_S) g_S^L\big(J_{L,c}(f)\big)_M \\
&= \sum_{S \in \mathcal{L}(M)} \sum_{L \in \mathcal{L}^0(S)} \partial_M^S(z_S) g_S^L\big(J_{L,c}(f)\big)_M.
\end{aligned}$$

We combine this with (9.7) (f being replaced by zf), and the differential equation (3.2) for $J_M(zf)$. We obtain

$$\begin{aligned}
&g_M^G\big(J_{G,c}(zf)\big) \\
&= \sum_{S \in \mathcal{L}(M)} \partial_M^S(z_S)\Big(J_S(f) - \sum_{L \in \mathcal{L}^0(S)} g_S^L\big(J_{L,c}(f)\big)\Big)_M \\
&= \sum_{S \in \mathcal{L}(M)} \partial_M^S(z_S) g_S^G\big(J_{G,c}(f)\big)_M \\
&= \sum_{\rho \in R_c(G)} \Big(\sum_{S \in \mathcal{L}(M)} \partial_M^S(z_S) g_S^G(\rho)_M\Big) f_G(\rho).
\end{aligned}$$

But

$$g_M^G\big(J_{G,c}(zf)\big) = \sum_{\rho \in R_c(G)} g_M^G(\rho)(zf)_G(\rho) = \sum_{\rho \in R_c(G)} g_M^G(\widehat{z}\rho) f_G(\rho).$$

Comparing the coefficients of $f_G(\rho)$ in the two expressions, we see that

$$g_M^G(\widehat{z}\rho) = \sum_{S \in \mathcal{L}(M)} \partial_M^S(z_S) g_S^G(\rho)_M, \qquad \rho \in R_c(G).$$

This is the equation (5.7) in the remaining case that $L = G$.

It remains only to check the symmetry conditions (5.8), (5.9) and $(5.9)^*$. Given an isomorphism θ: $G \to \theta G$ over $\mathbb{R}$, we need to prescribe the mapping

$$\widetilde{\chi}_{\theta M} : \mathcal{C}(\theta G) \longrightarrow \widehat{\mathcal{I}}_{\theta G}(\theta M, \theta G)$$

of Proposition 8.3 for the 4-tuple $(G_1, K_1, M_1, c_1) = (\theta G, \theta K, \theta M, \theta c)$ in terms of the chosen mapping $\widetilde{\chi}_M$ for (G, K, M, c). We do so in the obvious way, by setting

$$\widetilde{\chi}_{\theta M}(\theta f) = \theta \widetilde{\chi}_M(f), \qquad f \in \mathcal{C}(G).$$

This mapping depends of course on (G_1, K_1, M_1, c_1), but by the symmetry condition (8.13) for G, it is independent of the choice of θ. The conditions (8.11)–(8.13) for θG follow from (8.9) and the corresponding conditions for G. Having defined the mapping $\widetilde{\chi}_{\theta M}$, we then need only appeal to the earlier discussion of this section. If ρ belongs to $R_c(M)$, we obtain

$$\begin{aligned} J_{\theta M}(\theta\rho, \theta f) &= \langle \theta\rho, \widetilde{\chi}_{\theta M}(\theta f)\rangle \\ &= \langle \theta\rho, \theta\widetilde{\chi}_M(f)\rangle = J_M(\rho, f), \end{aligned}$$

from (9.1). This is the condition (5.8) in the remaining case that $L = M$. For $n \geq 0$, we also obtain

$$\begin{aligned} g_{\theta M}^{\theta G, n}(\theta\gamma, \theta\rho) &= \widetilde{K}_{\theta M}^{n}(\theta\gamma, f_{\theta\rho}^{N}) - \widetilde{\chi}_{M}^{n}(\theta\gamma, f_{\theta\rho}^{N}) \\ &= \widetilde{K}_{\theta M}^{n}(\theta\gamma, \theta f_{\rho}^{N}) - \widetilde{\chi}_{M}^{n}(\theta\gamma, \theta f_{\rho}^{N}) \\ &= \widetilde{K}_{M}^{n}(\gamma, f_{\rho}^{N}) - \widetilde{\chi}_{M}^{n}(\gamma, f_{\rho}^{N}) \\ &= g_M^{G,n}(\gamma, \rho), \end{aligned}$$

from (9.5), (6.10) and the definition of $\widetilde{\chi}_{\theta M}(\theta f)$ above. This is condition $(5.9)^*$ in the remaining case that $L = G$. Finally, we observe that

$$\begin{aligned} g_{\theta M}^{\theta G}(\theta\rho) &= \lim_{\substack{\longleftarrow \\ n}} g_{\theta M}^{\theta G, n}(\theta\rho) \\ &= \lim_{\substack{\longleftarrow \\ n}} \theta g_M^{G,n}(\rho) = \theta g_M^G(\rho). \end{aligned}$$

This is the third symmetry condition (5.9), in the remaining case $L = G$.

We have now established the last of the conditions of Theorems 5.1 and 5.1^*. This brings us to the end of the induction argument begun in §6, and completes the proof of the two theorems.

We observed in §5 that the objects we have now constructed are not unique. The definitions of this section do depend canonically on the mapping $\widetilde{\chi}_M$ of Proposition 8.3, which is in turn determined by the mapping $\widetilde{\chi}_{M,1}$ in (8.14). But $\widetilde{\chi}_{M,1}$ is uniquely determined only up to translation by the $\mathrm{Aut}(G, K, M, c)$-fixed linear transformation C in (8.15). The coefficients $c(\rho_M, \rho_G)$ in (5.14), which were used in Proposition 5.2 to describe the lack of uniqueness, are of course related to C. Suppose that the basis $R_c(M)$ is chosen so that the subset

$$R_{c,G\text{-harm}}(M) = R_c(M) \cap \mathcal{D}_{c,G\text{-harm}}(M)$$

is a basis of $\mathcal{D}_{c,G\text{-harm}}(M)$. It then follows that

$$c(\rho_M, \rho_G) = \langle {}^t C\rho_M, \rho_G\rangle, \qquad \rho_M \in R_{c,G\text{-harm}}(M),\ \rho_G \in R_{c,N_1}(G).$$

For general ρ_M and ρ_G, the coefficient $c(\rho_M, \rho_G)$ is determined from this special case by the relation (5.16).

§10. Invariant distributions $I_M(\rho, f)$

Weighted orbital integrals have the obvious drawback of not being invariant. Their dependence on the maximal compact subgroup K is also not entirely agreeable. However, there is a natural way to construct a parallel family of distributions with better properties. We shall show that these distributions satisfy the same formal germ expansions as the weighted orbital integrals.

As we recalled in §1, elements in $\mathcal{I}(G)$ can be regarded as functions

$$f_G : \pi \longrightarrow f_G(\pi) = \mathrm{tr}\big(\pi(f)\big), \qquad f \in \mathcal{C}(G),\ \pi \in \Pi_{\mathrm{temp}}(G),$$

on $\Pi_{\mathrm{temp}}(G)$ (the set of irreducible tempered representations of $G(\mathbb{R})$), rather than $\Gamma_{\mathrm{reg}}(G)$ (the set of strongly regular conjugacy classes in $G(\mathbb{R})$). The two interpretations are related by the formula

$$f_G(\pi) = \int_{\Gamma_{\mathrm{reg}}(G)} f_G(\gamma)|D(\gamma)|^{\frac{1}{2}}\Theta_\pi(\gamma)d\gamma, \qquad f \in \mathcal{C}(G),\ \gamma \in \Gamma_{\mathrm{reg}}(G),$$

where Θ_π is the character of π, and $d\gamma$ is a measure on $\Gamma_{\mathrm{reg}}(G)$ provided by the Weyl integration formula. We have also noted that any invariant, tempered distribution I on $G(\mathbb{R})$ factors through the space $\mathcal{I}(G)$. In other words, there is a continuous linear form $\widehat{I}$ on $\mathcal{I}(G)$ such that

$$I(f) = \widehat{I}(f_G), \qquad f \in \mathcal{C}(G).$$

This can be proved either by analyzing elements in $\mathcal{I}(G)$ directly as functions on $\Gamma_{\mathrm{reg}}(G)$ [B2] or by using the characterization [A5] of elements in $\mathcal{I}(G)$ as functions on $\Pi_{\mathrm{temp}}(G)$.

We fix a Levi subgroup $M \subset G$ and a maximal compact subgroup $K \subset G(\mathbb{R})$, as in §3. For each Levi subgroup $L \in \mathcal{L}(M)$, one can define a continuous linear transformation

$$\phi_L = \phi_L^G : \mathcal{C}(G) \longrightarrow \mathcal{I}(L)$$

in terms of objects that are dual to weighted orbital integrals. If f belongs to $\mathcal{C}(G)$, the value of $\phi_L(f)$ at $\pi \in \Pi_{\mathrm{temp}}(L)$ is the weighted character

$$\phi_L(f, \pi) = \mathrm{tr}\big(\mathcal{M}_L(\pi, P)\mathcal{I}_P(\pi, f)\big), \qquad P \in \mathcal{P}(L),$$

defined on p. 38 of [A7]. In particular, $\mathcal{I}_P(\pi)$ is the usual induced representation of $G(\mathbb{R})$, while

$$\mathcal{M}_L(\pi, P) = \lim_{\lambda \to 0}\left(\sum_{Q \in \mathcal{P}(L)} \mathcal{M}_Q(\lambda, \pi, P)\theta_Q(\lambda)^{-1}\right)$$

is the operator built out of Plancherel densities and unnormalized intertwining operators between induced representations, as on p. 37 of [A7]. Weighted characters behave in many ways like weighted orbital integrals. In particular, $\phi_L(f)$ depends on K, and transforms under conjugation of f by $y \in G(\mathbb{R})$ by a formula

$$\phi_L(f^y) = \sum_{Q \in \mathcal{F}(L)} \phi_L^{M_Q}(f_{Q,y}) \tag{10.1}$$

that is similar to (3.1).

The role of the mappings ϕ_L is to make weighted orbital integrals invariant. One defines invariant linear forms

$$I_M(\gamma, f) = I_M^G(\gamma, f), \qquad \gamma \in M_{G\text{-reg}}(\mathbb{R}),\ f \in \mathcal{C}(G),$$

on $\mathcal{C}(G)$ inductively by setting

$$J_M(\gamma, f) = \sum_{L \in \mathcal{L}(M)} \widehat{I}_M^L\big(\gamma, \phi_L(f)\big).$$

In other words,

$$I_M(\gamma, f) = J_M(\gamma, f) - \sum_{L \in \mathcal{L}^0(M)} \widehat{I}_M^L\big(\gamma, \phi_L(f)\big).$$

This yields a family of tempered distributions, which are parallel to weighted orbital integrals, but which are invariant and independent of K. (See, for example, [A7, §3].) We would like to show that they satisfy the formal germ expansions of Theorems 5.1 and 5.1*.

We fix the conjugacy class $c \in \Gamma_{ss}(M)$, as before. We must first attach invariant linear forms to the noninvariant distributions $J_M(\rho, f)$ in (5.1). Following the prescription above, we define invariant distributions

$$I_M(\rho, f) = I_M^G(\rho, f), \qquad \rho \in R_c(M),\ f \in \mathcal{C}(G),$$

inductively by setting

$$J_M(\rho, f) = \sum_{L \in \mathcal{L}(M)} \widehat{I}_M^L\big(\rho, \phi_L(f)\big).$$

In other words,

$$I_M(\rho, f) = J_M(\rho, f) - \sum_{L \in \mathcal{L}^0(M)} \widehat{I}_M^L\big(\rho, \phi_L(f)\big).$$

The invariance of $I_M(\gamma, f)$ follows inductively in the usual way from (3.1) and (10.1). As a general rule, the application of harmonic analysis improves one property only at the expense of another. In the case at hand, the price to pay for making $J_M(\rho, f)$ invariant is that the new distribution $I_M(\rho, f)$ is no longer supported on $\mathcal{U}_c(G)$.

We have in any case replaced the family (5.1) with a family

$$f \longrightarrow I_L(\rho, f), \qquad L \in \mathcal{L}(M),\ \rho \in R_c(L), \tag{10.2}$$

of invariant tempered distributions. These new objects do have many properties in common with the original ones. They satisfy the differential equation

$$I_L(\rho, zf) = I_L(z_L\rho, f), \tag{10.3}$$

for each $z \in \mathcal{Z}(G)$. They also satisfy the symmetry condition

$$I_{\theta L}(\theta\rho, \theta f) = I_L(\rho, f), \tag{10.4}$$

for any isomorphism θ: $G \to \theta G$ over $\mathbb{R}$. In addition, the distributions satisfy the transformation formula

$$I_L(\rho', f) = \sum_{\rho} a_L(\rho', \rho) I_L(\rho, f), \tag{10.5.1}$$

for $\{\rho'\}$ and $A_L = \{a_L(\rho', \rho)\}$ as in (5.4.1). We leave the reader to check that these properties are direct consequences of the corresponding properties in §5.

It follows from (5.3) that the series

$$g_M^L\big(I_{L,c}(f)\big) = \sum_{\rho\in R_c(L)} g_M^L(\rho) I_L(\rho, f)$$

converges in (the adic topology of) $\widehat{\mathcal{G}}_c(M, L)$. The continuity of the linear forms (10.2) implies, moreover, that the mapping

$$f \longrightarrow g_M^L\big(I_{L,c}(f)\big)$$

from $\mathcal{C}(G)$ to $\widehat{\mathcal{G}}_c(M, L)$ is continuous (in the complex topology of $\widehat{\mathcal{G}}_c(M, L)$). Finally, (5.4.2) and (10.5.1) yield the functorial property that for any L and f,

(10.5) $\qquad g_M^L\big(J_{L,c}(f)\big)$ is independent of the choice of basis $R_c(L)$.

The distributions (10.2) play the role of coefficients in a formal germ expansion of the function $I_M(\gamma, f)$. Following §5, we set

$$I_M^n(\gamma, f) = \sum_L \sum_\rho g_M^{L,n}(\gamma, \rho) I_L(\rho, f), \tag{10.6}$$

for any $n \geq 0$, and for fixed representatives $g_M^{L,n}(\gamma, \rho)$ of $g_M^{L,n}(\rho)$ in $\mathcal{F}_c^\alpha(M, L)$ as in (5.11). We then obtain the following corollary of Theorems 5.1 and 5.1^*.

Corollary 10.1. *We can choose the weight function α so that $\alpha(1)$ equals zero, and so that for any n, the mapping*

$$f \longrightarrow I_M(\gamma, f) - I_M^n(\gamma, f), \qquad f \in \mathcal{C}(G),$$

is a continuous linear mapping from $\mathcal{C}(G)$ to the space $\mathcal{F}_{c,n}^\alpha(V, G)$. In particular, $I_M(f)$ has a formal germ expansion given by the sum

$$\sum_{L\in\mathcal{L}(M)} g_M^L\big(I_{L,c}(f)\big) = \sum_{L\in\mathcal{L}(M)} \sum_{\rho\in R_c(L)} g_M^L(\rho) I_L(\rho, f). \tag{10.7}$$

Proof. The second assertion follows immediately from the first, in the same way that the corresponding assertion of Theorem 5.1 follows from Theorem 5.1^*. To establish the first assertion, we write

$$I_M(\gamma, f) - I_M^n(\gamma, f)$$

as the difference between

$$J_M(\gamma, f) - J_M^n(\gamma, f)$$

and

$$\sum_{L\in\mathcal{L}^0(M)} \Big(\widehat{I}_M^L\big(\gamma, \phi_L(f)\big) - \widehat{I}_M^{L,n}\big(\gamma, \phi_L(f)\big)\Big).$$

The assertion then follows inductively from Theorem 5.1^*. □

Corollary 10.1 tells us that the sum (10.7) represents the same element in $\widehat{\mathcal{G}}_c(M, G)$ as $I_M(f)$. In other words, the invariant distributions attached to weighted orbital integrals satisfy asymptotic expansions

$$I_M(\gamma, f) \sim \sum_L \sum_\rho g_M^L(\gamma, \rho) J_L(\rho, f).$$

The invariant distributions $I_M(\rho, f)$ ultimately depend on our choice of the mapping $\widetilde{\chi}_M$. It is interesting to note that this mapping has an invariant formulation, which leads to posteriori to a more direct construction of the distributions. To see this, we first set

$$I\widetilde{K}_M(f) = I_M(f) - \sum_{\{L\in\mathcal{L}(M):\, L\neq M,G\}} g_M^L\big(I_{L,c}(f)\big), \qquad f \in \mathcal{C}(G). \tag{10.8}$$

Let α be a fixed weight function that satisfies the conditions of Lemma 6.3. Then $f \to I\widetilde{K}_M(f)$ is a continuous linear transformation from $\mathcal{C}(G)$ to $\widehat{\mathcal{G}}_c^\alpha(M,G)$.

Lemma 10.2. *Suppose that f belongs to $\mathcal{C}(G)$. Then*

$$\widetilde{K}_M(f) - I\widetilde{K}_M(f) = g_M^M\big(J_{M,c}(f)\big) - g_M^M\big(I_{M,c}(f)\big). \tag{10.9}$$

Proof. The proof is similar to that of Lemmas 6.1 and 6.2, so we shall be brief. The left hand side of (10.9) equals

$$\sum_{L_1} \widehat{I}_M^{L_1}\big(\phi_{L_1}(f)\big) - \sum_{L,L_1} g_M^L\big(\widehat{I}_{L,c}^{L_1}(\phi_{L_1}(f))\big),$$

where the first sum is over Levi subgroups $L_1 \in \mathcal{L}^0(M)$, and the second sum is over pairs $L, L_1 \in \mathcal{L}(M)$ with

$$M \subsetneq L \subset L_1 \subsetneq G.$$

Taking the second sum over L_1 outside the sum over L, we obtain an expression

$$\sum_{L_1\in\mathcal{L}^0(M)} \left(\Big(\widehat{I}_M^{L_1}\big(\phi_{L_1}(f)\big) - \sum_{L\in\mathcal{L}^{L_1}(M)} g_M^L\big(\widehat{I}_{L,c}^{L_1}(\phi_{L_1}(f))\big)\Big) + g_M^M\big(\widehat{I}_{M,c}^{L_1}(\phi_{L_1}(f))\big)\right).$$

By Corollary 10.1, the formal germ

$$\widehat{I}K_M^{L_1}\big(\phi_{L_1}(f)\big) = \widehat{I}_M^{L_1}\big(\phi_{L_1}(f)\big) - \sum_{L\in\mathcal{L}^{L_1}(M)} g_M^L\big(\widehat{I}_{L,c}^{L_1}(\phi_{L_1}(f))\big)$$

vanishes for any L_1. The left hand side of (10.9) therefore equals

$$\sum_{L_1\in\mathcal{L}^0(M)} g_M^M\big(\widehat{I}_{M,c}^{L_1}(\phi_{L_1}(f))\big).$$

By definition, this in turn equals the right hand side of the required formula (10.9). $\square$

The lemma implies that the mapping

$$f \longrightarrow I\widetilde{K}_M(f) - \widetilde{K}_M(f), \qquad f \in \mathcal{C}(G),$$

takes values in the subspace $\widehat{\mathcal{I}}_c(M,G)$ of $\widehat{\mathcal{G}}_c^\alpha(M,G)$. We shall use this property to give invariant versions of the constructions of §8. For any $n \geq 0$, and N large relative to n, we can write $I\psi_M^n$ for the restriction of $I\widetilde{K}_M^n$ to the subspace $\mathcal{C}_{c,N}(G)$ of $\mathcal{C}(G)$. If f is a function in $\mathcal{C}_{c,N}(G)$, the (α, n)-jet

$$I\psi_M^n(f) = \psi_M^n(f) - \big(\widetilde{K}_M^n(f) - I\widetilde{K}_M^n(f)\big)$$

then belongs to the image of $\mathcal{I}_c^n(M,G)$ in $\mathcal{G}_c^{\alpha,n}(M,G)$. This yields the invariant analogue of Proposition 8.1. In particular, there is a uniquely determined, continuous linear mapping

$$I\chi_M^n : \mathcal{C}_{c,N}(G) \longrightarrow \mathcal{I}_c^n(M,G)$$

such that for any $f \in \mathcal{C}_{c,N}(G)$, the image of $I\chi_M^n(f)$ in $\mathcal{G}_c^{\alpha,n}(M,G)$ equals $I\psi_M^n(f)$. Following (8.7), we set

$$\langle \sigma, I\chi_M(f)\rangle = \langle \sigma, I\chi_M^n(f)\rangle, \qquad \sigma \in \mathcal{D}_c(M),\ f \in \mathcal{C}_{c,N}(G),$$

for any $n \geq \deg(\sigma)$ and N large relative to n. Then

$$\langle \sigma, I\chi_M(f)\rangle = \langle \sigma, \chi_M(f)\rangle - \langle \sigma, \widetilde{K}_M(f) - I\widetilde{K}_M(f)\rangle.$$

Given the mapping $\widetilde{\chi}_M$ of Proposition 8.3, we set

$$I\widetilde{\chi}_M(f) = \widetilde{\chi}_M(f) - \big(\widetilde{K}_M(f) - I\widetilde{K}_M(f)\big), \qquad f \in \mathcal{C}(G). \tag{10.10}$$

Then $I\widetilde{\chi}_M$ is a continuous linear mapping from $\mathcal{C}(G)$ to $\widehat{\mathcal{I}}_c(M,G)$ that satisfies the invariant analogue of the restriction property (8.11). Moreover, it follows easily from the lemma that $I\widetilde{\chi}_M$ also satisfies the analogues of (8.12) and (8.13). Conversely, suppose that $I\widetilde{\chi}_M$ is any continuous mapping from $\mathcal{C}(G)$ to $\widehat{\mathcal{I}}_c(M,G)$ that satisfies the invariant analogues of (8.11)–(8.13). Then the mapping $\widetilde{\chi}_M(f)$ defined by (10.10) satisfies the hypotheses of Proposition 8.3. Thus, instead of choosing the extension $\widetilde{\chi}_M$ of mappings $\{\chi_M^n\}$, as in Proposition 8.3, we could equally well choose an extension $I\widetilde{\chi}_M$ of invariant mappings $\{I\chi_M^n\}$. To see the relationship of the latter with our invariant distributions, we take any element $\rho \in R_c(M)$, and write

$$\begin{aligned}
&I_M(\rho,f) - J_M(\rho,f)\\
=&\big\langle \rho, g_M^M\big(I_{M,c}(f)\big)\big\rangle - \big\langle \rho, g_M^M\big(J_{M,c}(f)\big)\big\rangle\\
=&\big\langle \rho, I\widetilde{K}_M(f) - \widetilde{K}_M(f)\big\rangle\\
=&\big\langle \rho, I\widetilde{\chi}_M(f)\big\rangle - \big\langle \rho, \widetilde{\chi}_M(f)\big\rangle,
\end{aligned}$$

by Lemma 10.2 and the definition (10.10). It follows from the definition (9.1) that

$$I_M(\rho,f) = \big\langle \rho, I\widetilde{\chi}_M(f)\big\rangle, \qquad f \in \mathcal{C}(G). \tag{10.11}$$

The invariant distributions can therefore be defined directly in terms of the mapping $I\widetilde{\chi}_M$.

§11. Supplementary properties

There are further constraints that one could impose on the mapping $\widetilde{\chi}_M$ of Proposition 8.3 (or equivalently, the invariant mapping (10.10)). Any new constraint makes the construction more rigid. It puts extra conditions on the families of coefficients (5.14) and linear transformations (8.15), either of which describes the lack of uniqueness of the construction. A suitable choice of $\widetilde{\chi}_M$ will also endow our distributions and formal germs with new properties.

The most important property is that of parabolic descent. Suppose that M_1 is a Levi subgroup of M, chosen so that $\mathfrak{a}_{M_1}$ is orthogonal to the Lie algebra of K. Any element γ_1 in $M_{1,G\text{-reg}}(\mathbb{R})$ obviously maps to an element $\gamma = \gamma_1^M$ in $M_{G\text{-reg}}(\mathbb{R})$. The associated weighted orbital integral satisfies the descent formula

$$J_M(\gamma, f) = \sum_{G_1 \in \mathcal{L}(M_1)} d_{M_1}^G(M, G_1) J_{M_1}^{G_1}(\gamma_1, f_{Q_1}), \tag{11.1}$$

in the notation of [A4, Corollary 8.2]. The coefficient $d_{M_1}^G(M, G_1)$ is defined on p. 356 of [A4], while the section

$$G_1 \longrightarrow Q_1 = Q_{G_1}, \qquad G_1 \in \mathcal{L}(M_1),\ Q_{G_1} \in \mathcal{P}(G_1),$$

is defined on p. 357 of [A4]. We would like to establish similar formulas for our singular distributions and our formal germs.

Suppose that c is the image in $\Gamma_{ss}(M)$ of a class $c_1 \in \Gamma_{ss}(M_1)$. If L belongs to $\mathcal{L}(M)$, and L_1 lies in the associated set $\mathcal{L}^L(M_1)$, we shall denote the image of c_1 in $\Gamma_{ss}(L_1)$ by c_1 as well. For any such L and L_1, there is a canonical induction mapping $\sigma_1 \to \sigma_1^L$ from $\mathcal{D}_{c_1}(L_1)$ to $\mathcal{D}_c(L)$ such that

$$h_L(\sigma_1^L) = h_{L_1}(\sigma_1), \qquad \sigma_1 \in \mathcal{D}_{c_1}(L_1),\ h \in \mathcal{C}(L).$$

Since we can view $J_L(\cdot, f)$ as a linear form on $\mathcal{D}_c(L)$, the tempered distribution

$$J_L(\sigma_1^L, f), \qquad f \in \mathcal{C}(G),$$

is defined for any σ_1. We also write $\sigma \to \sigma_{L_1}$ for the adjoint restriction mapping from $\mathcal{D}_c(L)$ to $\mathcal{D}_{c_1}(L_1)$, relative to the bases $R_c(L)$ and $R_{c_1}(L_1)$. In other words,

$$\sum_{\rho \in R_c(L)} \phi_1(\rho_{L_1})\phi(\rho) = \sum_{\rho_1 \in R_{c_1}(L_1)} \phi_1(\rho_1)\phi(\rho_1^L),$$

for any linear functions ϕ_1 and ϕ on $\mathcal{D}_{c_1}(L_1)$ and $\mathcal{D}_c(L)$, respectively, for which the sums converge. (The restriction mapping comes from a canonical linear transformation $\widehat{\mathcal{I}}_c(L) \to \widehat{\mathcal{I}}_{c_1}(L_1)$ between the dual spaces of $\mathcal{D}_c(L)$ and $\mathcal{D}_{c_1}(L_1)$. Its basis dependent formulation as a mapping from $\mathcal{D}_c(L)$ to $\mathcal{D}_{c_1}(L_1)$ is necessitated by our notation for the formal germs $g_M^L(\rho)$.) We recall that as an element in $\widehat{\mathcal{G}}_c(M, L)$, $g_M^L(\rho)$ can be mapped to the formal germ $g_M^L(\rho)_{M_1}$ in $\widehat{\mathcal{G}}_{c_1}(M_1 \mid M, L)$.

Proposition 11.1. *We can choose the mapping $\widetilde{\chi}_M$ of Corollary 8.3 so that for any M_1 and c_1, the distributions (5.1) satisfy the descent formula*

$$J_L(\rho_1^L, f) = \sum_{G_1 \in \mathcal{L}(L_1)} d_{L_1}^G(L, G_1) J_{L_1}^{G_1}(\rho_1, f_{Q_1}), \qquad L_1 \in \mathcal{L}^L(M_1),\ \rho_1 \in R_{c_1}(L_1), \tag{11.2}$$

while the formal germs (5.2) satisfy the descent formula

$$g^L_M(\rho)_{M_1} = \sum_{L_1\in\mathcal{L}^L(M_1)} d^L_{M_1}(M, L_1) g^{L_1}_{M_1}(\rho_{L_1}), \qquad \rho \in R_c(L). \tag{11.3}$$

Proof. We have to establish the two formulas for any $L \in \mathcal{L}(M)$. We can assume inductively that for each M_1 and c_1, (11.2) holds for $L \neq M$, and (11.3) holds for $L \neq G$. In particular, both formulas hold for any L in the complement $\widetilde{\mathcal{L}}(M)$ of $\{G, M\}$ in $\mathcal{L}(M)$. We shall use this property to establish a descent formula for the formal germ

$$\widetilde{K}_M(f)_{M_1} = J_M(f)_{M_1} - \sum_{L\in\widetilde{\mathcal{L}}(M)} g^L_M\big(J_{L,c}(f)\big)_{M_1}.$$

The original identity (11.1) leads immediately to a descent formula

$$J_M(f)_{M_1} = \sum_{G_1\in\mathcal{L}(M_1)} d^G_{M_1}(M, G_1) J^{G_1}_{M_1}(f_{Q_1})$$

for the first term in the last expression for $\widetilde{K}_M(f)_{M_1}$. We apply (11.2) and (11.3) inductively to the summands in the second term

$$\sum_{L\in\widetilde{\mathcal{L}}(M)} g^L_M\big(J_{L,c}(f)\big)_{M_1}. \tag{11.4}$$

We obtain

$$\begin{aligned}
& g^L_M\big(J_{L,c}(f)\big)_{M_1} \\
&= \sum_{\rho\in R_c(L)} g^L_M(\rho)_{M_1} J_L(\rho, f) \\
&= \sum_{L_1\in\mathcal{L}^L(M_1)} d^L_{M_1}(M, L_1) \sum_{\rho\in R_c(L)} g^{L_1}_{M_1}(\rho_{L_1}) J_L(\rho, f) \\
&= \sum_{L_1\in\mathcal{L}^L(M_1)} d^L_{M_1}(M, L_1) \sum_{\rho_1\in R_{c_1}(L_1)} g^{L_1}_{M_1}(\rho_1) J_L(\rho^L_1, f) \\
&= \sum_{L_1\in\mathcal{L}^L(M_1)} \sum_{G_1\in\mathcal{L}(L_1)} d^L_{M_1}(M, L_1) d^G_{L_1}(L, G_1) \Big(\sum_{\rho_1\in R_{c_1}(L_1)} g^{L_1}_{M_1}(\rho_1) J^{G_1}_{L_1}(\rho_1, f_{Q_1}) \Big).
\end{aligned}$$

Therefore (11.4) equals the sum over $L \in \widetilde{\mathcal{L}}(M)$ of the expression

$$\sum_{L_1\in\mathcal{L}^L(M_1)} \sum_{G_1\in\mathcal{L}(L_1)} \big(d^L_{M_1}(M, L_1) d^G_{L_1}(L, G_1)\big) g^{L_1}_{M_1}\big(J^{G_1}_{L_1,c_1}(f_{Q_1})\big). \tag{11.5}$$

We can of course sum L over the larger set $\mathcal{L}(M)$, provided that we subtract the values of (11.5) taken when $L = M$ and $L = G$. If $L = M$, $d^L_{M_1}(M, L_1)$ vanishes unless $L_1 = M_1$, in which case $d^L_{M_1}(M, L_1) = 1$. The value of (11.5) in this case is

$$\sum_{G_1\in\mathcal{L}(M_1)} d^G_{M_1}(M, G_1) g^{M_1}_{M_1}\big(J^{G_1}_{M_1,c_1}(f_{Q_1})\big). \tag{11.6}$$

If $L = G$, $d^G_{L_1}(L, G_1)$ vanishes unless $G_1 = L_1$, in which case $d^G_{L_1}(L, G_1) = 1$. The value of (11.5) in this case is

$$\sum_{G_1\in\mathcal{L}(M_1)} d^G_{M_1}(M, G_1) g^{G_1}_{M_1}\big(J^{G_1}_{G_1,c_1}(f_{Q_1})\big). \tag{11.7}$$

Thus (11.4) equals the sum over $L \in \mathcal{L}(M)$ of (11.5) minus the sum of (11.6) and (11.7). The only part of (11.5) that depends on L is the product of coefficients in the brackets. We shall therefore take the sum over L inside the two sums over L_1 and G_1, which at the same time, we interchange. Then G_1, L_1 and L will be summed over $\mathcal{L}(M_1)$, $\mathcal{L}^{G_1}(M_1)$, and $\mathcal{L}(L_1)$ respectively. The resulting interior sum

$$\sum_{L \in \mathcal{L}(L_1)} d^L_{M_1}(M, L_1) d^G_{L_1}(L, G_1)$$

simplifies. According to [A4, (7.11)], this sum is just equal to $d^G_{M_1}(M, G_1)$, and in particular, is independent of L_1. We can therefore write (11.4) as the difference between the expression

$$\sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) \sum_{L_1 \in \mathcal{L}^{G_1}(M_1)} g^{L_1}_{M_1}\big(J^{G_1}_{L_1,c_1}(f_{Q_1})\big)$$

and the sum of (11.6) and (11.7). But (11.6) is equal to contribution to the last expression of $L_1 = M_1$, while (11.7) equals the contribution of $L_1 = G_1$. We conclude that (11.4) equals

$$\sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) \sum_{L_1 \in \widetilde{\mathcal{L}}^{G_1}(M_1)} g^{L_1}_{M_1}\big(J^{G_1}_{L_1,c_1}(f_{Q_1})\big).$$

We have established that $\widetilde{K}_M(f)_{M_1}$ equals

$$\sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) \Big(J^{G_1}_{M_1}(f_{Q_1}) - \sum_{L_1 \in \widetilde{\mathcal{L}}^{G_1}(M_1)} g^{L_1}_{M_1}\big(J^{G_1}_{L_1,c_1}(f_{Q_1})\big)\Big).$$

Since the expression in the brackets equals $\widetilde{K}^{G_1}_{M_1}(f_{Q_1})$, we obtain a descent formula

$$\widetilde{K}_M(f)_{M_1} = \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) \widetilde{K}^{G_1}_{M_1}(f_{Q_1}). \tag{11.8}$$

Suppose that $n \geq 0$, and that N is large relative to n. The mapping χ^n_M of Proposition 8.1 then satisfies

$$\begin{aligned}
\chi^n_M(f)_{M_1} &= \psi^n_M(f)_{M_1} = \widetilde{K}^n_M(f)_{M_1} \\
&= \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) \widetilde{K}^{G_1,n}_{M_1}(f_{Q_1}) \\
&= \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) \psi^{G_1,n}_{M_1}(f_{Q_1}) \\
&= \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) \chi^{G_1,n}_{M_1}(f_{Q_1}),
\end{aligned}$$

for any $f \in \mathcal{C}_{c,N}(G)$. This implies that

$$\langle \sigma, \chi_M(f) \rangle = \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) \langle \sigma_1, \chi^{G_1}_{M_1}(f_{Q_1}) \rangle, \qquad f \in \mathcal{C}_{c,N}(G),$$

for any induced element $\sigma = \sigma_1^M$ with $\sigma_1 \in \mathcal{D}_{c_1}(M_1)$, and for N large relative to σ_1.

We are now in a position to choose the mapping

$$\widetilde{\chi}_M : \mathcal{C}(G) \longrightarrow \widehat{\mathcal{I}}_c(M, G)$$

of Proposition 8.3. More precisely, we shall specify that part of the mapping that is determined by its proper restrictions $\widetilde{\chi}_M(f)_{M_1}$. We do so by making the inductive definition

$$\langle\sigma, \widetilde{\chi}_M(f)\rangle = \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1)\langle\sigma_1, \widetilde{\chi}^{G_1}_{M_1}(f_{Q_1})\rangle, \qquad f \in \mathcal{C}(G), \tag{11.9}$$

for any properly induced element

$$\sigma = \sigma_1^M, \qquad \sigma_1 \in \mathcal{D}_{c_1}(M_1),\ M_1 \subsetneq M,$$

in $\mathcal{D}_c(M)$. The right hand side of this expression is easily seen to depend only on σ, as opposed to the inducing data (M_1, σ_1). In fact, using the grading (1.5) of $\mathcal{D}_c(M)$, we can choose (M_1, σ_1) so that σ_1 belongs to $\mathcal{D}_{c_1,\mathrm{ell}}(M_1)$. The condition (8.11) of Proposition 8.3 follows from the formula above for $\langle\sigma, \chi_M(f)\rangle$. The conditions (8.12) and (8.13) follow inductively from the corresponding conditions for the terms $\langle\sigma_1, \widetilde{\chi}^{G_1}_{M_1}(f_{Q_1}\rangle$. The formula (11.9) thus gives a valid definition of the linear form $\widetilde{\chi}_M(f)$ on the subspace $\mathcal{D}_{c,\mathrm{par}}(M)$ spanned by elements in $\mathcal{D}_c(M)$ that are properly induced. For elements σ in the complementary subspace $\mathcal{D}_{c,\mathrm{ell}}(M)$, we remain free to define $\langle\sigma, \widetilde{\chi}_M(f)\rangle$ in any way that satisfies the conditions (8.12)–(8.13) of Proposition 8.3.

Having chosen $\widetilde{\chi}_M(f)$, we have only to apply the appropriate definitions. The first descent formula (11.2), in the remaining case that $L = M$, follows as directly from (11.9) and (9.1). Notice that (11.2) implies a similar formula

$$g^M_M\big(J^G_{M,c}(f)\big)_{M_1} = \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) g^{M_1}_{M_1}\big(J^{G_1}_{M_1,c_1}(f_{Q_1})\big)$$

for the formal germ $g^M_M\big(J^G_{M,c}(f)\big)$. Notice also that

$$K_M(f)_{M_1} = \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) K^{G_1}_{M_1}(f_{Q_1}),$$

since both sides vanish by Theorem 5.1. Combining these two observations with (11.8), we see that

$$g^G_M(f_{G,c})_{M_1} = \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) g^{G_1}_{M_1}(f_{Q_1,c_1}).$$

Now as a linear form in f, each side of this last formula is a linear combination of distributions $f_G(\rho)$ in the basis $R_c(M)$. We can therefore compare the coefficients of $f_G(\rho)$. The resulting identity is the second descent formula (11.3), in the remaining case that $L = G$. This completes the proof of the proposition. □

Corollary 11.2. *Suppose that the mapping $\widetilde{\chi}_M$ is chosen as in the proposition. Then for any M_1 and c_1, the invariant distributions (10.2) satisfy the descent formula*

$$I_L(\rho_1^L, f) = \sum_{G_1 \in \mathcal{L}(L_1)} d^G_{L_1}(L, G_1)\widehat{I}^{G_1}_{L_1}(\rho_1, f_{G_1}), \qquad L_1 \in \mathcal{L}^L(M_1),\ \rho_1 \in R_{c_1}(L_1). \tag{11.10}$$

Proof. We can assume inductively that (11.10) holds for any $L \in \mathcal{L}(M)$ with $L \neq M$, so it will be enough to treat the case that $L = M$. This frees the symbol L for use in the definition

$$I_M(\rho_1^L, f) = J_M(\rho_1^L, f) - \sum_{L \in \mathcal{L}^0(M)} \widehat{I}^L_M\big(\rho_1^L, \phi_L(f)\big)$$

from §10. We apply (11.2) to the first term $J_M(\rho_1^L, f)$. To treat the remaining summands $\widehat{I}_M^L\big(\rho_1^L, \phi_L(f)\big)$, we combine an inductive application of (11.10) to $I_M^L(\rho_1^L)$ with the descent formula

$$\phi_L(f)_{L_1} = \sum_{G_1 \in \mathcal{L}(L_1)} d_{L_1}^G(L, G_1)\phi_{L_1}^{G_1}(f_{Q_1}), \qquad L_1 \in \mathcal{L}^L(M_1),$$

established as, for example, in [A2, (7.8)]. We can then establish (11.10) (in the case $L = M$) by following the same argument that yielded the descent formula (11.8) in the proof of the proposition. (See also the proof of [A4, Theorem 8.1].) □

For the conditions of Proposition 11.1 and its corollary to hold, it is necessary and sufficient that the mapping $\widetilde{\chi}_M = \widetilde{\chi}_M^G$ satisfy its own descent formula. That is,

$$\widetilde{\chi}_M(f)_{M_1} = \sum_{G_1 \in \mathcal{L}(M_1)} d_{M_1}^G(M, G_1)\widetilde{\chi}_{M_1}^{G_1}(f_{Q_1})$$

for each M_1 and c_1. This in turn is equivalent to asking that the corresponding invariant mapping $I\widetilde{\chi}_M = I\widetilde{\chi}_M^G$ satisfy the descent formula

$$I\widetilde{\chi}_M(f)_{M_1} = \sum_{G_1 \in \mathcal{L}(M_1)} d_{M_1}^G(M, G_1)\widehat{I}\widetilde{\chi}_{M_1}^{G_1}(f_{G_1}),$$

again for each M_1 and c_1. Recall that $\widetilde{\chi}_M(f)$ can be identified with a $W(M)$-fixed linear form on $\mathcal{D}_c(M)$. Its value at any element in $\mathcal{D}_c(M)$ is determined by the descent condition and the differential equation (8.12), once we have defined $\widetilde{\chi}_M(f)$ as a linear form on the subspace

$$\mathcal{D}_{c,\mathrm{ell},G\text{-harm}}(M) = \mathcal{D}_{c,\mathrm{ell}}(M) \cap \mathcal{D}_{c,G\text{-harm}}(M).$$

The mapping $\widetilde{\chi}_M$ is then uniquely determined up to an $\mathrm{Aut}(G, K, M, c)$-fixed linear transformation

$$C : \mathcal{C}(G)/\mathcal{C}_{c,N}(G) \longrightarrow \mathcal{D}_{c,\mathrm{ell},G\text{-harm}}(M)^*.$$

There is another kind of descent property we could impose on our distributions and formal germs. This is geometric descent with respect to c, the aim of which would be to reduce the general study to the case of $c = 1$. One would try to find formulas that relate the objects attached to (G, M, c) with correspond objects for $(G_c, M_c, 1)$. This process has been carried out for p-adic groups. Geometric descent formulas for distributions were given in Theorem 8.5 of [A3] and its corollaries, while the descent formula for p-adic germs was in [A3, Proposition 10.2]. These formulas have important applications to the stable formula. In the archimedean case, however, geometric descent does not seem to play a role in the trace formula. Since it would entail a modification of our construction in the case of $c \neq 1$, we shall not pursue the matter here.

Finally, it is possible to build the singular weighted orbital integrals of [A3] into the constructions of this paper. Suppose that γ_c is a conjugacy class in $M(\mathbb{R})$ that is contained in $\mathcal{U}_c(M)$, and has been equipped with $M(\mathbb{R})$-invariant measure. The associated invariant integral gives a distribution

$$h \longrightarrow h_M(\gamma_c), \qquad h \in \mathcal{C}(M),$$

in $\mathcal{D}_c(M)$. We write $\mathcal{D}_{c,\mathrm{orb}}(M)$ for the subspace of $\mathcal{D}_c(M)$ spanned by such distributions. Any element in $\mathcal{D}_{c,\mathrm{orb}}(M)$ is known to be a finite linear combination of distributions $h \to h_M(\sigma)$, for triplets $\sigma = \big(T, \Omega, \partial(u)\big)$ in $S_c(M)$ such that u is M_c-harmonic. Since $W(M_c, T)$ is contained in $W(G, T)$, any M_c-harmonic element is automatically G-harmonic. The space $\mathcal{D}_{c,\mathrm{orb}}(M)$ is therefore contained in $\mathcal{D}_{c,G\text{-harm}}(M)$. The point is that one can define a *canonical* distribution $J_M(\sigma, f)$, for any σ in $\mathcal{D}_{c,\mathrm{orb}}(M)$ [A3, (6.5)]. This distribution is supported on $\mathcal{U}_c(G)$, and satisfies the analogue

$$J_M(\sigma, f^y) = \sum_{Q\in\mathcal{F}(M)} J_M^{M_Q}(\sigma, f_{Q,y}), \qquad f \in C_c^\infty(G),\ y \in G(\mathbb{R}),$$

of (5.5). It follows from (5.13) that $J_M(\sigma, f)$ can be chosen to represent an element in the family (5.1) (and in particular, is a tempered distribution). Otherwise said, the constructions of [A3] provide a canonical definition for a part of the operator $\widetilde{\chi}_M$ of Proposition 8.3. They determine the restriction of each linear form $\widetilde{\chi}_M(f)$ to the subspace $\mathcal{D}_{c,\mathrm{orb}}(M)$ of $\mathcal{D}_c(M)$. The conditions of Proposition 8.3 and [A3, (6.5)] therefore reduce the choice of $\widetilde{\chi}_M$ to that of an $\mathrm{Aut}(G, K, M, c)$-fixed linear transformation that fits into a diagram

$$\mathcal{C}_{c,N_1}(G) \hookrightarrow \mathcal{C}(G) \xrightarrow{\widetilde{\chi}_M} \mathcal{D}_{c,G\text{-harm}}(M)^* \twoheadrightarrow \mathcal{D}_{c,\mathrm{orb}}(M)^*,$$

in which the composition of any two arrows is predetermined. The mapping $\widetilde{\chi}_M$ is thus uniquely determined up to an $\mathrm{Aut}(G, K, M, c)$-fixed linear transformation

$$C:\ \mathcal{C}(G)/\mathcal{C}_{c,N_1}(G) \longrightarrow \big(\mathcal{D}_{c,G\text{-harm}}(M)/\mathcal{D}_{c,\mathrm{orb}}(M)\big)^*.$$

However, the last refinement of our construction is not compatible with that of Proposition 11.1. This is because an induced distribution $\rho = \rho_1^M$ in $\mathcal{D}_c(M)$ may be orbital without the inducing distribution ρ_1 being so. The conditions of Proposition 11.1 and of [A5] are thus to be regarded as two separate constraints. We are free to impose either one of them on the general construction of Proposition 8.3, but not both together. The decision of which one to choose in any given setting would depend of course upon the context.

References

[A1] J. Arthur, *The characters of discrete series as orbital integrals,* Invent. Math. **32** (1976), 205–261.

[A2] ———, *The trace formula in invariant form,* Ann. of Math. **114** (1981), 1–74.

[A3] ———, *The local behaviour of weighted orbital integrals,* Duke Math. J. **56** (1988), 223–293.

[A4] ———, *The invariant trace formula I. Local theory,* J. Amer. Math. Soc. **1** (1988), 323–383.

[A5] ———, *The trace Paley-Wiener theorem for Schwartz functions,* Contemp. Math. **177** (1994), 171–180.

[A6] ———, *On local character relations,* Selecta Math. **2** (1996), 501–579.

[A7] ———, *Canonical normalization of weighted characters and a transfer conjecture,* C.R. Math. Rep. Acad. Sci. Canada **20** (1998), 33–52.

[A8] ———, *On the transfer of distributions: singular orbital integrals,* in preparation.

[BV] D. Barbasch and D. Vogan, *The local structure of characters,* J. Funct. Anal. **37** (1980), 27–55.

[B1] A. Bouaziz, *Intégrales orbitales sur les algèbres de Lie réductives,* Invent. Math. **115** (1994), 163–207.

[B2] ———, *Intégrales orbitales sur les groupes de Lie réductives,* An. Scient. Éc. Norm. Sup. **27** (1994), 573–609.

[D] J. Dieudonné, *Foundations of Modern Analysis,* Academic Press, 1969.

[H1] Harish-Chandra, *Invariant eigendistributions on a semisimple Lie group,* Trans. Amer. Math. Soc. **119** (1965), 457–508.

[H2] ———, *Two theorems on semisimple Lie groups,* Ann. of Math. **83** (1966), 74–128.

[H3] ———, *Discrete series for semisimple Lie groups, II. Explicit determination of the characters,* Acta. Math. **116** (1966), 1–111.

[H4] ———, *Harmonic analysis on real reductive groups I. The theory of the constant term,* J. Funct. Anal. **19** (1975), 104–204.