The Embedded Eigenvalue Problem for Classical Groups

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This paper is dedicated to Freydoon Shahidi on the occasion of his sixtieth birthday.

Abstract. We report briefly on an endoscopic classification of representations by focusing on one aspect of the problem, the question of embedded Hecke eigenvalues.

1. The problem for G

By "eigenvalue", we mean the family of unramified Hecke eigenvalues of an automorphic representation. The question is whether there are any eigenvalues for the discrete spectrum that are also eigenvalues for the continuous spectrum. The answer for classical groups has to be part of any general classification of their automorphic representations.

The continuous spectrum is to be understood narrowly in the sense of the spectral theorem. It corresponds to representations in which the continuous induction parameter is unitary. For example, the trivial one-dimensional automorphic representation of the group $SL(2)$ does not represent an embedded eigenvalue. This is because it corresponds to a value of the one-dimensional induction parameter at a nonunitary point in the complex domain. For general linear groups, the absence of embedded eigenvalues has been known for some time. It is a consequence of the classification of Jacquet-Shalika [**JS**] and Moeglin-Waldspurger [**MW**]. For other classical groups, the problem leads to interesting combinatorial questions related to the endoscopic comparison of trace formulas.

We shall consider the case that G is a (simple) quasisplit symplectic or special orthogonal group over a number field F. Suppose for example that G is split and of rank n. The continuous spectrum of maximal dimension is then parametrized by n-tuples of (unitary) idele class characters. Is there any n-tuple whose unramified Hecke eigenvalue family matches that of an automorphic representation π in the discrete spectrum of G? The answer is no if π is required to have a global Whittaker model. This follows from the work of Cogdell, Kim, Piatetskii-Shapiro and Shahidi

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[**CKPS**]. Any such π will automatically have a local Whittaker model at each place. However, it is by no means clear that π must also have a global Whittaker model. In fact, the general existence of global Whittaker models appears to be dependent on some a priori classification of the full discrete spectrum of G.

Our discussion of the embedded eigenvalue problem can therefore be regarded as a short introduction to the larger question of the endoscopic classification of representations. It represents an attempt to isolate a manageable part of a broader topic, which at the same time illustrates some of the basic techniques. These techniques rest on a comparison of trace formulas on different groups.

2. A distribution and its stabilization

It is the *discrete part* of the trace formula that carries the information about automorphic representations. This is by definition the linear form

$$(1) \qquad I^G_{\mathrm{disc}}(f) = \sum_M |W(M)|^{-1} \sum_{w \in W(M)_{\mathrm{reg}}} |\det(w-1)|^{-1} \mathrm{tr}\big(M_P(w)\mathcal{I}_P(f)\big),$$

for a test function $f \in C^\infty_c\big(G(\mathbb{A})\big)$ on $G(\mathbb{A})$. We recall that M ranges over the finite set of conjugacy classes of Levi subgroups of G, that

$$W(M) = \mathrm{Norm}_G(A_M)/M$$

is the Weyl group of M over F, and that $W(M)_{\mathrm{reg}}$ is the set of elements $w \in W(M)$ such that the determinant of the associated linear operator

$$(w-1) = (w-1)_{\mathfrak{a}^G_M}$$

is nonzero. As usual,

$$\mathcal{I}_P(f) = \mathcal{I}_P(0,f), \qquad P \in \mathcal{P}(M),$$

is the representation of $G(\mathbb{A})$ on the Hilbert space

$$\mathcal{H}_P = L^2_{\mathrm{disc}}\big(N_P(\mathbb{A})M(\mathbb{Q})A_M(\mathbb{R})^0\backslash G(\mathbb{A})\big)$$

induced parabolically from the discrete spectrum of M, while

$$M_P(w): \ \mathcal{H}_P \longrightarrow \mathcal{H}_P$$

is the global intertwining operator attached to w. Recall that

$$A^+_{M,\infty} = (R_{F/\mathbb{Q}}A_M)(\mathbb{R})^0$$

is a central subgroup of $M(\mathbb{A})$ such that the quotient

$$M(F)A^+_{M,\infty}\backslash M(\mathbb{A})$$

has finite invariant volume.

This is the core of the trace formula. It includes what one hopes ultimately to understand, the automorphic discrete spectrum

$$\mathcal{H}_G = L^2_{\mathrm{disc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$$

of G. Indeed, the term with $M = G$ is simply the trace of the right convolution operator of f on this space. The summands for smaller M represent contributions of Eisenstein series to the trace formula. They are boundary terms, which arise from the truncation methods required to deal with the noncompactness of the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$. The operators $M_P(w)$ are of special interest, being at the heart of the theory of Eisenstein series. It was their study that led to the Langlands-Shahidi method, and much recent progress in the theory of automorphic L-functions.

With its classical ingredients, the expression for $I^G_{\mathrm{disc}}(f)$ is remarkably simple. There are of course other terms in the trace formula, some of which are quite complex. We shall not discuss them here. Our purpose will be rather to see what can be established for the spectral information in $I^G_{\mathrm{disc}}(f)$, knowing that the complementary terms have already been taken care of.

To have a chance of understanding the terms in the formula for $I^G_{\mathrm{disc}}(f)$, we really need something to compare them with. A solution of sorts is provided by the *stabilization* of $I^G_{\mathrm{disc}}(f)$. This is an innocuous looking expansion

$$I^G_{\mathrm{disc}}(f) = \sum_{G'} \iota(G,G')\widehat{S}^{G'}_{\mathrm{disc}}(f^{G'}) \tag{2}$$

of $I^G_{\mathrm{disc}}(f)$ into stable distributions $S^{G'}_{\mathrm{disc}}$ on endoscopic groups G', with coefficients $\iota(G,G')$ that are defined by simple formulas. The sum is actually over the isomorphism classes of elliptic endoscopic data G' of G. For example, if G is the split adjoint group $SO(2n+1)$, the dual group $\widehat{G}$ equals $Sp(2n,\mathbb{C})$. We then have

$$\widehat{G}' = Sp(2m,\mathbb{C}) \times Sp(2n-2m,\mathbb{C})$$

and

$$G' = SO(2m+1) \times SO(2n-2m+1).$$

In particular, the sum in (2) is parametrized in this case by integers that range from 0 to the greatest integer in $\frac{1}{2}n$.

The mapping

$$f \longrightarrow f^{G'}, \qquad\qquad f \in C^\infty_c\big(G(\mathbb{A})\big),$$

in (2) is the Langlands-Shelstad transfer of functions. With Ngo's recent proof of the fundamental lemma [**N**], it is now known that this correspondence takes $C^\infty_c\big(G(\mathbb{A})\big)$ to the space $C^\infty_c\big(G'(\mathbb{A})\big)$ of test functions on $G'(\mathbb{A})$, as originally conjectured by Langlands. The general resolution of the problem is a culmination of work by many people, including Langlands [**L**], Shelstad [**S**], Langlands-Shelstad [**LS**], Goresky-Kottwitz-MacPherson [**GKM**], Waldspurger [**W1**], [**W3**], and Lauman-Ngo [**LN**], as well as Ngo. We recall that it is a local question, which has to be formulated for each completion F_v of F. It was first treated for archimedean v, in [**S**]. The fundamental lemma is required explicitly for the places v that are unramified (relative to f), and implicitly as a hypothesis in the solution [**W1**] for general p-adic v.

The formula (2) was established in [**A3**], following partial results [**L**] and [**K**] obtained earlier. It was predicated on a generalization of the fundamental lemma that applies to unramified *weighted* orbital integrals. This has now been established by Chaudouard and Laumon [**CD**], building on the techniques of Ngo. The stabilization formula (2) is therefore unconditionally valid.

We note that the proof of (2) is indirect. It is a consequence of a stabilization that must be established directly for all of the other terms in the trace formula. For example, the papers [**L**] and [**K**] can be regarded as stabilizations of, respectively, the regular elliptic and the singular elliptic terms. In general, the terms that are complementary to those in $I^G_{\mathrm{disc}}(f)$ each come with their own individual set of problems, all of which must be taken care of. This accounts for the difficulty of the proof of (2).

As we have said, the linear forms $S^{G'}_{\mathrm{disc}}$ in (2) are stable distributions on the groups $G'(\mathbb{A})$. (The symbol $\widehat{S}'$ is understood to be the pullback of S' to the space of stable orbital integrals on $C^\infty_c\big(G'(\mathbb{A})\big)$, a space in which the correspondence

$f \to f^{G'}$ takes values.) However, there is nothing in the formula (2) that tells us anything concrete about these objects. We can regard (2) as simply an inductive definition

$$S^G_{\mathrm{disc}}(f) = I^G_{\mathrm{disc}}(f) - \sum_{G' \neq G} \iota(G, G') \widehat{S}^{G'}_{\mathrm{disc}}(f^{G'})$$

of a stable distribution on $G(\mathbb{A})$ in terms of its analogues for groups G' of smaller dimension. It does tell us that the right hand side, defined inductively on the dimension of G in terms of the right side of (1), is stable in f. This is an interesting fact, to be sure. But it is not something that by itself will give us concrete information about the automorphic discrete spectrum of G. To use (2) effectively, we must combine it with something further.

3. Its twisted analogue for $GL(N)$

The extra ingredient is the twisted trace formula for $GL(N)$, and its corresponding stablization. To describe what we need, we write

$$\widetilde{G} = GL(N) \rtimes \theta,$$

for the standard outer automorphism

$$\theta(x) = {}^t x^{-1}, \qquad x \in GL(N),$$

of $GL(N)$. Then $\widetilde{G}$ is the nonidentity component of the semidirect product

$$\widetilde{G}^+ = \widetilde{G}^0 \rtimes \langle \theta \rangle = GL(N) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

With this understanding, the twisted trace formula requires little change in notation. Its discrete part can be written in a form

$$(\tilde{1}) \qquad I^{\tilde{G}}_{\mathrm{disc}}(\widetilde{f}) = \sum_M |\widetilde{W}(M)|^{-1} \sum_{w \in \widetilde{W}(M)_{\mathrm{reg}}} |\det(w-1)|^{-1} \mathrm{tr}\big(M_P(w)\mathcal{I}_P(\widetilde{f})\big)$$

that matches (1). In particular, M ranges over the set of conjugacy classes of Levi subgroups in the connected group $\widetilde{G}^0 = GL(N)$, and P represents a parabolic subgroup of $\widetilde{G}^0$ with Levi component M. The only changes from (1) are that the test function $\widetilde{f} \in C^\infty_c\big(\widetilde{G}(\mathbb{A})\big)$ and the Weyl set

$$\widetilde{W}(M) = \mathrm{Norm}_{\widetilde{G}}(M)/M$$

are taken relative to the component $\widetilde{G}$, and that $\mathcal{I}_P$ stands for a representation induced from P to $\widetilde{G}^+$. As before, $M_P(w)$ is the global intertwining operator attached to w. (See [**CLL**] and [**A1**].) The last step in the proof of the general (invariant) twisted trace formula has been the archimedean twisted trace Paley-Wiener theorem, established recently by Delorme and Mezo [**DM**].

The stabilization of $I^{\tilde{G}}_{\mathrm{disc}}(\widetilde{f})$ takes the form

$$(\tilde{2}) \qquad I^{\tilde{G}}_{\mathrm{disc}}(\widetilde{f}) = \sum_G \iota(\widetilde{G}, G) \widetilde{S}^G_{\mathrm{disc}}(\widetilde{f}^G),$$

where the symbols S^G_{disc} represent stable distributions defined inductively by (2), and $\iota(\widetilde{G}, G)$ are again explicit coefficients. The sum is over isomorphism classes of elliptic twisted endoscopic data G for $\widetilde{G}$. For example, if $N = 2n+1$ is odd, the component

$$\widetilde{G} = GL(2n+1) \rtimes \theta$$

has a "dual set"

$$\widehat{G} = GL(2n+1, \mathbb{C}) \rtimes \widehat{\theta}.$$

We then have

$$\widehat{G} = Sp(2m, \mathbb{C}) \times SO(2n-2m+1, \mathbb{C})$$

and

$$G = SO(2m+1) \times Sp(2n-2m).$$

In general, a twisted endoscopic datum G entails a further choice, that of a suitable L-embedding

$$\xi_G : \; {}^L G \longrightarrow GL(N, \mathbb{C})$$

of the appropriate form of the L-group of G into $GL(N, \mathbb{C})$. However, if we forget this extra structure, we see in this case that G is just a group parametrized by an integer that ranges from 0 to n.

The mapping

$$\widetilde{f} \longrightarrow \widetilde{f}^G, \qquad \widetilde{f} \in C_c^\infty\big(\widetilde{G}(\mathbb{A})\big),$$

in $(\tilde{2})$ is the Kottwitz-Langlands-Shelstad correspondence of functions. The long-standing conjecture has been that it takes $C_c^\infty\big(\widetilde{G}(\mathbb{A})\big)$ to $C_c^\infty\big(G(\mathbb{A})\big)$. With the recent work of Ngo [N] and Waldspurger [W1]–[W3], this conjecture has now been resolved. The resulting transfer of functions becomes the fundamental starting point for a general stabilization of the twisted trace formula.

The actual identity $(\tilde{2})$ is less firmly in place. The twisted generalization of the weighted fundamental lemma does follow from the work of Chaudouard and Laumon, and of Waldspurger. However, the techniques of [**A3**] have not been established in the twisted case. Some of these techniques will no doubt carry over without much change. However, there will be others that call for serious refinement, and perhaps also new ideas. Still, there is again reason to be hopeful that a general version of $(\tilde{2})$ can be established in the not too distant future. We shall assume its stated version for $GL(N)$ in what follows.

Taken together, the stabilizations (2) and $(\tilde{2})$ offer us the possibility of relating automorphic representations of a classical group G with those of a twisted general linear group $\widetilde{G}$. As we have noted, the identity (2) represents an inductive definition of a stable distribution on $G(\mathbb{A})$ in terms of unknown spectral automorphic data (1) for G. The identity $(\tilde{2})$ provides a relation among the distributions in terms of known spectral automorphic data $(\tilde{1})$ for $GL(N)$.

This is not to say that the subsequent analysis is without further difficulty. It in fact contains many subtleties. For example, there is often more than one unknown stable distribution S^G_{disc} on the right hand side of the identity $(\tilde{2})$. The problem is more serious in case $N = 2n$ is even, where there are data G with dual groups $Sp(2n, \mathbb{C})$ and $SO(2n, \mathbb{C})$ that are both distinct and simple. This particular difficulty arises again and again in the analysis. Its constant presence requires a sustained effort finally to overcome.

4. Makeshift parameters

The comparison of (2) and $(\tilde{2})$ requires a suitable description of the automorphic discrete spectrum of the group $\widetilde{G}^0 = GL(N)$. Let $\Psi_2(N)$ be the set of formal tensor products

$$\psi = \mu \boxtimes \nu, \qquad N = mn,$$

where μ is a unitary cuspidal automorphic representation of $GL(m)$ and ν is the irreducible representation of the group $SL(2,\mathbb{C})$ of dimension n. The cuspidal representation μ comes with what we are calling an "eigenvalue". This, we recall, is the Hecke family

$$c(\mu) = \{c_v(\mu) = c(\mu_v) : \ v \notin S\}$$

of semisimple conjugacy classes in $GL(m,\mathbb{C})$ attached to the unramified constituents μ_v of v. To the tensor product ψ, we attach the "eigenvalue"

$$c(\psi) = c(\mu) \otimes c(\nu).$$

This is the family of semisimple conjugacy classes

$$c_v(\mu) \otimes \nu \begin{pmatrix} q_v^{\frac{1}{2}} & 0 \\ 0 & q_v^{-\frac{1}{2}} \end{pmatrix} = c_v(\mu) q_v^{\frac{n-1}{2}} \oplus \cdots \oplus c_v(\mu) q_v^{-\frac{n-1}{2}}, \qquad v \notin S,$$

in $GL(N,\mathbb{C})$. It follows from [**JS**] and [**MW**] that there is a bijection $\psi \to \pi_\psi$ from $\Psi_2(N)$ onto the set of unitary automorphic representations π_ψ in the discrete spectrum of $GL(N)$ (taken modulo the center) such that

$$c(\psi) = c(\pi_\psi).$$

More generally, one can index representations in the broader automorphic spectrum by sums of elements in $\Psi_2(N_i)$. Let $\Psi(N)$ be the set of formal direct sums

$$\psi = \ell_1\psi_1 \boxplus \cdots \boxplus \ell_r\psi_r, \tag{3}$$

for positive integers ℓ_i and distinct elements $\psi_i = \mu_i \boxtimes \nu_i$ in $\Psi_2(N_i)$, whose ranks $N_i = m_i n_i$ satisfy

$$N = \ell_1 N_1 + \cdots + \ell_r N_r = \ell_1 m_1 n_1 + \cdots \ell_r m_r n_r.$$

For any ψ, we attach the "eigenvalue"

$$c(\psi) = \ell_1 c(\psi_1) \oplus \cdots \oplus \ell_r c(\psi_r),$$

of semisimple conjugacy classes

$$c_v(\psi) = \underbrace{c_v(\psi_1) \oplus \cdots \oplus c_v(\psi_1)}_{\ell_1} \oplus \cdots \oplus \underbrace{c_v(\psi_r) \oplus \cdots \oplus c_v(\psi_r)}_{\ell_r}$$

in $GL(N,\mathbb{C})$. It then follows from Langlands' theory of Eisenstein series that there is a bijection $\psi \to \pi_\psi$ from $\Psi(N)$ to the set of unitary representations π_ψ in the full automorphic spectrum of $GL(N)$ such that

$$c(\psi) = c(\pi_\psi).$$

The elements in $\Psi(N)$ are to be regarded as makeshift parameters. They are basically forced on us in the absence of the hypothetical automorphic Langlands group L_F. Recall that L_F is supposed to be a locally compact group whose irreducible, unitary, N-dimensional representations parametrize the unitary, cuspidal automorphic representation of $GL(N)$.

If we had the group L_F at our disposal, we could identify elements in our set $\Psi(N)$ with (equivalence classes of) N-dimensional representations

$$\psi : \ L_F \times SL(2,\mathbb{C}) \longrightarrow GL(N,\mathbb{C})$$

whose restrictions to L_F are unitary. This interpretation plays a conjectural role in the representation theory of the quasisplit group G. Regarding G as an elliptic twisted endoscopic datum for $GL(N)$, and $\Psi(N)$ as the set of N-dimensional representations of $L_F \times SL(2,\mathbb{C})$, we would be able to introduce the subset of mappings ψ in $\Psi(N)$ that factor through the embedded L-group

$$\xi_G : \ {}^LG \longrightarrow GL(N,\mathbb{C}).$$

Any such ψ would then give rise to a complex reductive group, namely the centralizer

$$S_\psi = S_\psi^G = \mathrm{Cent}\big(\mathrm{Im}(\psi), \widehat{G}\big)$$

in $\widehat{G} \subset {}^LG$ of its image in ${}^LG$. The finite quotient

$$\mathcal{S}_\psi = S_\psi / S_\psi^0 Z(\widehat{G})^{\Gamma_F}, \qquad \Gamma_F = \mathrm{Gal}(\overline{F}/F), \tag{4}$$

of S_ψ is expected to play a critical role in the automorphic representation theory of G.

5. The groups $\mathcal{L}_\psi$

The first challenge is to define the centralizers S_ψ and their quotients $\mathcal{S}_\psi$ without having the group L_F. For any makeshift parameter ψ as in (3), we can certainly form the contragredient parameter

$$\begin{aligned}\psi^\vee &= \ell_1\psi_1^\vee \boxplus \cdots \boxplus \ell_r\psi_r^\vee \\ &= \ell_1(\mu_1^\vee \boxtimes \nu_1) \boxplus \cdots \boxplus \ell_r(\mu_r^\vee \boxtimes \nu_r).\end{aligned}$$

The subset

$$\widetilde{\Psi}(N) = \big\{\psi \in \Psi(N) : \ \psi^\vee = \psi\big\}.$$

of self-dual parameters in $\Psi(N)$ consists of those ψ for which the corresponding automorphic representation π_ψ is θ-stable. The idea is to attach a makeshift group $\mathcal{L}_\psi$ to any ψ. The group $\mathcal{L}_\psi$ will then be our substitute for L_F. We shall formulate it as an extension of the Galois group Γ_F by a complex connected reductive group.

The main problem in the construction of $\mathcal{L}_\psi$ is to deal with the basic case that $\psi = \mu$ is cuspidal. Since ψ is assumed to lie in $\widetilde{\Psi}(N)$, μ equals $\mu^\vee$. It therefore represents a self dual cuspidal automorphic representation of $GL(N)$. At this point we have to rely on the following theorem.

THEOREM 1. *Suppose that μ is a self-dual, unitary, cuspidal automorphic representation of $GL(N)$. Then there is a unique elliptic, twisted endoscopic datum $G = G_\mu$ for $GL(N)$ that is simple, and such that*

$$c(\mu) = \xi_{G_\mu}\big(c(\pi)\big),$$

for a cuspidal automorphic representation π of $G(\mathbb{A})$.

The theorem asserts that there is exactly one G for which there is a cuspidal "eigenvalue" that maps to the "eigenvalue" of μ in $GL(N)$. Its proof is deep. In working on the general classification, one assumes inductively that the theorem holds for the proper self-dual components μ_i of a general parameter ψ. The resolution of this (and other) induction hypotheses then comes only at the end of the entire argument. However, we shall assume for the discussion here that the theorem is valid without restriction. In the case that $\psi = \mu$, this allows us to define

$$\mathcal{L}_\psi = {}^LG_\mu.$$

We then write $\widetilde{\psi}$ for the L-homomorphism $\xi_\mu = \xi_{G_\mu}$ of this group into $GL(N,\mathbb{C})$.

Consider now an arbitrary parameter $\psi \in \widetilde{\Psi}(N)$ of the general form (3). Since ψ is self-dual, the operation $\mu \to \mu^\vee$ acts as an involution on the cuspidal components μ_i of ψ. If i is an index with $\mu_i^\vee = \mu_i$, we introduce the group $G_i = G_{\mu_i}$ provided by the theorem, as well as the L-homomorphism

$$\xi_i = \xi_{\mu_i} : \ {}^L G_i \longrightarrow GL(m_i, \mathbb{C}).$$

If j parametrizes an orbit $\{\mu_j, \mu_j^\vee\}$ of order two, we set $G_j = GL(m_j)$, and we take

$$\xi_j : \ {}^L\big(GL(m_j)\big) \longrightarrow GL(2m_j, \mathbb{C})$$

to be the homomorphism that is trivial on Γ_F, and that restricts to the embedding

$$g \longrightarrow \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}$$

of $GL(m_j,\mathbb{C})$ into $GL(2m_j,\mathbb{C})$. We define our general makeshift group $\mathcal{L}_\psi$ to be the fibre product

$$\mathcal{L}_\psi = \prod_{k \in \{i,j\}} \big({}^L G_k \longrightarrow \Gamma_F\big)$$

of these L-groups over Γ_F. The various homomorphisms ξ_k can then be combined in the natural way with the corresponding representations

$$\nu_k : \ SL(2,\mathbb{C}) \longrightarrow GL(n_k, \mathbb{C})$$

to give a homomorphism

$$\widetilde{\psi} : \ \mathcal{L}_\psi \times SL(2,\mathbb{C}) \longrightarrow GL(N,\mathbb{C}).$$

We regard $\widetilde{\psi}$ as an equivalence class of N-dimensional representations of the group $\mathcal{L}_\psi \times SL(2,\mathbb{C})$.

Suppose that G represents a simple twisted endoscopic datum for $GL(N)$. We define $\widetilde{\Psi}(G)$ to be the subset of parameters $\psi \in \widetilde{\Psi}(N)$ such that $\widetilde{\psi}$ factors through the image of ${}^L G$ in $GL(N,\mathbb{C})$. For any $\psi \in \widetilde{\Psi}(G)$, we then have an L-embedding

$$\widetilde{\psi}_G : \ \mathcal{L}_\psi \times SL(2,\mathbb{C}) \longrightarrow {}^L G$$

such that

$$\xi_G \circ \widetilde{\psi}_G = \widetilde{\psi}.$$

We are treating $\widetilde{\psi}$ as an equivalence class of N-dimensional representations. This means that $\widetilde{\psi}_G$ is determined only up to the group $\mathrm{Aut}_{\widetilde{G}}(G)$ of L-automorphisms of ${}^L G$ induced by the stabilizer in $GL(N,\mathbb{C})$ of its image. Nevertheless, we can still write

$$S_\psi = S_\psi^G = \mathrm{Cent}\big(\mathrm{Im}(\widetilde{\psi}_G), \widehat{G}\big)$$

and

$$\mathcal{S}_\psi = S_\psi / S_\psi^0 Z(\widehat{G})^{\Gamma_F},$$

where $\widetilde{\psi}_G$ stands for some L-homomorphism in the associated $\mathrm{Aut}_{\widetilde{G}}(G)$-orbit. Since $\mathcal{S}_\psi$ is a finite abelian group (a 2-group actually), it is uniquely determined by ψ up a unique isomorphism.

The parameters $\psi \in \widetilde{\Psi}(G)$, along with the groups $\mathcal{L}_\psi$ and the associated centralizer groups S_ψ and $\mathcal{S}_\psi$, were described in §30 of [**A4**]. They will be discussed in greater detail in Chapter 1 of [**A5**]. The deeper properties of the hypothetical

Langlands group L_F probably mean that its existence will be one of the last theorems to be proved in the subject. However, if L_F does exist, its expected properties imply that the family of $\mathrm{Aut}_{\tilde{G}}(G)$-orbits of homomorphisms

$$L_F \times SL(2,\mathbb{C}) \longrightarrow {}^L G$$

is in natural bijection with the set $\widetilde{\Psi}(G)$ we have just defined. Moreover, this bijection identifies the corresponding centralizers S_ψ and their quotients $\mathcal{S}_\psi$. It is also compatible with the localization

$$\psi \longrightarrow \psi_v$$

of parameters, something we will not discuss here.

This all means that our makeshift groups $\mathcal{L}_\psi$ capture the information from L_F that is relevant to the endoscopic classification of representations of G. In other words, the groups $\mathcal{L}_\psi$ are as good as the Langlands group for the purposes at hand, even though they vary with ψ. They are used in [**A5**] to formulate the classification of automorphic representations of G.

6. The ψ-components of distributions

The next step is to isolate the ψ-components of the terms in the expansions (1), (2), $(\tilde{1})$ and $(\tilde{2})$. Recall that a parameter $\psi \in \widetilde{\Psi}(N)$ comes with an "eigenvalue" $c(\psi)$. If D is a distribution that occurs in one of these expansions, its ψ-component D_ψ is a "ψ-eigendistribution", relative to the convolution action of the unramified Hecke algebra on the test function f (or $\widetilde{f}$). We thus obtain two expansions

$$(1)_\psi \qquad I^G_{\mathrm{disc},\psi}(f) = \sum_M |W(M)|^{-1} \sum_{w \in W(M)_{\mathrm{reg}}} |\det(w-1)| \mathrm{tr}\big(M_{P,\psi}(w)\mathcal{I}_{P,\psi}(f)\big)$$

and

$$(2)_\psi \qquad I^G_{\mathrm{disc},\psi}(f) = \sum_{G'} \iota(G,G')\widehat{S}^{G'}_{\mathrm{disc},\psi}(f^{G'})$$

of the ψ-component $I^G_{\mathrm{disc},\psi}(f)$. Similarly, we obtain two expansions $(\tilde{1})_\psi$ and $(\tilde{2})_\psi$ for the ψ-component $I^{\tilde{G}}_{\mathrm{disc},\psi}(\widetilde{f})$ of $I^{\tilde{G}}_{\mathrm{disc}}(\widetilde{f})$. The problem is to compare explicitly the terms in these two identities.

We are trying to describe these matters in the context of the embedded eigenvalue problem. According to general conjecture, a parameter $\psi \in \widetilde{\Psi}(G)$ would be expected to contribute to the discrete spectrum of G if and only if the group

$$\bar{S}_\psi = S_\psi / Z(\widehat{G})^{\Gamma_F}$$

is finite. In other words, the component group

$$\mathcal{S}_\psi = \pi_0(\bar{S}_\psi)$$

that is supposed to govern spectral multiplicities is actually equal to $\bar{S}_\psi$. If we apply this inductively to a Levi subgroup M of G, we see that ψ contributes Eisenstein series of rank k to the spectrum of G if and only if the rank of $\bar{S}_\psi$ equals k. The problem then is to show that if $\bar{S}_\psi$ is *not* finite, it does not contribute to the discrete spectrum of G. That is, there is no automorphic representation π of G in the discrete spectrum with $c(\psi) = c(\pi)$.

One has thus to show that if $\bar{S}_\psi$ is infinite, the term in $(1)_\psi$ with $M = G$ vanishes. However, we know nothing about this term. We can say (by induction) that ψ contributes to the term corresponding to a *unique* proper M. We would first try to express this term as concretely as possible. We would then want to express the terms on the right hand side of $(2)_\psi$ in such a way that their sum could be seen to cancel the term of M in $(1)_\psi$. This would tell us that the term of G in $(1)_\psi$ vanishes, as desired. But the distributions in $(2)_\psi$ are by no means explicit. They consist of the stable linear form $S^G_{\mathrm{disc},\psi}(f)$, about which we know very little, and its analogues for proper endoscopic groups G', which are at least amenable to induction. To deal with $S^G_{\mathrm{disc},\psi}(f)$, we have to compare the right hand side of $(2)_\psi$ (as G varies) with the right hand side of $(\tilde{2})_\psi$. We would then have to compare $(\tilde{2})_\psi$ with the expression on the right hand side of $(\tilde{1})_\psi$, about which we do know something (because it pertains to $GL(N)$).

7. Statement of theorems

It is a rather elaborate process. We shall describe the theorems that lead to a resolution of the problem. Our statements of these theorems will have to be somewhat impressionistic, since we will not take the time to describe all their ingredients precisely. We refer the reader to the forthcoming volume [**A5**] for a full account.

Theorem 2 (Stable Multiplicity Formula). *Suppose that $\psi \in \widetilde{\Psi}(G)$. Then the term in $(2)_\psi$ corresponding to $G' = G$ satisfies an explicit formula*

$$S^G_{\mathrm{disc},\psi}(f) = m_\psi |\mathcal{S}_\psi|^{-1} \sigma(\bar{S}^0_\psi)\varepsilon_\psi(s_\psi) f^G(\psi),$$

where $m_\psi \in \{1,2\}$ equals the number of $\widehat{G}$-orbits in the $\mathrm{Aut}_{\tilde{G}}(G)$-orbit of embeddings $\widetilde{\psi}_G$, $\varepsilon(s_\psi) = \pm 1$ is a sign defined in terms of values at $s = \frac{1}{2}$ of global ε-factors attached to ψ, and $\sigma(\bar{S}^0_\psi)$ is the number attached to the complex connected group $\bar{S}^0_\psi$ in Theorem 4 below.

The last term $f^G(\psi)$ in the formula is harder to construct. It represents the pullback to $G(\mathbb{A})$ of the twisted character

$$\mathrm{tr}\big(\pi_\psi(\widetilde{f})\big), \qquad \widetilde{f} \in C_c^\infty\big(\widetilde{G}(\mathbb{A})\big),$$

on $GL(N,\mathbb{A})$. (We use the theory of Whittaker models for $GL(N)$ to extend the θ-stable representation π_ψ to the component

$$\widetilde{G}(\mathbb{A}) = GL(N,\mathbb{A}) \rtimes \theta$$

on which $\widetilde{f}$ is defined.) The construction is essentially local. Since the criterion of Theorem 1 that determines the subset $\widetilde{\Psi}(G)$ of $\widetilde{\Psi}(N)$ to which ψ belongs is global, the definition of $f^G(\psi)$ requires effort. It is an important part of the proof of Theorem 2.

The formula of Theorem 2 is easily specialized to the other summands in $(2)_\psi$. For any G', it gives rise to a sum over the subset $\widetilde{\Psi}(G',\psi)$ of parameters $\psi' \in \widetilde{\Psi}(G')$ that map to ψ. The formulas so obtained can then be combined in the sum over G'. The end result is an explicit expression for the right hand side of $(2)_\psi$ in terms of the distributions

$$f^{G'}(\psi'), \qquad f \in C_c^\infty\big(G(\mathbb{A})\big),\ \psi' \in \widetilde{\Psi}(G',\psi),$$

and combinatorial data attached to the (nonconnected) complex reductive group $\overline{S}_\psi$.

THEOREM 3. *Suppose that $\psi \in \widetilde{\Psi}(G)$ contributes to the induced discrete spectrum of a proper Levi subgroup M of G, and that w lies in $W(M)_{\mathrm{reg}}$. Then there is a natural formula for the corresponding distribution*

$$\mathrm{tr}\big(M_{P,\psi}(w)\mathcal{I}_{P,\psi}(f)\big)$$

in $(1)_\psi$ in terms of

(i) *the distributions*

$$f^{G'}(\psi'), \qquad \psi' \in \widetilde{\Psi}(G',\psi),$$

(ii) *the order of poles of global L-functions at $s = 1$, and*

(iii) *the values of global ε-factors at $s = \frac{1}{2}$.*

In this case, we have not tried to state even a semblance of a formula. However, the resulting expression for the sum in $(1)_\psi$ will evidently have ingredients in common with its counterpart for $(2)_\psi$ discussed above. It will also have two points of distinction. In $(1)_\psi$ there will be only one vanishing summand (other than the summand of G we are trying to show also vanishes). Furthermore, the summand of M contains something interesting beyond the distribution above, the coefficient

$$|\det(1-w)|^{-1}.$$

One sees easily that the distribution of Theorem 3 vanishes unless w has a representative in the subgroup $\overline{S}_\psi$ of $\widehat{G}/Z(\widehat{G})^{\Gamma_F}$. We can therefore analyze the combinatorial properties of the coefficients in the context of this group.

Suppose for a moment that S is any connected component of a general (nonconnected) complex, reductive algebraic group S^+. Let T be a maximal torus in the identity component $S^0 = (S^+)^0$ of this group. We can then form the Weyl set

$$W = W(S) = \mathrm{Norm}_S(T)/T,$$

induced by the conjugation action of elements in S on T. Let W_{reg} be the set of elements w in W that are regular, in the sense that as a linear operator on the real vector space

$$\mathfrak{a}_T = \mathrm{Hom}(X(T),\mathbb{R}),$$

the difference $(1-w)$ is nonsingular. We define the sign $\varepsilon^0(w) = \pm 1$ of an element $w \in W$ to be the parity of the number of positive roots of (S^0, T) mapped by w to negative roots. Given these objects, we attach a real number

$$i(S) = |W|^{-1} \sum_{w \in W_{\mathrm{reg}}} \varepsilon^0(w)|\det(w-1)|^{-1}$$

to S.

As is often customary, we write S_s for the centralizer in S^0 of a semisimple element $s \in S$. This is of course a complex reductive group, whose identity component we denote by S_s^0. We then introduce the subset

$$S_{\mathrm{ell}} = \big\{s :\ |Z(S_s^0)| < \infty\big\},$$

where $Z(S_1)$ denotes the center of any given complex connected group S_1. The set $\mathrm{Orb}(S_{\mathrm{fin}}, S^0)$ of orbits in S_{ell} under conjugation by S^0 is finite.

THEOREM 4. *There are unique constants $\sigma(S_1)$, defined whenever S_1 is a complex connected reductive group, such that for any S the number*

$$e(S) = \sum_{s \in \mathrm{Orb}(S_{\mathrm{ell}}, S^0)} |\pi_0(S_s)|^{-1}\sigma(S_s^0)$$

equals $i(S)$, *and such that*

$$\sigma(S_1) = \sigma(S_1/Z_1)|Z_1|^{-1},$$

for any central subgroup $Z_1 \subset Z(S_1)$ *of* S_1.

The numbers $i(S)$ and $e(S)$ of the theorem are elementary. However, they bear an interesting formal resemblance to the deeper expansions on the right hand sides of $(1)_\psi$ and $(2)_\psi$ respectively. In particular, the data in $(2)_\psi$ are vaguely endoscopic. I have sometimes wondered whether Theorem 4 represents some kind of broader theory of endoscopy for Weyl groups.

The proof of the theorem is also elementary. It was established in §8 of [**A2**]. We have displayed the result prominently here because of the link it provides between Theorems 2 and 3, or rather between the expressions for the right hand sides of $(1)_\psi$ and $(2)_\psi$ that these theorems ultimately yield. We have discussed these expressions in only the most fragmentary of terms. We add here only the following one-line summary. If the summand of G in $(1)_\psi$ is put aside, the two expressions are seen to match, up to coefficients that reduce respectively to the numbers $i(S)$ and $e(S)$ attached to the components S of the group $\bar{S}_\psi$. Theorem 4 then tells us that the right hand of $(2)_\psi$ equals the difference between the right side of $(1)_\psi$ and the summand of G in $(1)_\psi$. Since the left hand sides of $(1)_\psi$ and $(2)_\psi$ are equal, the summand of G does vanish for any $\psi \in \widetilde{\Psi}(G)$ with $\bar{S}_\psi$ infinite, as required. We thus obtain the following theorem.

THEOREM 5. *The automorphic discrete spectrum of G has no embedded eigenvalues.*

This is the result we set out to describe. As we have said, it is part of a general classification of the automorphic representations of G. The reader will have to refer to [A4, §30] and [A5] for a description of the classification. However, the theorems discussed here are at the heart of its proof.

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