Zeroes in the Bakhshali manuscript

by Bill Casselman

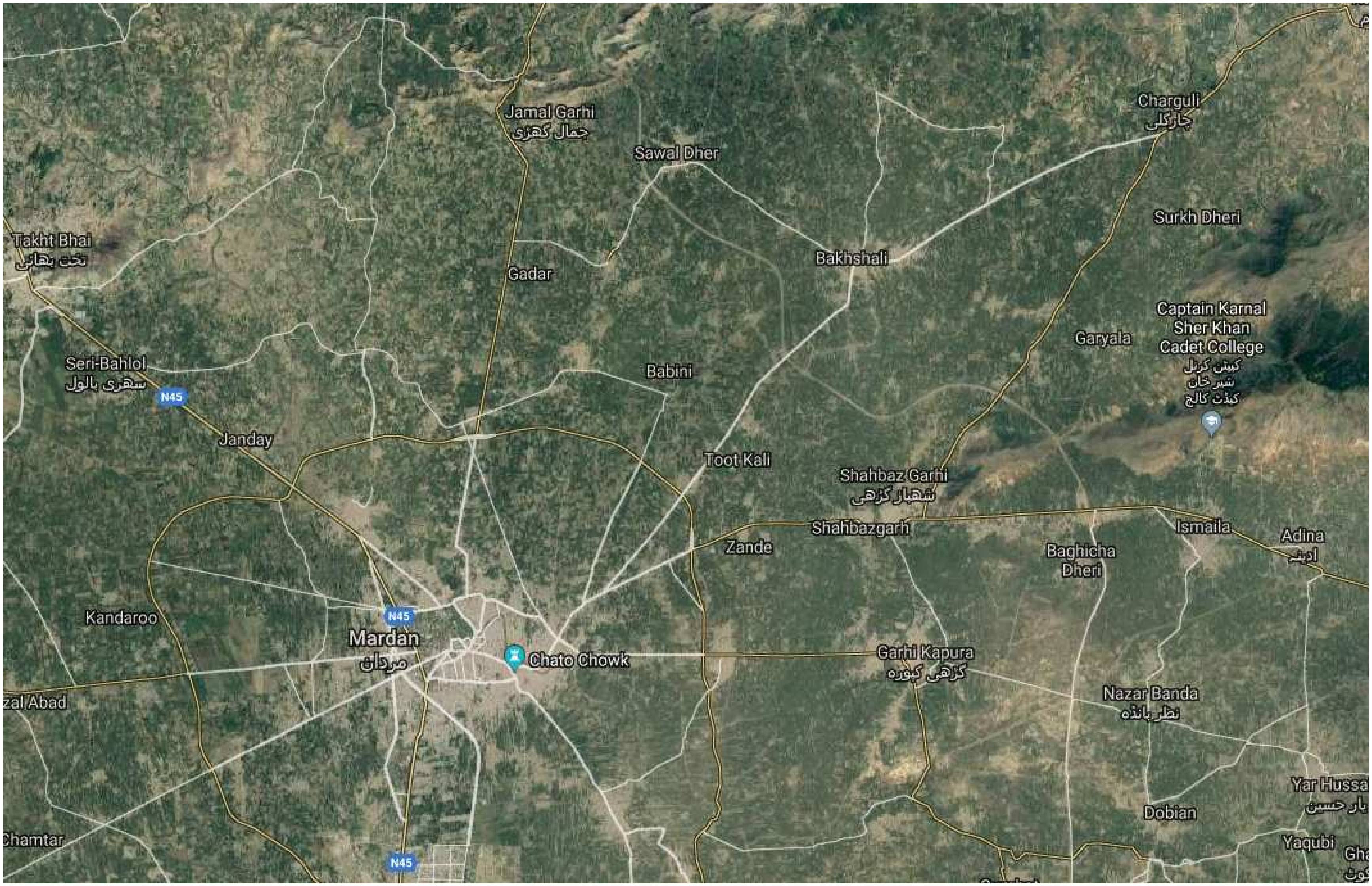
I cannot post this talk because of copyright restrictions, but there is much overlap with

`http://www.ams.org/publicoutreach/feature-column/fc-2018-06`

The Bakhshali manuscript amounts to 70 ‘pages’ of mathematics written in ink on birch bark. It was unearthed in 1881 by a local resident near the village of Bakhshali, a few kilometres from Mardan, in what was then part of British India, but now part of Pakistan.

It is a mostly a collection of problems in algebra and their solutions, with little connection to each other and no apparent overall theme. It is presumably an amalgamation of earlier works.

There is no known direct predecessor and no later referencs to it. It is not unrelated to other Indian mathematics of the period, but has a number of unique features. For one thing, it is by far the oldest extant mathematics manuscript. Birch bark does not survive long in a damp climate.



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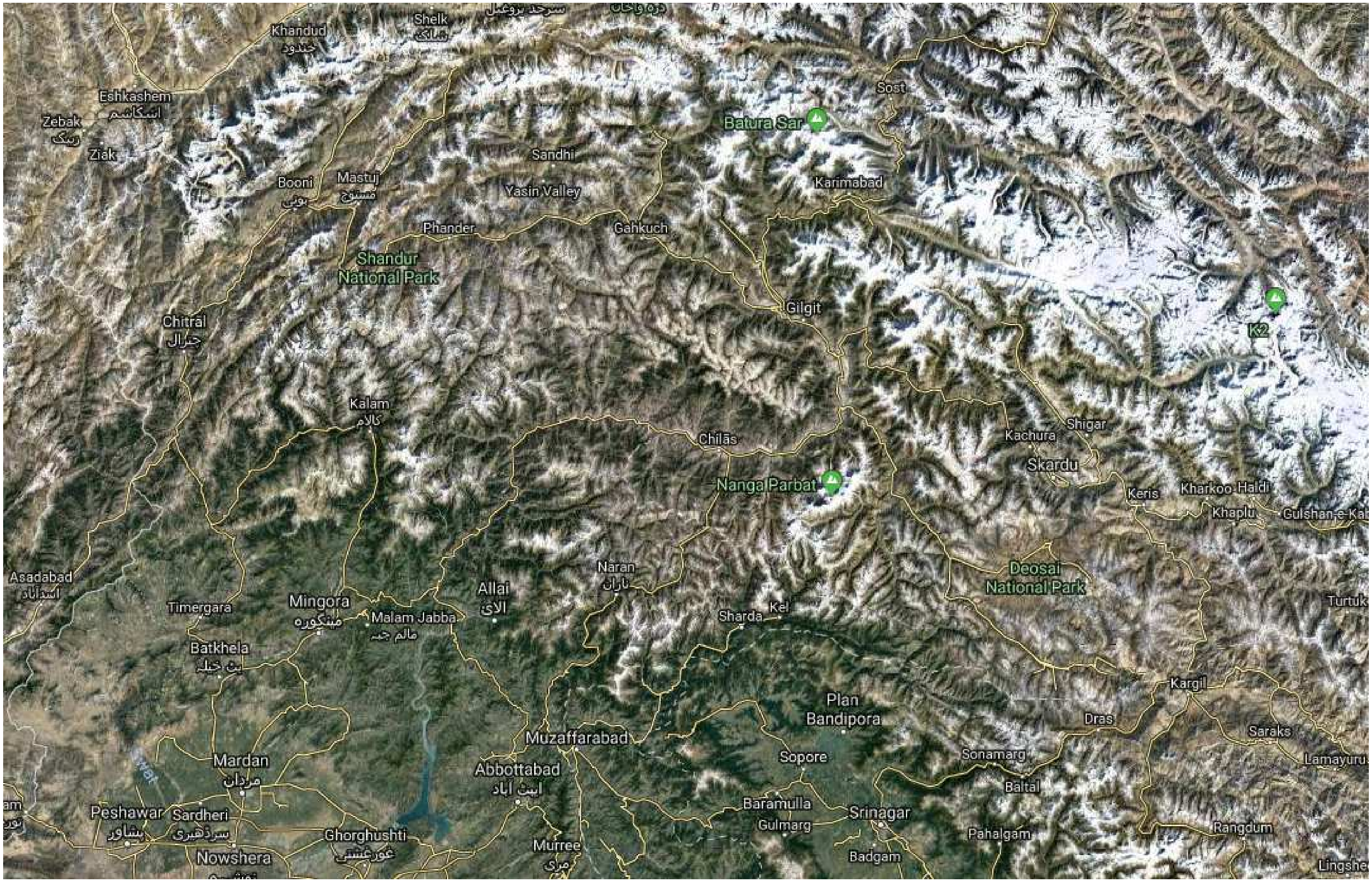

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Sopore
Sonamarg
Lamayuru
Abbottabad
Baltal
Peshawar
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Baramulla
Srinagar
Gulmarg
Ghorghushti
Pahalgam
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Murree
Nowshera
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Кӯлоб
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Хорог
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Corridor
Nature Refuge
Qala-e
Mafushad
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和田地区
Taleqan
تالقان
Khanabad
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Chitral
چترال
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Skardu
Bagram
Kabul
کابل
Jalalabad
جلال آباد
Mingora
Allai
Naran
Abbottabad
ایبٹ آباد
Peshawar
پشاور
Islamabad
اسلام آباد
Kohat
Rawalpindi
راولپنڈی
Gardez
Khost
خوست
Bannu
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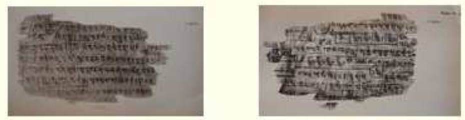
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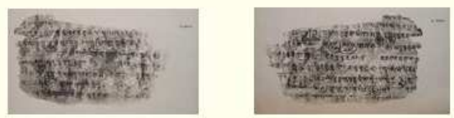
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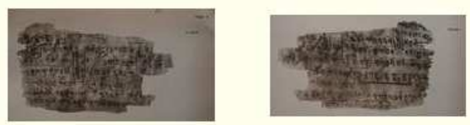
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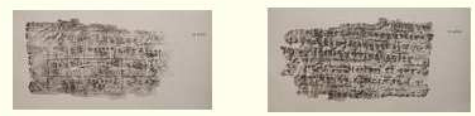
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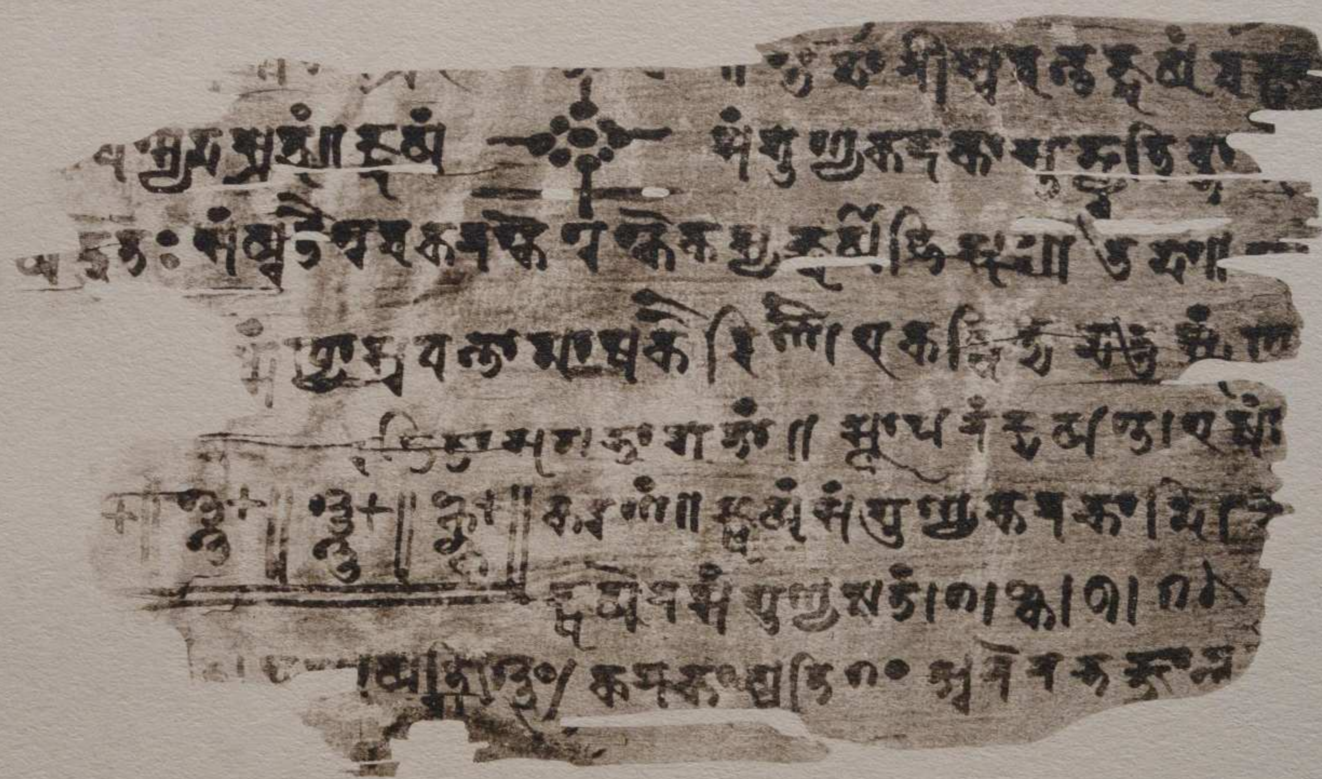

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THE BAKHSHĀLĪ MANUSCRIPT

A Study in Mediæval Mathematics

BY

G. R. KAYE

CALCUTTA: GOVERNMENT OF INDIA
CENTRAL PUBLICATION BRANCH
1927

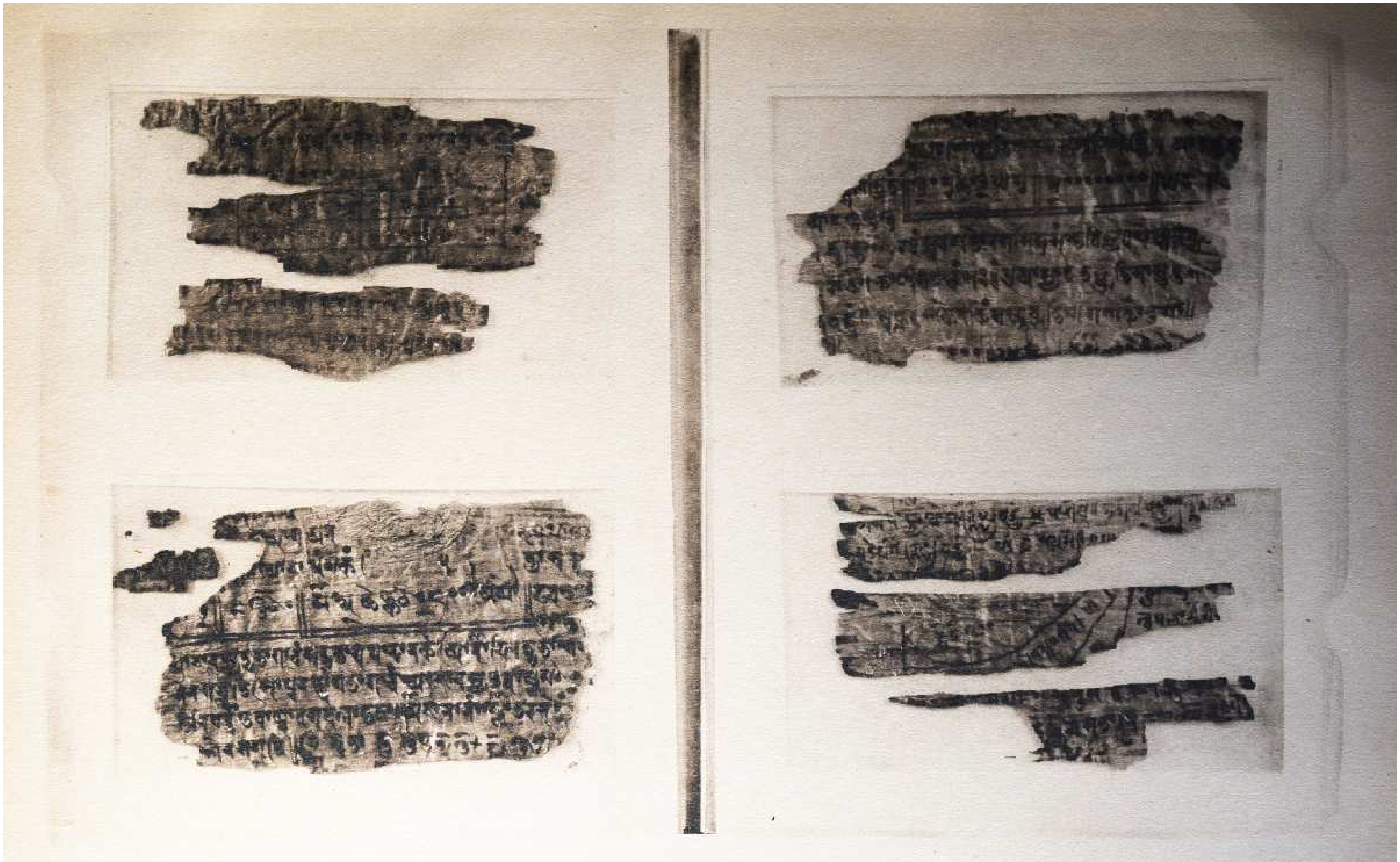
Price Rs.

Literature:

Hoernle (1887), Kaye (1927), Datta (1929), Hayashi (1995)

`http://www.math.ubc.ca/ cass/bakhshali/`

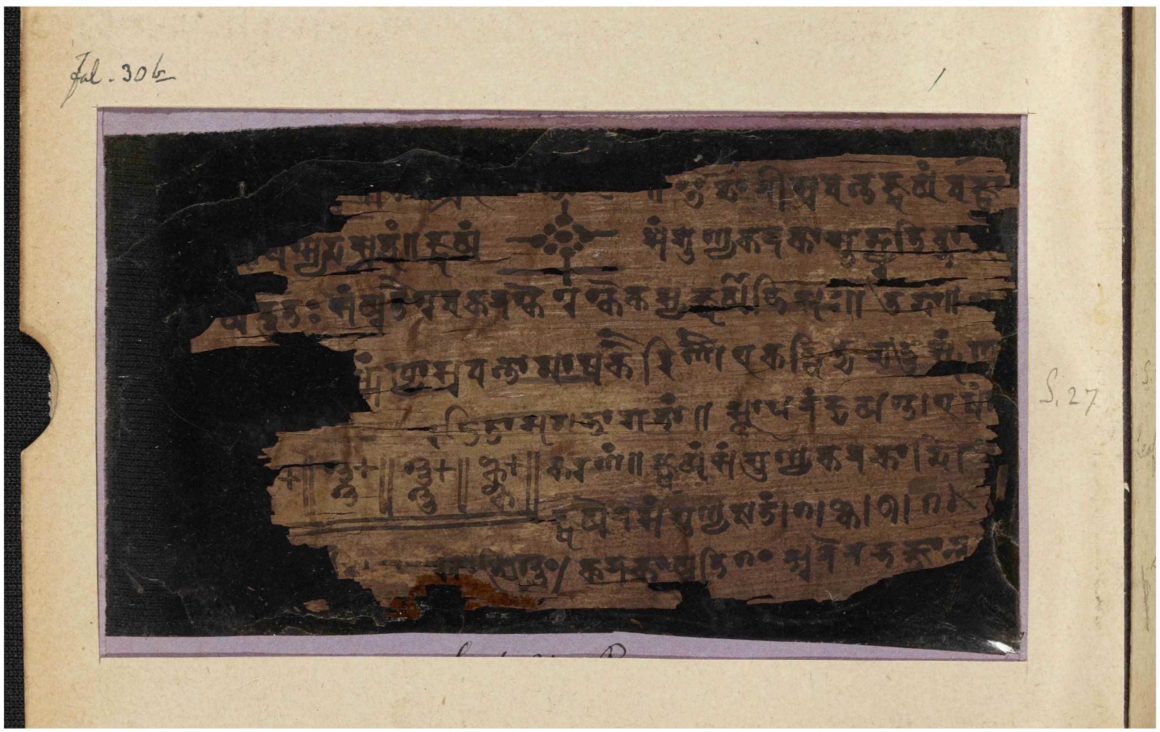
After the photographs of the manuscript were taken for Kaye's edition, all but one of them was encased in mica sheets and made into an album. The following image, taken from Kaye's book, shows what a typical pair of facing pages looked like around 1927.



THE BAKHSHĀLĪ MANUSCRIPT AS PRESERVED IN THE BODLEIAN LIBRARY
Folios
35 VERSO | 37 RECTO
36 VERSO | 38 RECTO

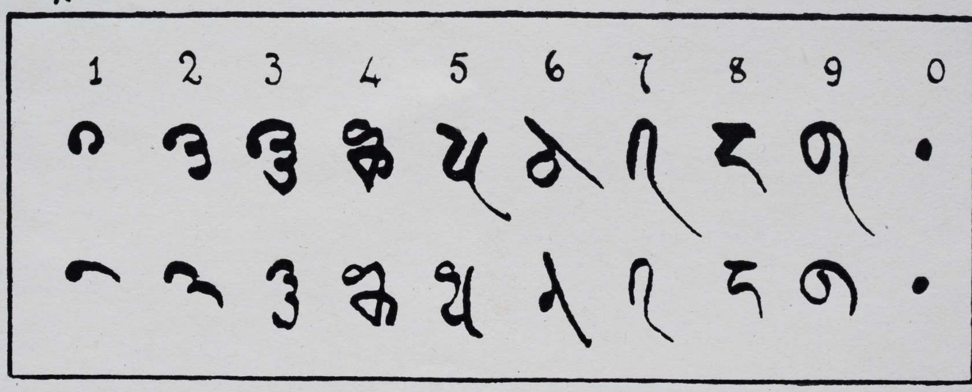
A recent image from a publicity release by the Bodleian Library:

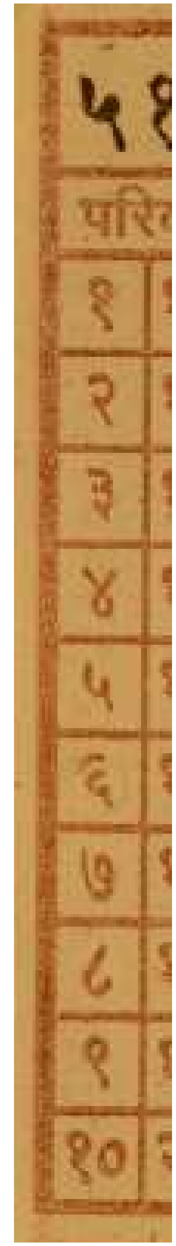
The mica sheets were an unfortunate choice. The bark has stuck to them and broken up into hundreds and hundreds of tiny fragments. In addition, the bark has darkened considerably. It is now impossible to restore the manuscript in a good way, and at first sight you might think that taking new photographs of the manuscript would be out of the question. However, modern technology is wonderful, and the good news is that it is apparently possible to take good photographs of the pages even though they are extremely dark and encased in mica. You can see this in one of the recent photographs issued by the Bodleian Library.

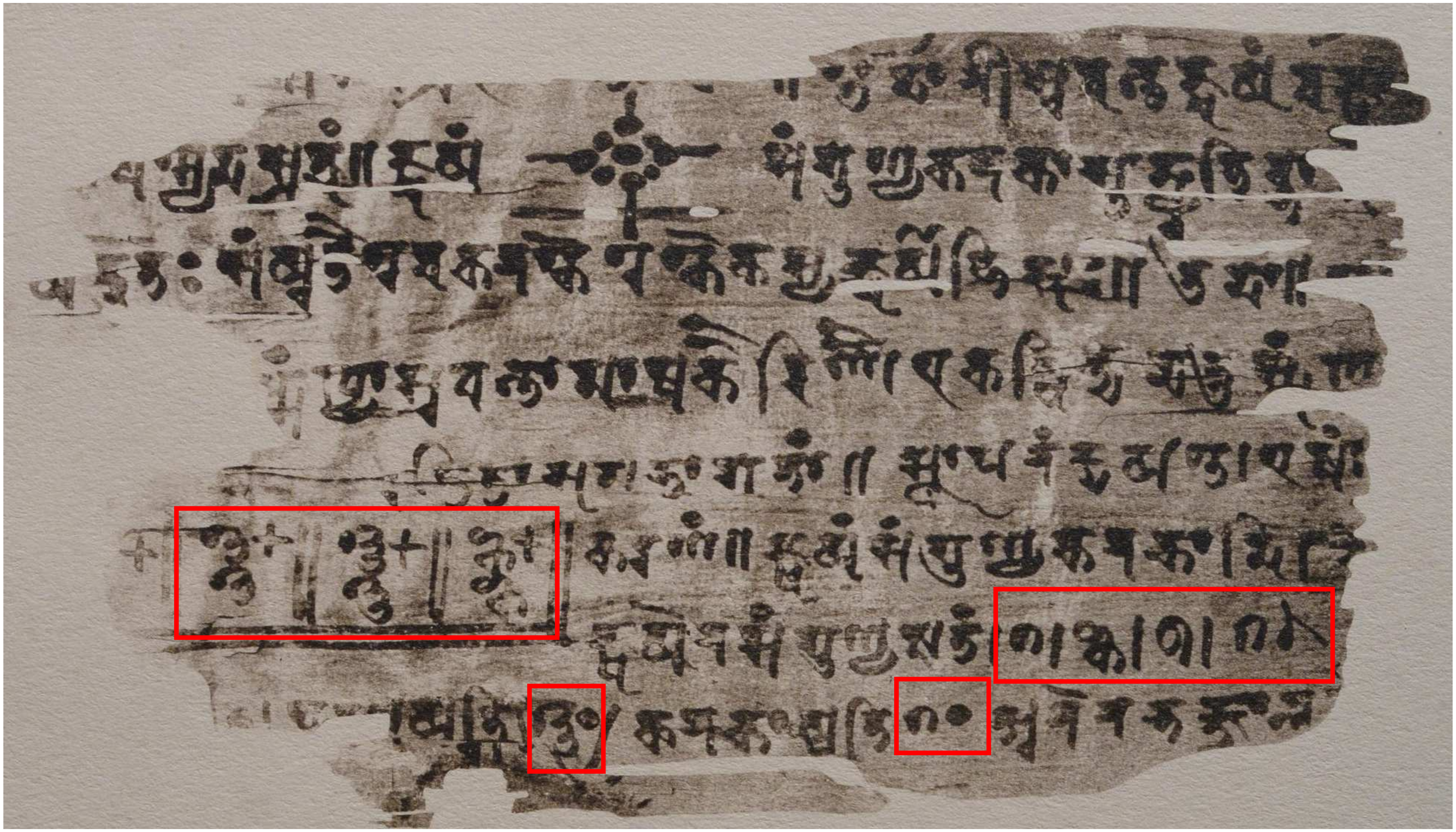


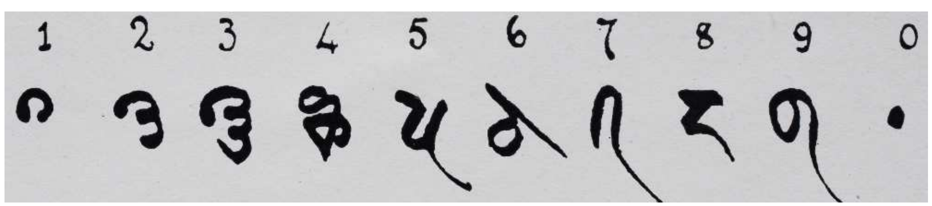
One can hope, although probably in vain, that the Bodleian Library will photograph and post the entire collection.

7. Numetals

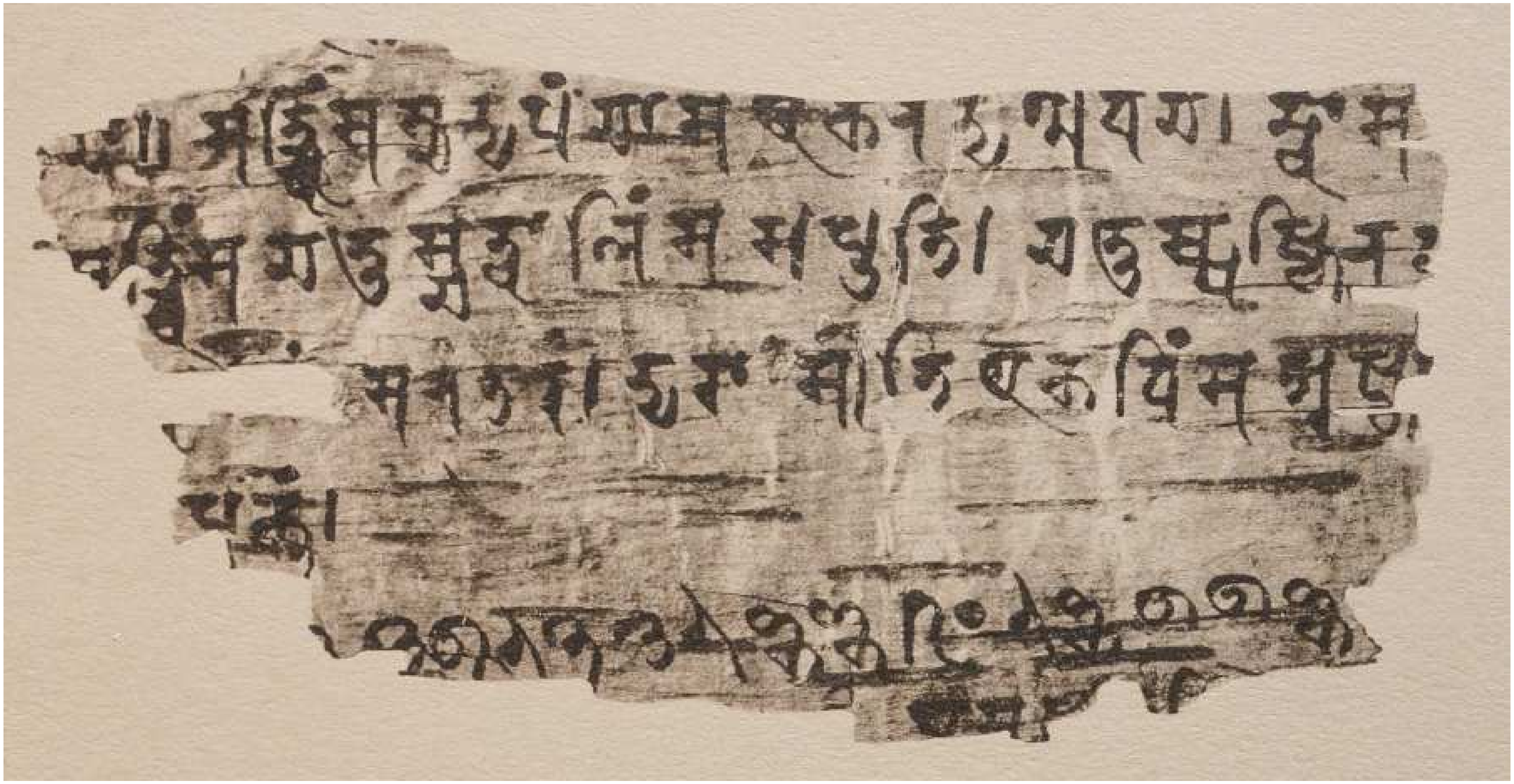




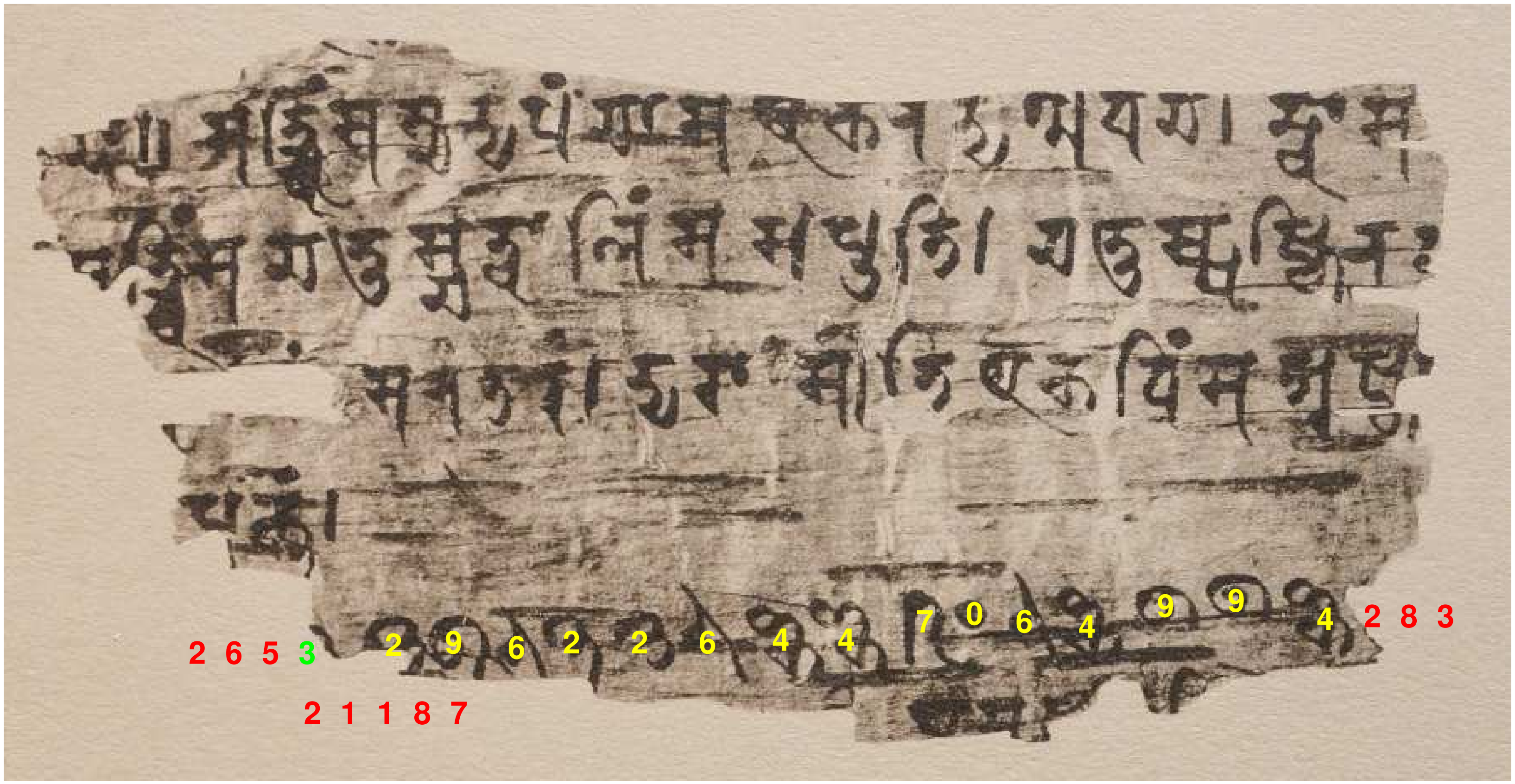



1 2 3 4 5 6 7 8 9 0

The manuscript contains, in written form, some very large numbers. The largest one is $26\,53\,29\,62\,26\,44\,70\,64\,99\,42\,83\,21\,18\,7$, and it appears in a rather strange way. It is along the bottom of 58r:



Well, *some* **of it is along the bottom of 58r. Where do the rest of the digits come from?**



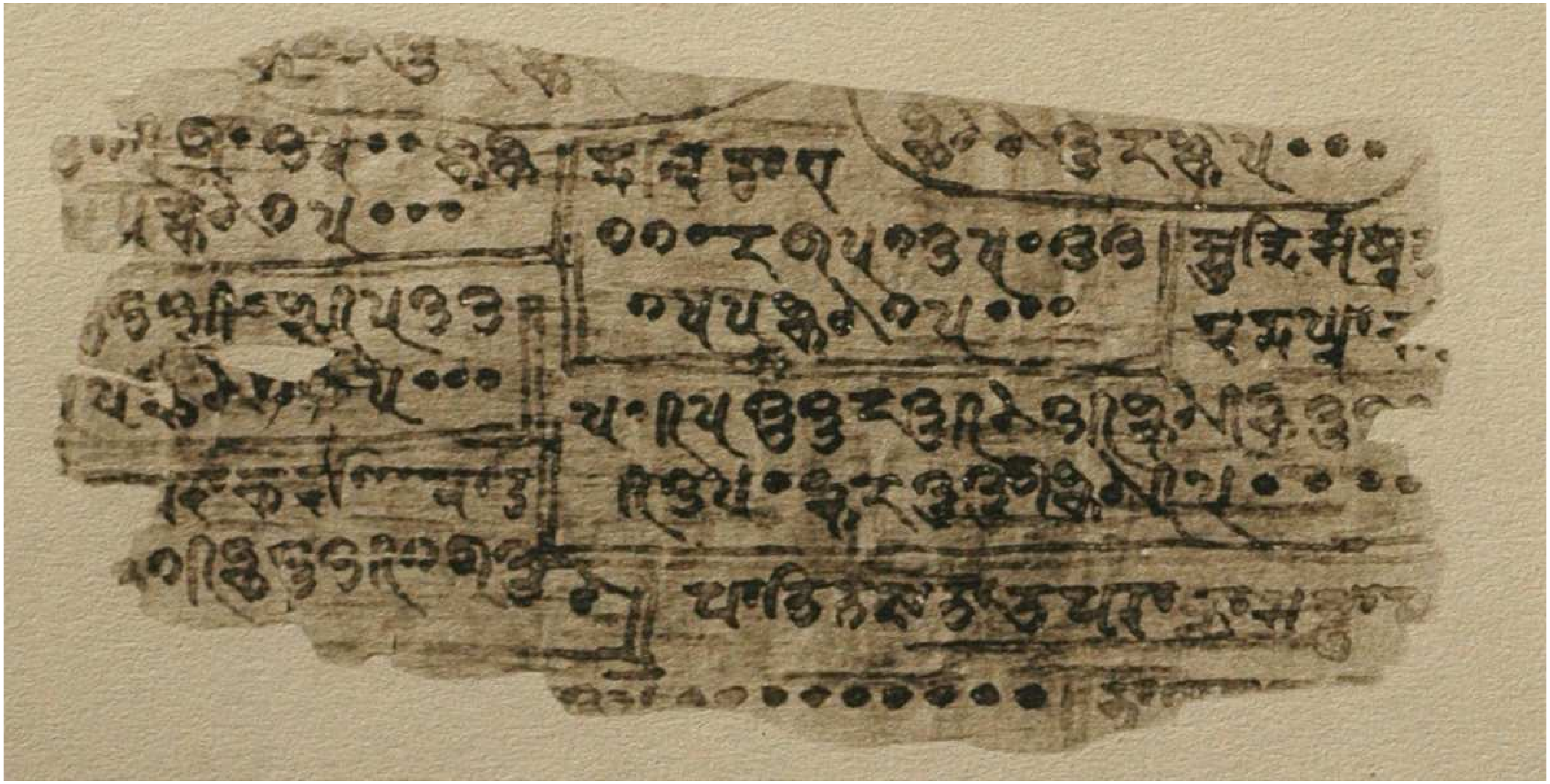
I have displayed in yellow the numerals that one sees clearly, in green those that one sees part of, and in red those one conjectures. What is going on is perhaps a unique feature of Indian mathematics.

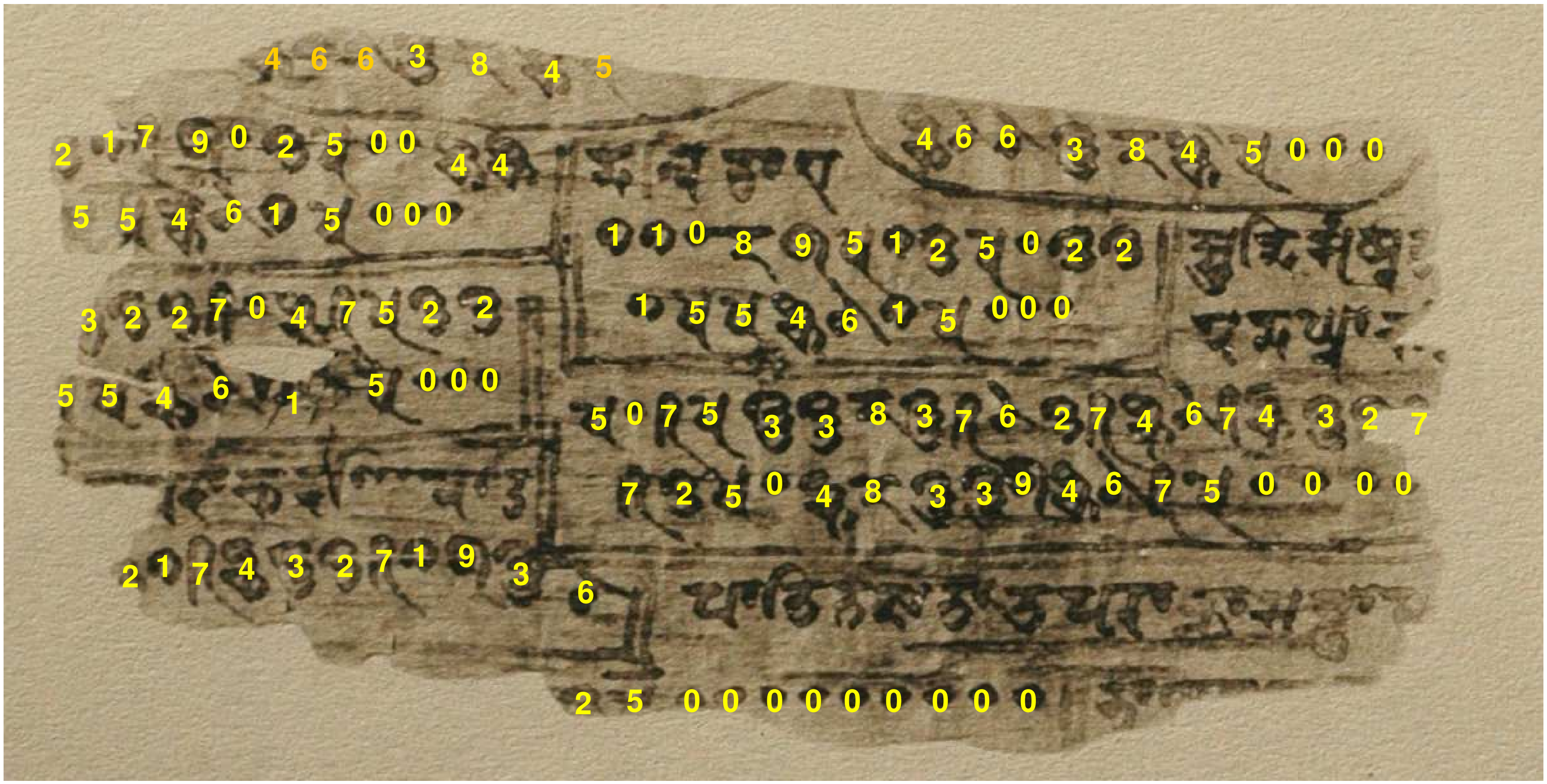
Almost all of ancient Indian literature is in verse, and of course numerals do not exactly scan. Numbers are often written as words. Towards the top of 58r a sequence of words is found that records a single number as pairs of digits - for example, Sanskrit versions of 'six-and-twenty' for 26 and 'one-less-than-thirty' for 29—and the words at the top of the page read (but also with a few items guessed at)

$$26\ 53\ 29\ 62\ 26\ 44\ 70\ 64\ 99\ 42\ 83\ 21\ 18\ 7\,.$$

There is no mathematics at all! Why this particular number and no other? Where this page comes from is a mystery. As I have said, the manuscript is a collection of miscellaneous things.

My favourite page is 46r:





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5 5 4 6 1 5 0 0 0
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3 2 2 7 0 4 7 5 2 2
1 5 5 4 6 1 5 0 0 0
5 5 4 6 1 5 0 0 0
5 0 7 5 3 3 8 3 7 6 2 7 4 6 7 4 3 2 7
7 2 5 0 4 8 3 3 9 4 6 7 5 0 0 0 0
2 1 7 4 3 2 7 1 9 3 6
2 5 0 0 0 0 0 0 0 0 0


4 4 3 5 8 0 5 0 0 0 8 8
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5 0 7 5 3 3 8 3 7 6 2 7 2 5 0 0 0 0 0 0 0 0 0
7 2 5 0 4 8 3 3 9 4 6 7 5 0 0 0 0 0 0

What’s going on here?

The basic pattern in the manuscript is this: (1) A rule (called a *sutra*) for solving a mathematical problem is stated or at least alluded to; (2) it is followed by some examples of how the solution is carried out; (3) just after each example, some kind of check is run on the solution. The *sutra* is stated in words instead of an algebraic formula. There can be a great deal of arithmetic involved, but in general things are explained in such a way that one might as well be using algebraic variables.

One category of problems that occurs strikes us as very strange.

In elementary mathematics classes, one is often given an arithmetic progression in terms of (i) its initial term a, (ii) the common difference between terms Δ, amd (iii) a certain number of terms in the series. One is then asked to give the sum of the series. The formula for the sum is

$$S = \left[\frac{(n-1)\Delta}{2} + a \right] n \, .$$

But one can also formulate an inverse problem: given the sum S of the series, *how many terms are there in the progression?* **This comes down to solving for n in the formula above. This gives a quadratic equation, which one can solve to see that**

$$n = \frac{\sqrt{8\Delta S + (2a-\Delta)^2} - (2a - \Delta)}{2\Delta} \, .$$

For example, if I tell you that $\Delta = 3$, $a = 2$, and $S = 442$, then $2a - \Delta = 1$ and you can calculate

$$n = \frac{\sqrt{8 \cdot 3 \cdot 442 + 1^2} - 1}{2 \cdot 3} = 17\,.$$

Furthermore—and this is very important—you can check to see that this correct by computing

$$\left(\frac{(17-1) \cdot 3}{2} + 2\right) 17 = 442\,.$$

But in the Bakhshali manuscript there generally arises a weird version of this problem—the number of terms, which must be an integer if the problem is to make any real sense, is rarely found to be one. In fact, in the Bakhshali manuscript it is $never$ one. The motivation for these problems is rather mysterious.

Finding the value of n involves finding the square root of an integer. Now since the final n is not an integer, the exact square root cannot be a rational number. You can only find an approximation to it. One of the most intriguing features of the manuscript is precisely its technique for finding approximations to square roots of integers that are not perfect squares. It almost seems that the person who made up these problems is far more interested in demonstrating his superior square-root approximation techniques than in posing problems with any real meaning.

This is made more interesting because it was only somewhat later that Arabian mathematicians invented decimal fractions (like $1.414213562\ldots$), so that |sl all work was to be done with ordinary fractions (like $665857/470832$). What is very impressive to us is that the approximation technique involves some serious rational arithmetic. It is even more impressive that checking the correctness of the solution, involves some *extremely* serious rational arithmetic, as we shall see.

This sort of question about arithmetic progressions with non-integral n occurs elsewhere in Indian mathematical treatises. Dealing with such very large numbers as the Bakhshali manuscript does is very, very rare. I am not aware of anything like this in other cultures until mathematicians much later started to compute approximations to π.

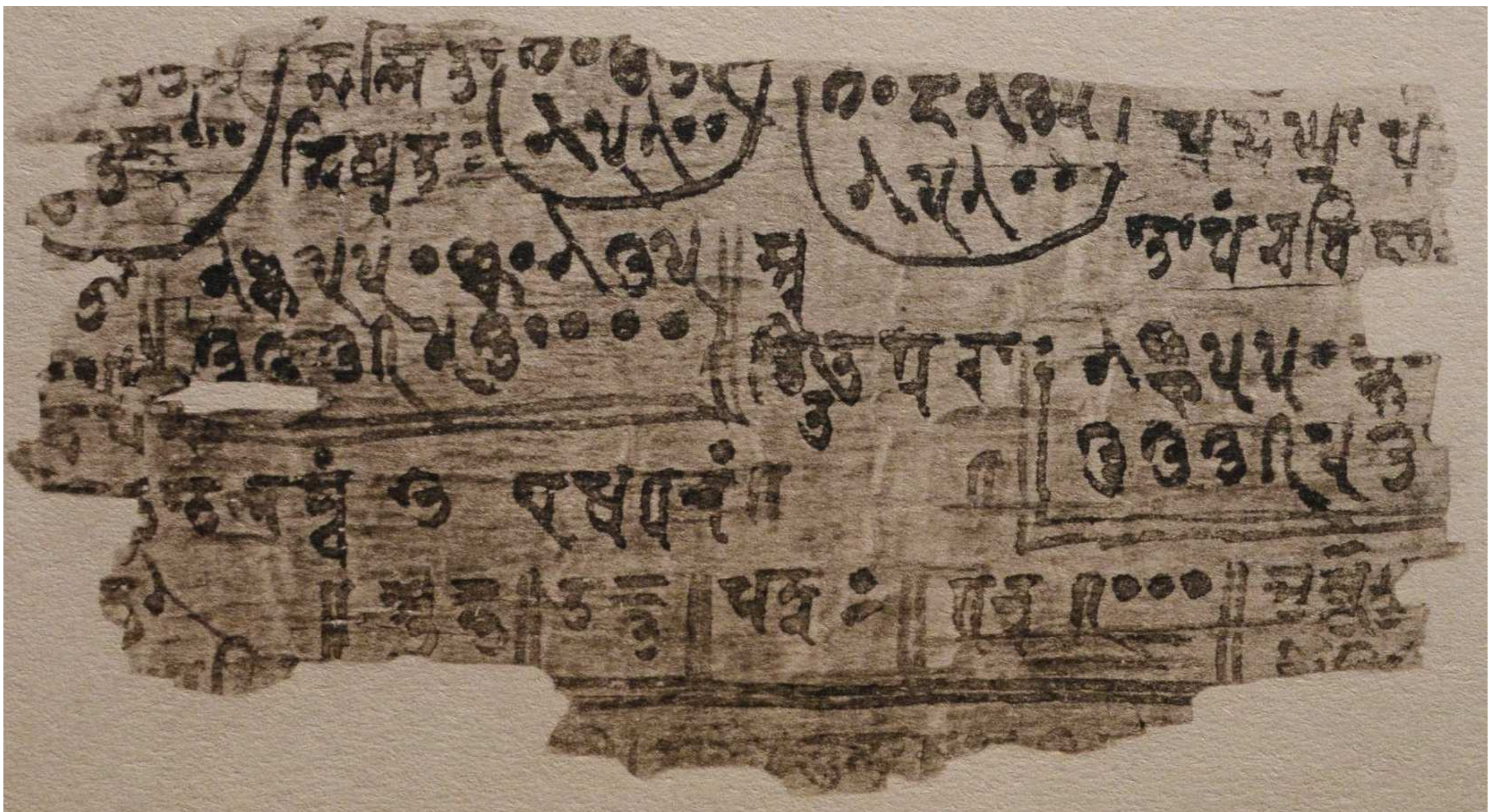
In the Bakhshali manuscript, this sort of problem occurs on the pages 65v, 56vr, 64rv, 57vr, 45rv, and 46r (this illustrates that Hayashi's order is different from Kaye's). Unfortunately, these pages are damaged, and all lack certain parts of the computations. However, there remains enough that one can plausibly fill in the missing parts. This was attempted already by the first editor of the manuscript, G. R. Kaye, who understood the basic techniques involved but missed out on important details. These were added very soon after Kaye by the Indian historian of mathematics Bibhutibhusan Datta, and then finally the task was finished in the edition of Takao Hayashi. All in all, an impressive accomplishment.

Our basic formula is

$$n = \frac{\sqrt{8\Delta S + (2a - \Delta)^2} - (2a - \Delta)}{2\Delta}\,.$$

The problem that produces page 46r begins on 45r, and has as data

$$a = 3/2, \quad \Delta = 3/2, \quad S = 7000\,.$$



We then have $2a - \Delta = 3/2$ **and get**

$$n = \frac{\sqrt{8 \cdot (3/2) \cdot 7000 + (3/2)^2} - (3/2)}{2 \cdot 3/2}$$
$$= \frac{(\sqrt{336009}/2) - (3/2)}{3}.$$

$$n = \frac{(\sqrt{336009}\,/2) - (3/2)}{3}\,.$$

Now 336009 is not a perfect square. We would express it as $579.662833\ldots$ (the advantage of decimal fractions is that estimates of relative size are immediate). This option was not available to the author of the Bakhshali manuscript. Instead, with a little work he would see that

$$336009 = 579^2 + 768 = 580^2 - 391$$

and deduce only that $\sqrt{336009} = 579 +$ something between 0 and 1 . So we want to find

$$\sqrt{579^2 + 768}\,.$$

What next? Nearly all civilizations had from early days a reasonable approximate formula for square roots, and the Indians were no exception. The common formula asserts that

$$\sqrt{N^2 + A} \sim N + \frac{A}{2N} \, .$$

Since the days of Isaac Newton, we would derive it as the first terms in the series expansion $N\sqrt{1 + A/N^2}$.

We don't have any idea how this approximation was first found in India, but what is intriguing is that the Bakhshali manuscript knows of an even better one. The elementary formula says that is we have an approximation p_1 to $\sqrt{K}$ and $K = p_1^2 + E_1$ with E_1 small so that p_1 is a first approximation to $\sqrt{K}$, then

$$p_2 = p_1 + \frac{E_1}{2p_1}$$

is a better approximation.

In modern terms, the Bakhshali manuscript applies this yet one more time to get the approximation

$$p_3 = p_2 + \frac{E_2}{2p_2}$$

with $E_2 = K - p_2^2 = -(E_1/p_1)^2$**. The manuscript doesn't say anything about derivation, it just tells you to calculate the following in order approximate** $\sqrt{K}$ **for** $K = p_1^2 + E$**:**

$$p_2 = p_1 + \frac{E_1}{2p_1}$$
$$p_3 = p_2 - \frac{(E_1/2p_1)^2}{2p_2}\ .$$

In our case $K = 336009 = 569^2 + 768$**, so**

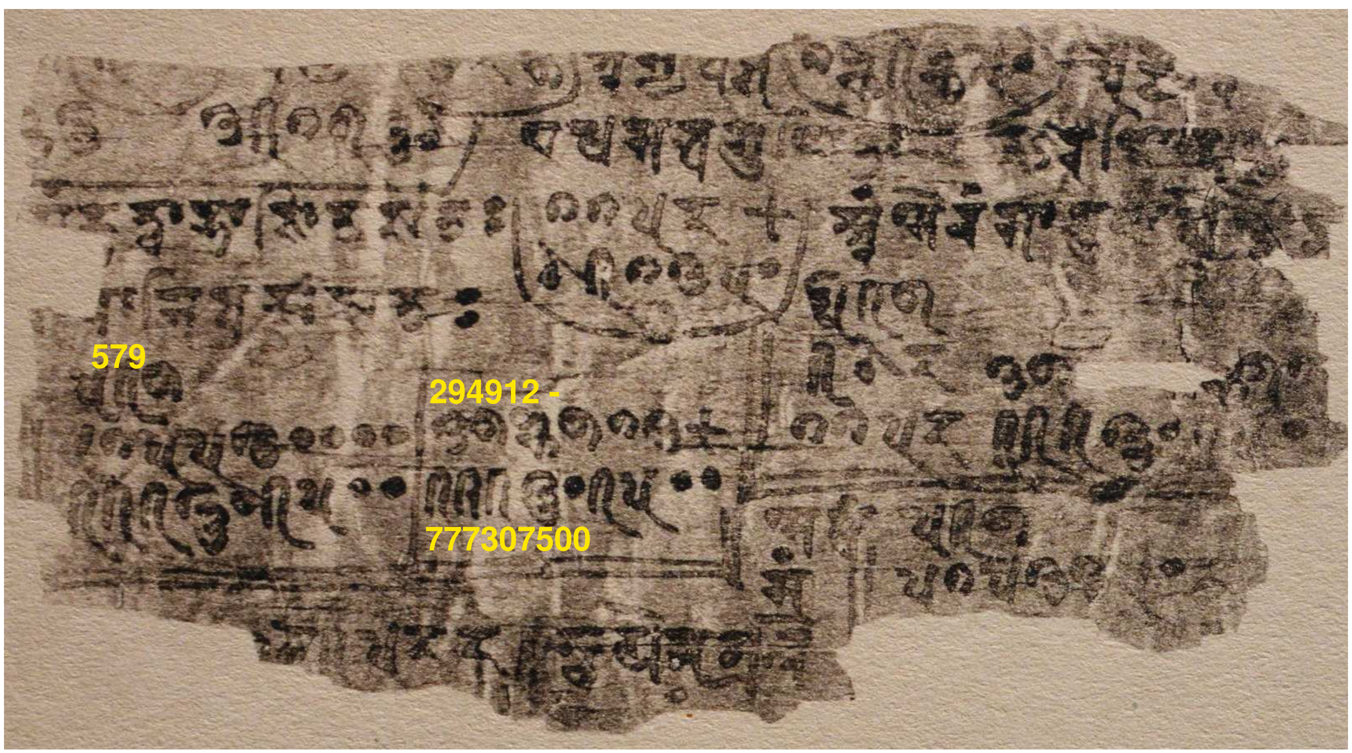
$$p_1 = 569, \quad E_1 = 768$$

$$p_2 = 569 + \frac{768}{2 \cdot 569} = \frac{111875}{193}$$

$$E_1/2p_1 = \frac{768}{1138} = \frac{128}{193}$$

$$(E_1/2p_1)^2 = \frac{294912}{777307500}$$

$$p_3 = p_2 - \frac{(E_1/2p_1)^2}{2p_2} = \frac{12516007433}{43183750}$$



We finally arrive at the impressive evaluation

$$\begin{aligned} n_3 &= \frac{(p_3/2) - (3/2)}{3} \\ &= \frac{448244345088}{4663845000} \\ &= \frac{6225615904}{64775625}\,. \end{aligned}$$

When tracking the computations in the manuscript, it is often necessary not to reduce fractions.

It is right at this point that 46r takes up the story. It is wholly concerned with checking the solution, i.e. checking that

$$S = \left(\frac{(n-1)\Delta}{2} - a \right) n \,.$$

There are couple of things to note, however. (1) Checking isn't going to work, because we are not using an exact value for $\sqrt{K'}$ but only an approximation. (2) The number n is quite large, and when we calculate S we are going to find ourselves multiplying two large numbers to get an even larger one. To be precise, the numerator of n has 12 digits, so in the process of calculating S we can expect to see around 24 digits.

To deal with the first problem, we have to take into account the error between our approximation p_3 to $\sqrt{K}$ and the exact value. In fact, it comes to knowing the difference $K - p_3^2$, which happens to be a by-product of the work. As for the second, it is true that we wind up looking at a fraction with incredibly large numerator and denominator, but the manuscript has a few tricks to make things work out. A kind of miracle occurs:

$$\frac{50753383762725000000000}{7250483394675000000} = 7000\,,$$

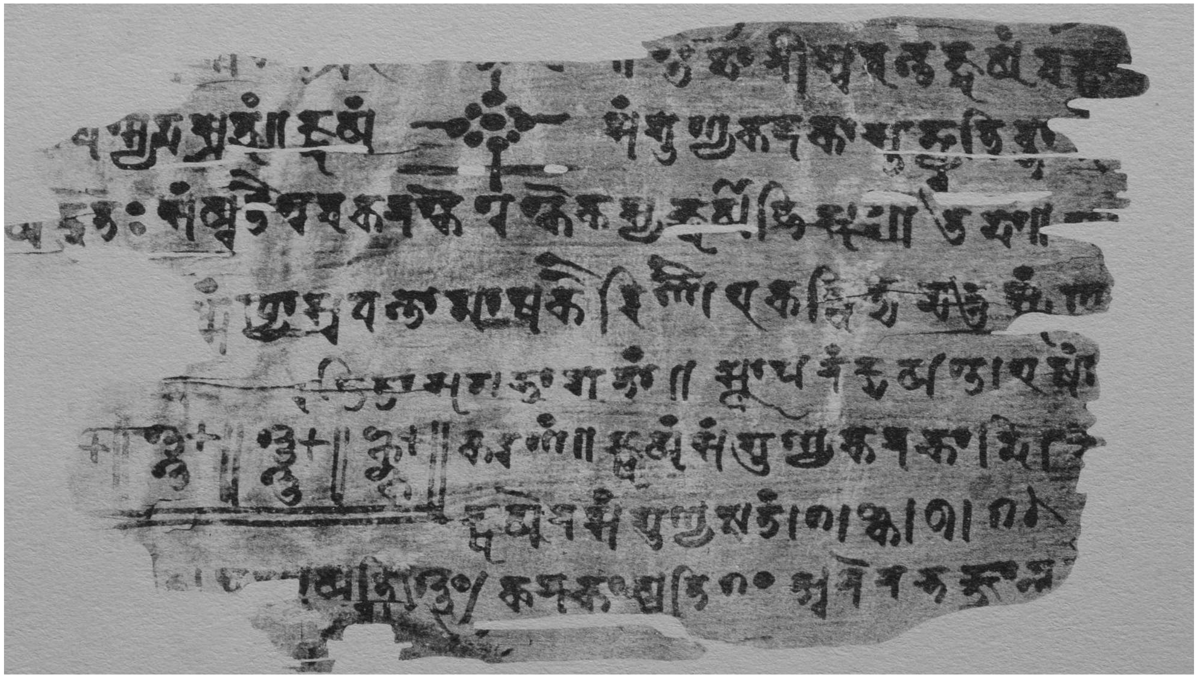
which is the value of the original S.

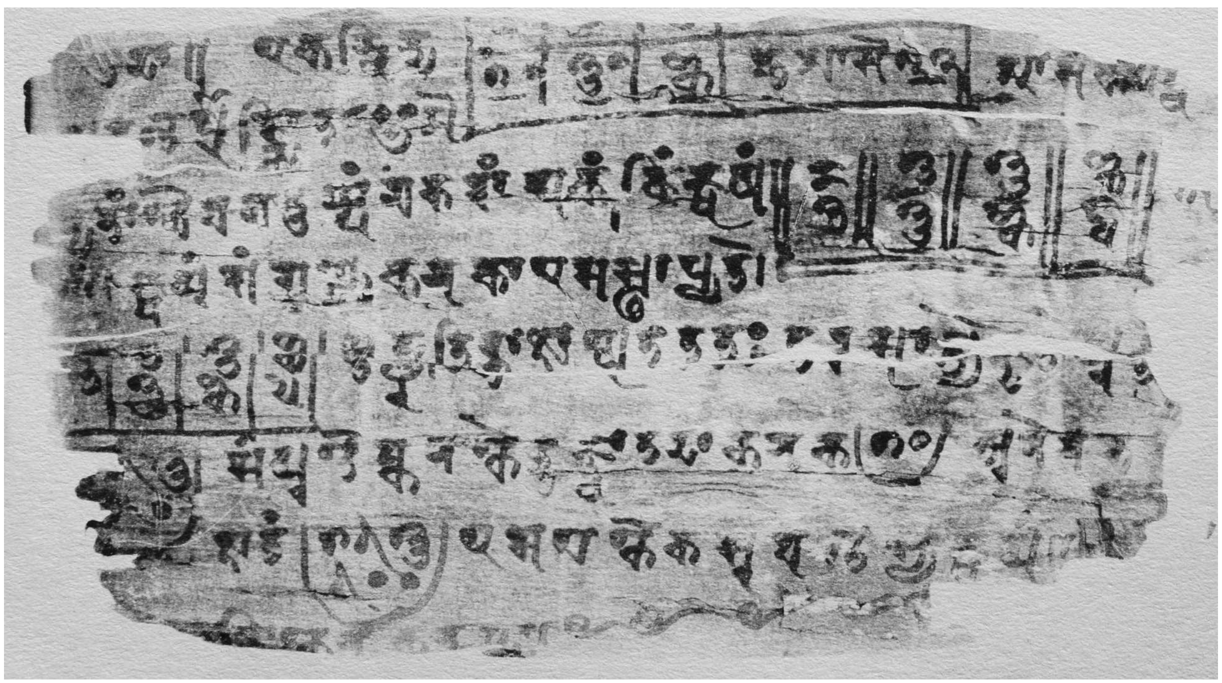
Tracking the author of the manuscript through all the necessary computations, first Datta and then Hayashi managed to fill in many of the missing parts of pages in the manuscript. In particular, 46r becomes as we have seen

I'm not sure which I admire more, the original author or those (Kaye, Datta, Hayashi) who managed to reconstruct the manuscript. In any case, stunning performances.

When was the manuscript written?

This has had controversial answers ever since Hoernle's original estimate. The only estimate now agreed on by everyone is, somewhere between 300 C.E. and 1200 C.E.





• August Hoernle, On the Bakhshali manuscript, Alfred Hölder, Vienna, 1887.

Hoernle, who at this time worked for the Government of India, was the first expert to see the manuscript.

• G. R. Kaye, The Bakhshali Manuscript: a study in medieval mathematics, the Archaeological Survey of India, Calcutta, 1927.

Many of Kaye's opinions are clearly wrong, but parts of this are readable and interesting. His book contains the only good publicly available images of the pages of the manuscript.

• Bibhutibhusan Datta, 'The Bakhshali mathematics', *The Bulletin of the Calcutta Mathematical Society* 21, 1-60.

This criticizes Kaye's edition severely. It is also a good introduction to the Bakhshali manuscript, even if occasionally itself in error as well as incomplete.

• Takao Hayashi, The Bakhshali Manuscript: an ancient Indian mathematical treatise, Groningen, 1995.

This is the definitive account.

- https://www.sciencemag.org/sites/default/files/Bakhshali%20Research%20Statement_13%209%2017_FINAL.pdf

A press release from the Bodleian Library. This includes photographs of one of the folia (to which I imagine they have applied some image manipulation to enhance contrast), as well as a photograph of the album of mica-encased bark fragments. I have shown these above.

This release contains some extremely foolish comments by people who should have known better.

- **Kim Plofker, Agathe Keller, Takao Hayashi, Clemency Montelle, and Dominik Wujastyk, 'The Bakhshali Manuscript: A response to the Bodleian Library's radiocarbon dating',** *History of Science in South Asia* **5 (2017):**

- **The Wikipedia entry on birch bark manuscripts:**
  https://en.wikipedia.org/wiki/Birch_bark_manuscript

- **J. L. Berggren, Episodes in the history of medieval Islam, Springer, 2003.**

Section 2.3 has a brief and valuable account of the invention of decimal fractions.