

# Hyperboloids of revolution and the hexagons of Pascal and Brianchon

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Translated informally from

*Mémoire sur l'hyperboloïde de  
révolution et sur les hexagones  
de Pascal et de M. Brianchon*

Nouveaux mémoires de l'Académie royale des sciences  
et belles-lettres de Bruxelles **3** (1826)

1. Theorem. *Whatever the respective positions of a right cone and a plane in space, it is always necessary that they intersect non-trivially. One may consider two spheres which, touching the cone in its interior, touch also the cutting plane. Then the two points of contact of the plane and the spheres are the foci of the conic section..*

Through the axis of the cone we construct a plane perpendicular to the plane of the section. It will cut the cone in  $AS$  and  $BS$ , the two spheres in the circles  $C$  and  $c$  tangent to them, and the plane of the section in the line  $Ff$  tangent to the two circles in  $F$  and  $f$ , which will be the points of contact of the spheres and of the plane section.

The two spheres will touch the cone at two parallel circles, perpendicular both of them to the plane  $ASB$ , and whose traces on the plane are  $ab$  and  $AB$ .

With all this given let us consider in addition an edge  $ST$  along the cone; this edge will touch the two spheres at  $T$  and  $t$  on the circumference of the circles  $ATB$  and  $atb$ , and the distance  $Tt$  will be evidently equal to  $Aa$ .

This edge will cut also the plane section in a point  $m$ , whose projection is at  $m'$  on the trace of the major axis of the section, and if one constructs the lines  $mf$ ,  $mF$  they will be tangent, one to the sphere  $C$ , the other to the sphere  $c$ ; but  $mt$  is also tangent to the sphere  $c$ , so  $mt$  and  $mf$  are equal, and by the same reason one has also  $mT = mF$ . From which it follows that  $mT + mt$  or  $tT$  or  $Aa$  is equal to the sum of the segments  $mF$  and  $mf$  between  $F$  and  $f$ , at the point  $m$  of the curve; but since the point  $m$  is arbitrary and  $Aa$  is constant one sees that this property takes place for all points of the section; thus the curve is an ellipse with foci  $F$  and  $f$ .

One can demonstrate in exactly the same fashion the same thing for the parabola and the hyperbola, thus we consider our general theorem as proven.

2. One could easily reverse this reasoning and show that through every conic section of which one fixes a focus, one can always make pass a cone tangent to each sphere tangent itself to the plane of the section of its focus; let, for example,  $Dd$  be the major axis of the given curve,  $f$  its focus,  $DSd$  of a plane perpendicular to the plane of the curve; one traces on the plane an arbitrary circle tangent to the axis  $Dd$  at  $f$ , the point  $S$  will be evidently the apex of a right cone which passes the given curve, which is easy to prove; for by what we have already seen, its intersection with the plane of this curve will have  $f$  for focus, and  $Dd$  for major axis; that's all that's needed to be the same as the given curve.

3. One sees that the number of cones which one can determine in this way is infinite. First, what they have in common is that all their summits lie in the same plane. Next, one always has  $DS = Da + aS$ ,  $dS = db + bS$ ,  $Da = Df$ ,  $db = df$ ,  $Sa = Sb$ , from which one gets

$$DS - dS = Df - fd = Ff;$$

from which it follows that all the apices of the cones which pass through the ellipse, are on a hyperbola which has for foci the ends of the major axis of the ellipse, and for major axis the eccentricity of the ellipse. A similar theorem is true for the parabola and the hyperbola, and can be demonstrated easily. This property of conic sections has been demonstrated by my knowledgeable colleague M. Quetelet, who, I believe, recognized it first.

4. If one places a straight line of indefinite extent in such a way that it be in contact with a sphere, and then makes the sphere rotate around one of its diameters (the axis of rotation) which remains fixed, it will carry with it the line which we assume to remain fixed to it, and in this movement describes the surface called the *hyperboloid of revolution*.

The hyperboloid of revolution contains the right cone as a special case; for that one only need choose the line so it cuts the axis of rotation. Thus one might conjecture that all conic sections might be considered as coming from the hyperboloid; one can demonstrate this in other fashion.

5. On the extension of the rotation axis, and before beginning the rotation, one can suppose the centres of a lot of spheres, all tangent to the *generating* line; next, putting the system in rotation all these spheres will be tangent to the hyperboloid generated, in circles perpendicular to the rotation axis.

6. If by the point of contact of the primitive sphere and the generator, one constructs another line equally tangent to the sphere, and making with the rotation axis a supplementary angle, with respect to the angle made by the original line, this second line will generate in the rotation a surface absolutely identical to the first.

One may therefore conceive the hyperboloid of revolution as composed of a system of lines, all making the same angles, but in the sense inverse to the axis of the surface. Calling one system *direct*, the other *inverse*, one can establish that:

7. Through every point of the hyperboloid passes two generators, one direct and the other inverse.

8. Every direct generator meets every inverse generator, and reciprocally.

9. No direct generator ever meets another direct generator; and conversely, the inverse generators never meet.

10. Theorem. *Two tangent spheres to the hyperboloid being given, if one constructs a plane  $P$  tangent at  $F$  and  $f$  to these two spheres, it cuts the hyperboloid in a curve analogue to conic sections, with foci  $F$  and  $f$ .*

Let  $O$  and  $o$  be the centres of the two spheres. These two spheres will touch the hyperboloid in two circles  $C$  and  $c$ . Through any point  $m$  of the intersection of the hyperboloid and the plane  $P$ , construct an arbitrary generator. It will be tangent somewhere at  $T$  and  $t$  to the two spheres, and the points  $T$  and  $t$  are located on  $C$  and  $c$ .

Now at  $m$  construct the lines  $mF$  and  $mf$ . The first will be evidently  $mT$ , the second  $mt$ , since the first two are tangents from  $m$  to the sphere  $O$ , and the second two to the sphere  $o$ .

Now, one of two things has to happen: either the point  $m$  will be contained between  $C$  and  $c$ , in which case all the other points of the section will also be there; or the point will be outside this interval between the two circles, in which case the other points of the circle will be outside it also.

In the first case one will have:

$$mF - mf = mT - mt = Tt,$$

and in the second

$$mF - mf = mT - mt = Tt.$$

But  $Tt$ , that is to say the length of the portion of the generator between the two circles  $C$  and  $c$ , is constant whatever  $m$  may be; thus in the first case the curve is an ellipse with foci  $F$  and  $f$ ; and in the second an hyperbola with foci  $F$  and  $f$ .

11. It can also happen that only one sphere can be constructed tangent to the cutting plane, thus giving rise to a parabola. The argument in this case is just a special case of the other two.

12. Theorem. *Through every conic section one can construct a hyperboloid of revolution.*

Through the extremity of the major axis construct an arbitrary line out of the plane of the curve. Through each of its foci construct a sphere tangent to the plane of the curve and this line. Make the line and the two spheres turn around the diameter common to the spheres, and you will form an hyperboloid whose section will have two foci and an end of a major axis, common with the proposed curve. That's all we need to establish equality of the two curves.

In considering the parabola as an ellipse with infinite eccentricity, the reasoning above applies word for word. Thus, every conic section may be considered as if it belonged to the hyperboloid.

The preceding Theorem 10 is susceptible of the following interesting extension:

An arbitrary plane  $P$  and a hyperboloid of revolution being given, in making a sphere tangent to the hyperboloid take all possible positions, it will finish by cutting somewhere the plane  $P$ ; then *it will be possible to construct through the section of the hyperboloid two right cones tangent to the sphere. Next, if through one of the extremities of the diameter perpendicular to  $P$ , one constructs two straight lines to the vertices of these cones, they will cut the plane  $P$  in the two foci of the section.*

This theorem can be deduced so simply from what we have already seen, that I don't think it is necessary to give its proof. One can draw from it several curious corollaries, but they will present themselves to those who are curious about this sort of thing.

**13.** Imagine on the hyperboloid the traces of six generators, three direct and three inverse. Each generator will be cut in three points by the three generators of the opposite system, which gives in all eighteen points of intersection, which amount to nine distinct ones: among these points, take six arbitrarily, but such that no more than two lie on one line. we form in this way a *skew hexagon* whose sides are formed alternately of segments from the direct and inverse systems. Let these sides be

$$d, i, d', i', d'', i'' .$$

The letter  $d$  refers to *direct*,  $i$  to *inverse*. The segment  $d$  is contiguous with  $i$ , etc. The angles between these sides will be  $\widehat{di}$ , etc. The vertices will be  $di$ , etc.

**14.** Before going further, I must define *opposite* angles and sides. The opposite sides are those which, in the order of sides, are separated by two other sides; thus  $d$  and  $i'$  are opposite; the opposite angles are those formed by sides respectively opposite; thus  $\widehat{di}$  is opposite to  $\widehat{i'd''}$ . The diagonals are those lines joining the summits of opposite angles.

**15.** Theorem. *In the skew hexagon traced on the hyperboloid, the planes of opposite angles meet two by two in three lines, all lying in the same plane.*

In effect, the two opposite sides  $d$  and  $i'$ , since they are from the direct and inverse systems, meet in a point  $di'$ .

Similarly  $i''$  and  $d'$  will meet in  $i''d'$ . Thus the plane of the angle  $\widehat{i''d}$  and that of the opposite angle  $i'd'$  will both pass through the points  $di'$ ,  $d'i''$ ; their intersection will also pass through these points.

One sees also that the intersection of the planes of the opposite angles  $\widehat{di}$  and  $\widehat{i'd''}$  will pass through  $di'$  and  $id''$ .

And finally, the intersection of the planes of the two last opposite angles  $id'$  and  $d''i''$  will pass through  $id''$  and  $d'i''$ .

The first and the second therefore have the common point  $di'$ . The second and the third the point  $id''$ , the third and first  $i''d'$ . These three lines are thus all contained in the same plane passing through  $di'$ ,  $id''$ ,  $i''d'$ .

**16.** Through a conic section let us pass a hyperboloid of revolution. Take on the curve six arbitrary points  $d, i, d', i', d'',$  and  $i''$ , and let pass through these points as many generators of the same index, as in §13. The traces of the angles of the hexagon, traced thus on the hyperboloid, being taken in the plane of the curve, will form a hexagon inscribed in this curve, and whose opposite sides correspond to the opposite angles of the skew hexagon. The intersections of these sides, extended if necessary, will find themselves thus at points where the plane of the curve is cut by the lines of intersection of the planes of opposite angles of the skew hexagon. But these last lines being thus in the same plane, their intersections with the plane of the curve will be in a straight line. Thus:

*The three points resulting from the intersections of three pairs of opposite sides of an arbitrary hexagon inscribed in a conic section will always lie in a straight line.*

**17.** This beautiful Theorem due to Pascal, and which one has up to now seen proven in diverse ways, but practically always by using analysis, seems to me to have been discovered by a method much like that we have

just seen. In effect, when one examines carefully the line of argument, one sees that it requires neither calculation nor knowledge of surfaces of degree two, and that in addition it is very similar to other arguments of Pascal's time. It seems that Pascal wished to start with this property for his theory of conic sections. It is in fact easy to deduce from it all the properties of these curves, in a very elegant manner. But this is not the place to do it.

**18.** *If in the skew hexagon one constructs the three diagonals joining the three opposite vertices, they intersect in a single point.*

Consider the two opposite angles  $\widehat{i''d}$ , and  $\widehat{i'd'}$ . Through the two generators of opposite type, construct the plane  $\widehat{i''d'}$ , which is possible, this plane will contain the diagonal which joins the vertices of the angles.

We use the same notation to express the plane of two lines as for expressing their angle. This causes no trouble, and expresses well what one wants to say, since in effect the two sides of an angle determine the position of the plane absolutely.

Similarly the plane  $\widehat{di'}$  also contains this diagonal.

Similarly for  $\widehat{di}$  and  $\widehat{i'd''}$ . One sees that the diagonal which joins their vertices finds itself on the two planes  $\widehat{di'}$ , and  $\widehat{i'd''}$ .

And finally the diagonal from  $\widehat{id'}$  to  $\widehat{d''i''}$  is on the planes  $\widehat{id''}$  and  $\widehat{d'i''}$ .

The first and the second diagonal are therefore in the same plane  $\widehat{di'}$ , the second and the third on the other plane  $\widehat{id''}$ , the third and the first on the third plane  $\widehat{d'i''}$ . These three diagonals are therefore the aretes of a trihedral angle formed by the three planes  $\widehat{di'}$ ,  $\widehat{id''}$ ,  $\widehat{d'i''}$ . All three therefore pass through one vertex, which is what was to be proven.

**19.** One knows that through every plane section of an hyperboloid one can pass a cone tangent to the surface of the hyperboloid. Now suppose one is given any conic section whatever. Construct a hyperboloid of revolution passing through it; construct the tangent cone to this surface, and consider the whole system which we can describe as a perspective operation, the eye being at the vertex of the cone, the plane of the curve being the tableau.

Take arbitrarily on the curve six points  $d, i, d', i', d'', i''$ . Make pass through these six points six generators  $d$ , etc. We get again the skew hexagon of §13.

Observing that if one constructs through one of these points a plane tangent to the hyperboloid, it will pass through the vertex of the tangent cone which we have constructed, thus through the generator which passes at the point of contact, we convince ourselves easily that the perspective of each generator on the plane section will be tangent to this section. Thus, the perspective of the skew hexagon will be a hexagon circumscribing the curve, which will have for diagonals the perspective of the diagonals of the skew hexagon. But those cross in a single point, so it is the same for their perspectives. Therefore:

*In a hexagon circumscribing a conic section, the three opposite diagonals all intersect in a single point.* This is the theorem of M. Brianchon.

**20.** The proof of this theorem and that of Pascal, was the aim of this note; I won't say anything about several interesting corollaries which one can use for the solution of a great number of the most beautiful theorems of geometry. I wish only that the simplicity and the ease with which we have arrived at the demonstration of our two theorems, can engage some geometer in this agreeable and flavourful part of mathematics.