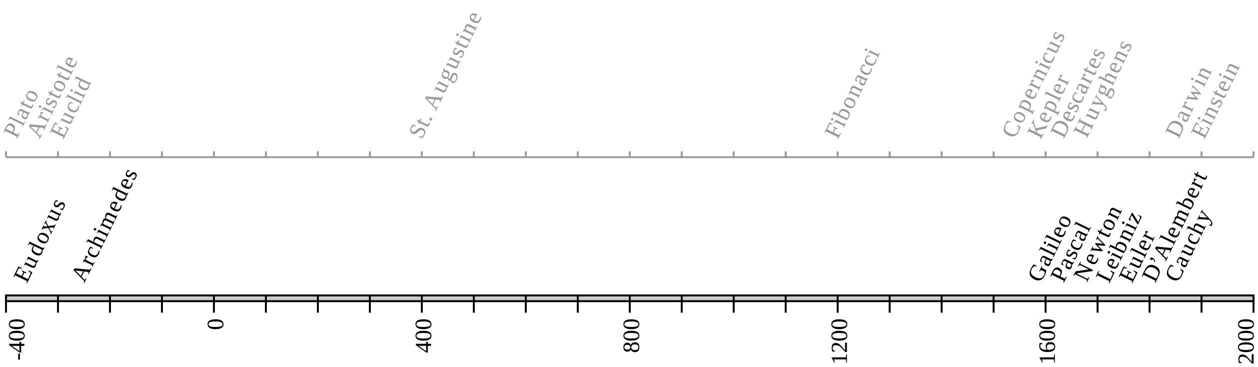
Mathematics 102 — Fall 1999

Introduction

Some geometric problems, which we now solve routinely by Calculus, were also solved with great ingenuity by the ancient Greeks roughly between 360–260 B.C.E., and through methods which had a decisive influence on modern techniques. Nonetheless, Calculus is essentially a European invention of the 17th century. It originated with the investigations of Galileo into the motion of falling objects a few years after 1600, and most of the main outlines of the subject had been filled in by 1680, as far as practical matters were concerned, with the work of first Newton and then Leibniz. After the basic tools were put together in this period, the two major developments were (1) its application to increasingly sophisticated problems, such as the details of planetary motion and structural dynamics, and (2) a much better understanding of its logical foundations, which were still pretty shaky when Newton and Leibniz had finished their research. New applications of Calculus continue to be found even in our time, including to problems in the fields of economics and biology, but its mathematical foundations were complete by about 1850.

It might help to give you sense of the passing of time if we show you a timeline, with a few important mathematicians' names attached to it, even though these names will not mean much to you now. For comparison, we list also a few other and perhaps more familiar scientists we are likely to encounter in our story.



Rogues' gallery - - the principal inventors of Calculus

No completely convincing simple explanation has been offered for why Greek investigations leading to Calculus came to a dead halt in the third century before the current era, but that this was exactly when the 'practical' Romans began to take over the Mediterranean from the 'theoretical' Greeks is a suspicious circumstance. The termination of Greek work on Calculus is well symbolized by the death of Archimedes at Syracuse, on the island of Sicily, at the hands of a Roman soldier.

On the other hand, the renaissance of mathematical investigation in the time of Galileo took place at the tail end of the more widespread Renaissance of interest in ancient culture. This reinforces the minority opinion that mathematics ought to be considered part of, and indeed one of the most impressive of, human cultural achievements. Calculus is ubiquitous in modern technology, and the rise of Calculus coincides also with the rise of modern science. Another striking observation reinforced by the figure is how thin the skin of technology is on the flesh of human history.

The timeline displays only the progress of the mathematics of Calculus, as opposed to its use in science. We will tell you later about the history of applications, as we introduce them.

One consequence of the way things happened is that nearly all of the mathematics that you are going to see in this course is over 300 years old! You might think that such ancient material would be very elementary, and that time and familiarity would have made it trivial for us to teach the subject and for you to learn it, but that is unfortunately not the case. It is our belief, however, is that we can use what we know about the historical development of Calculus to explain it in a more comprehensible way. The problems this development met, and the way in which they were surmounted, can still serve to help illuminate the subject.

The beginning of physics

We are going to begin by looking at an idealized version of one of Galileo's early experiments, part of a series which might be said to have founded modern physics. It is the one where he deduces the laws of gravitational fall near the Earth's surface from data on objects rolling down an inclined ramp. Some features of this experiment are well known, but only recently have his own hand-written working notes been analyzed in detail. Some surprises were found.

We shall keep recalling this experiment from time to time in this course to introduce various themes. We are not going to pretend to give an entirely factual account of what Galileo did, because as usual in the history of science the original path Galileo followed meandered around a lot before arriving at its destination, and although it would be interesting and valuable to see what the meanders of a genius are like that would take us too much time. So we are going to tell our story as if it were entirely true, even though we know it is not. You can look in the books referred to later on if you want to find out more exactly what happened.

We are interested in understanding what happens when an object falls. Of course you probably know more or less what your text books tell you, but we are going to pretend not to know the end of the story; there will still be a few surprises.

In order to appreciate the account it is important to try to imagine yourself in Galileo's circumstances. Four hundred years later, when his pioneering work has become a standard part of every physics class, this is not easy.

• *What was he hoping to learn from his experiment?* Not quite what we would expect, probably. What he wanted to learn about was what happened to the speed of an object as it fell. There was even in his time a long tradition of philosophical inquiry on this question, going back to Aristotle, who lived in the middle of the forth century B.C.E. There were many opinions—surprisingly many—in Galileo's time on what took place during free fall. Of course it was plain that, since the object started out at rest and clearly moved rather rapidly after a while, at some point in between its speed had to be changing. But it was very very difficult for people at that time to imagine how speed could change continuously and gradually if at all. Perhaps they were put off by a collection of paradoxes proposed by the Greek philosopher Zeno. Of course nowadays we all live quite happily with the idea of continuous acceleration, and are happy to learn about the true motion of a falling object without having to investigate it personally. But to give you an idea of how unintuitive our current knowledge is, we should tell you that Galileo himself at one time seems to have thought that all the acceleration of any falling object took place in a short time interval after it started falling. If you think about it you might be hard pressed to contradict this by direct evidence. Another opinion at the time was that the speed did increase, but that it did so in discrete steps. It was also commonly thought that the speed of a falling object was proportional to its weight. Again, until experiments were performed carefully, it was hard to contradict this—especially since it is so obvious that very light objects certainly fall slowly, and heavy ones much faster. When Galileo (later in life) showed that two reasonably heavy objects, one however much heavier than the other, fell to Earth from a great height at almost—but not quite—exactly the same moment, his opponents made much of the fact that the lighter one did fall visibly earlier than the other. (Regarding this, NASA's film of an experiment involving a feather and a heavy object performed on the surface of the Moon is certainly worth viewing.)

• *Galileo's experimental equipment was severely limited.* The only clock he had access to, for example, was one which measured time according to the amount of water flowing out of a tank. Stopping and starting the flow from the tank may have been done simply by pressing and releasing a finger, so accuracy was definitely limited. He got around what might be called his hardware deficiency, at any rate, in an ingenious manner we shall describe later.

• To continue this analogy: *even more remarkable for us to understand, is the low quality of the software available to him.* We cannot resist a few remarks about the state of the art.

(1) Although Arabic numerals had been introduced to Europe several hundred years earlier and were commonly used in his era, primarily for the purposes of financial calculations, decimal fractions had only been introduced about 25 years earlier, by the Dutch scientist Simon Stevins. *Galileo never used them*—all he calculations of his that we know of use only integers. He does refer to continuously varying magnitudes, but never seems to express them as numbers. He follows in fact the same awkward conventions that Euclid did two thousand years before. Euclid's classic text *Elements of Geometry* had been one of the very first scientific books to be published—in 1482—and increasingly accurate editions were produced throughout the 16th century.

(2) Algebra was invented and even perfected in Galileo's lifetime, but he seems never to have used it. He expresses equations in words, again following Euclid. The Greeks had formulated algebraic concepts in geometric terms, and they had great difficulty in handling algebraic problems which could not be so formulated. It was only in about 1550, for example, that handling polynomials of degree more than two, such as the cubic polynomial $x^3 + x + 1$, became convenient.

(3) Nobody seems to have made a precise quantitative definition of speed, at least when it varied. This is not too surprising, since in fact it required the techniques of Calculus to do so. The definition of instantaneous speed was not really made correctly (by the Frenchman D'Alembert) until more than a century after Galileo died. But it was Galileo's experiments which began the inquiry.

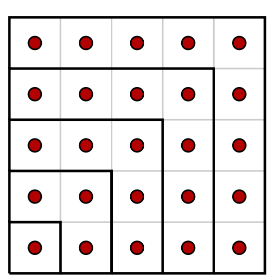
(4) There had been much qualitative discussion of speed, all somewhat imprecise, long before Galileo, but *he was apparently the first person ever to try seriously to measure it!* This is very hard for us to comprehend. Indeed, Galileo is arguably the very first person ever to perform a precise quantitative experiment of any kind, in the modern sense. Mathematics had played a role in astronomy for thousands of years, but of course astronomical experiments are rather difficult to carry out. In Galileo's time, it was commonly believed that astronomical phenomena had no particular relationship with terrestrial ones. It wasn't really until the time of Newton that it was understood that the laws of Nature extended uniformly to all of space and even perhaps to all of time. This developing realization was the one basic concept of science that caused consternation all throughout the 17th century, and, to tell the truth, still causes consternation in parts of Kansas.

An example of geometric algebra

We can give you an example of the sort of algebra the Greeks did know by looking at the sequence of positive integers we get by summing the first few odd numbers together, getting the new sequence

$$1, \quad 1 + 3 = 4, \quad 1 + 3 + 5 = 9, \quad \ldots .$$

It seems likely from what we see here that *the sum of the first n odd numbers together is always equal to n^2*. If this is true, it is both simple and striking enough that there ought to be a simple way to see why it is true. You ought in fact to be completely convinced by the following figure:



In words: we can partition an $n \times n$ square directly into n pieces of sizes equal to 1, 3, 5, ... up to the n-th odd number. It would be difficult to imagine a more satisfactory argument.

We shall next see that it is precisely this fact which Galileo used to discover what happened when an object falls.

Exercise 1. *The n-th even number is $2n$ for $n = 1, 2, 3, \ldots$. What is the n-th odd number?*

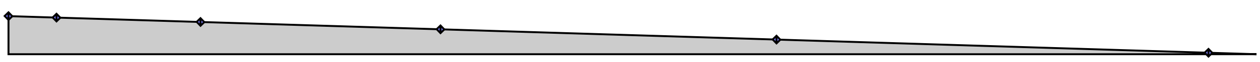
Descent

Dropped objects in free fall move very rapidly. In order to slow things down so that measurements could be made more accurately, Galileo had the clever idea of letting objects descend along an inclined ramp of low angle. Since sliding objects generate friction, he looked at rolling heavy balls. (It may have been realized only much later that this has a serious effect on the experiment.)

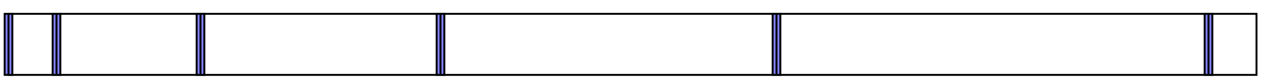
Exercise 2. *Drag up from your brain a bit of high school physics. In one English sentence, why should the speed of a ball rolling down a ramp be any different from that of a sliding one (even assuming that friction can be ignored)?*

All this is well known, although even twenty years ago it was denied by some scholars that he had ever looked at real objects rolling down real inclines, and just performed a few thought experiments. But it has been realized only recently that he had an even cleverer idea, one that in effect allowed him to freeze the location of the object at surprisingly uniform time intervals, just as a stroboscope would do. He did this by stretching taut wires across the ramp at certain intervals along it. As the object rolled down, it would make a slight bump as it passed over each of these, and the wires could be adjusted with extraordinary accuracy to make the time intervals between bumps all equal. It has been suggested, for example, that he sang along (what kind of song? one wonders) to get the rhythm right. Galileo had come across in this very first physics experiment a basic fact of life for both psychologists and experimenters: *it is almost easier to compare two quantities of roughly the same size as it is to measure either of them directly.* At any rate, here is a typical pattern of wires Galileo might get:

Side view

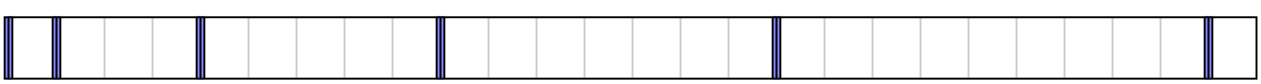


Top view



It is at least immediate from this that the speed continues to increase as time passes, since the distance travelled in successive time intervals is clearly getting larger and larger. The next step Galileo took was to measure quantitatively how the speed was increasing. To understand why he made the measurement he did, it helps to recall that he didn't make measurements in any system of units, but that following Euclid he usually measured things by calculating ratios of quantities of a similar nature. Therefore, it was perfectly natural for him to calculate for each successive time interval the ratio of the distance travelled to the distance travelled in the first time interval. You can see what he found if we just add a few lines to the previous picture:

Top view



Galileo at this point apparently had no preconception about what he would learn from his experiment, and was astounded at the simplicity of what he got: *Let y_1 be the distance gone in the first interval. Then for each interval including the first, calculate the ratio of distance gone to y_1. These ratios are the same as the sequence of odd integers* $1, 3, 5, 7, \ldots$. Or, *the ratio of the distance gone in the n-interval to that gone in the first is equal to the n-th odd integer.* I repeat that as far as we know Galileo expected something very complicated to be going on. He had no reason at all to think as we do, with several hundred years of experience in science, that the laws of Nature ought to be in any way simple. For him to see these simple odd integers appearing out of his data must have sent him dancing around the room. To whatever song he had been singing.

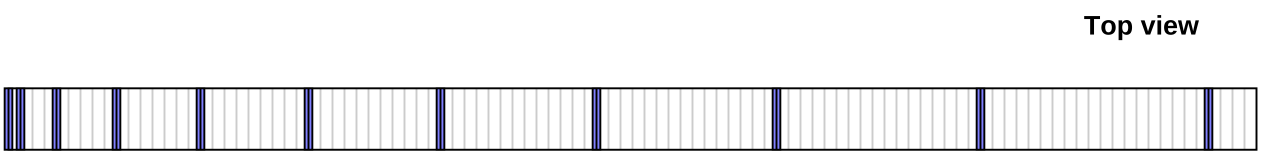
To summarize: Galileo set out to discover roughly how speed changed as an object fell. His discovery about the ratios of successive distances travelled told him roughly that the speed increased as time went on, but it didn't directly tell him anything about precise values for speeds. In other words, his experiment did not answer his original question. In compensation, he did perceive a simple pattern in his data, and from this he did learn something he hadn't expected—a simple rule for telling how position changed with time. How does that come about? Let t_1 be the interval of time between successive bumps. After an initial time interval t_1, the object has gone a distance y_1. In the next interval, it goes a distance $3y_1$, and at the end of that interval it has gone a total distance $y_1 + 3y_1 = 4y_1$. Similarly, at the end of three time intervals it has gone a total distance $(1 + 3 + 5)y_1 = 9y_1$, and at the end of n intervals it has gone a total distance $(1 + 3 + 5 + \cdots + (2n - 1))y_1$ which we know from our previous observations to be $n^2 y_1$. Galileo, having a fondness for ratios of similar qualities, would say that at the end of the n-th time interval $t_n = nt_1$ the distance gone satisfies the equation of ratios

$$\frac{y_n}{y_1} = \left(\frac{t_n}{t_1}\right)^2 .$$

Using algebra, we can write this as

$$y_n = ct_n^2, \quad c = y_1/t_1^2 .$$

We can now go a little further. Suppose we cut the time interval in half to an interval $t_1' = t_1/2$. In order for Galileo to do this, he would just add wires in between the original ones. Here is the new pattern he would get if he did that:



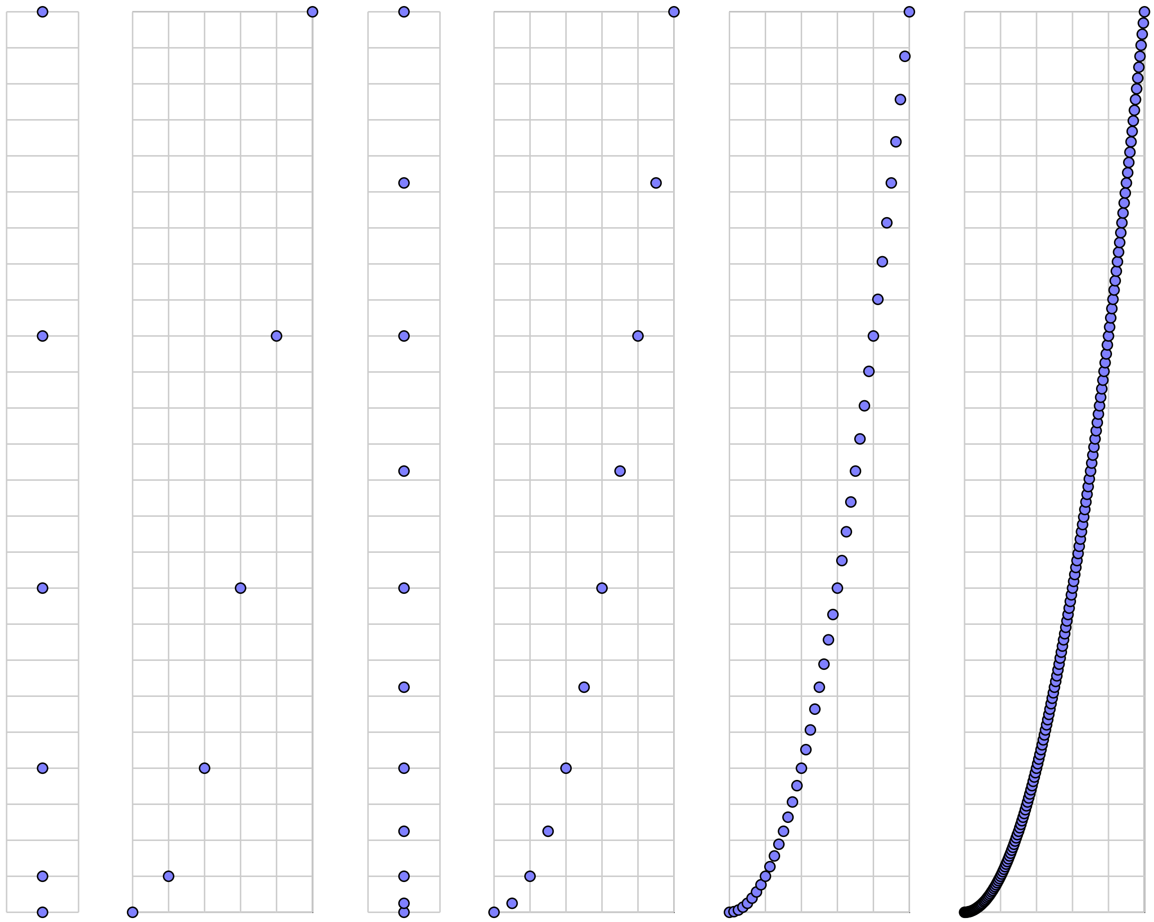
The pattern of odd integers is exactly the same, but at a different scale. The equation now satisfied by he positions is

$$y_n' = c(t_n')^2, \quad c = y_1'/(t_1')^2$$

where now y_n' ranges over the new positions, and t_n' over the new times. The important thing here to notice is that the new constant is the same as the old one, since

$$t_1' = t_1/2, \quad y_1' = y_1/4, \quad y_1'/(t_1')^2 = \big(y_1/4\big)/(t_1/2)^2 = y_1/t_1^2 .$$

We can again cut the time interval in half, as many times as we want—or at least until there is no room to add wires. The best way to understand what happens is now to introduce another software tool that Galileo didn't have available—a graphical way to present the data. We plot our original data points (t_n, y_n) in a picture, with time represented on a horizontal axis and distance on a vertical one. Then we do the same after cutting the time in two a few times.



Of course it is irresistible to *extrapolate* from the data we have, and conjecture that at any time t the total distance gone by time t is

$$y = ct^2$$

where c is the same constant we have seen before.

Summary

The formula $y = ct^2$ was the first result of many that were to come in the development of physics in the 17th century, and the first instance where mathematics and experiment combined to give a precise quantitative idea of an underlying mathematical simplicity in nature. There were a few more questions to be answered soon by Galileo himself. (1) He came across the formula for position almost by accident. What remained to be found was a formula for speed, which is what he had set out to discover. (2) He apparently felt that the formula for position was still too complicated to reveal an underlying simplicity in the laws of nature, and he started thinking about how to find a simple rule to cover the situation. (Actually, these first two problems are united, as we shall see later.) (3) Another natural question is how the constant c varies with circumstances, for example with the angle of inclination of the plane or the mass of the object. If this angle is taken to be $90°$, we are looking at bodies in free fall. It was known to at least a few scientists of Galileo's time, but still controversial, that the constant c did not in fact vary with the size of the object, as long as the effects of friction were negligible. This of course is so well known now as to be almost obvious, but in fact there is something subtle going on here which received proper understanding only in the work of Einstein in the early 20th century. We shall say something more about this later on.

The reasoning that led to Galileo's formula provides an illuminating example—a model, really—of the way in which mathematics and experiment can interact to help us understand the ways things are. A limited amount of data were collected. A mathematical pattern was noticed, and from this (the odd-integer rule) a basic formula (for position as a function of time) was conjectured, on purely mathematical grounds. Mathematics is thus a kind of *logical cement* that binds experimental data together. This will be a constant theme in this course.

Exercise 3. *The planet Krypton is small and dense, and the way in which falling objects behave is not quite so simple as it is on Earth. Here is the record of a fall from $h = 5$ metres on Krypton:*

t	h
0.0	5.0000
0.1	4.9878
0.2	4.9510
0.3	4.8895
0.4	4.8030
0.5	4.6910
0.6	4.5530
0.7	4.3881
0.8	4.1955
0.9	3.9740
1.0	3.7221
1.1	3.4381
1.2	3.1197
1.3	2.7642
1.4	2.3682
1.5	1.9272

(a) Plot these data on a graph. Draw a smooth curve through the plotted points. Estimate as carefully as you can where the object is at time $t = 0.95$; $t = 1.05$; $t = 0.99$; $t = 1.01$; (b) Make an estimate as to how fast the object is travelling at $t = 1$. Give reasons for your estimate. (c) In one sentence: how can you tell this is not taking place on Earth?

Exercise 4. *Find a formula for $1 + 2 + \cdots + n$ by a geometric argument about the $n \times n$ square.*

Exercise 5. *What is $1 + 4 + 7 + \cdots + 1000$?*

Exercise 6. *What is a formula for the n-th term in the sequence*

$$1,\ 4,\ 7,\ 10, \ldots$$

Find a formula for the sum of the first n terms. Give reasons—either geometric or algebraic—for your formula.

Exercise 7. *The algebraic reason that the sum of the first odd numbers is equal to n^2 is that $(n+1)^2 = n^2 + 2n + 1$. (a) Explain what this formula has to do with the picture we saw earlier. Explain with a similar picture why this formula is true. (b) What is the corresponding formula for $(n+1)^3$? $(n+1)^4$?*

Exercise 8. *What are the next three numbers in the sequence*

$$1,\ 7,\ 19,\ 37,\ \ldots$$

Find a formula for the sum of the first n numbers in it.

Further reading

1. Stillman Drake, *Galileo at Work*, University of Chicago, 1978.

2. Stillman Drake, *Galileo: Pioneer Scientist*, University of Toronto, 1990.

3. Feathers falling on the moon can be seen at

`http://lava.larc.nasa.gov/ABSTRACTS/LV-1998-00046.html`

4. Images of Galileo's manuscripts can be seen at

`http://www.mpiwg-berlin.mpg.de/texts/Galileo.Nuncius.html`