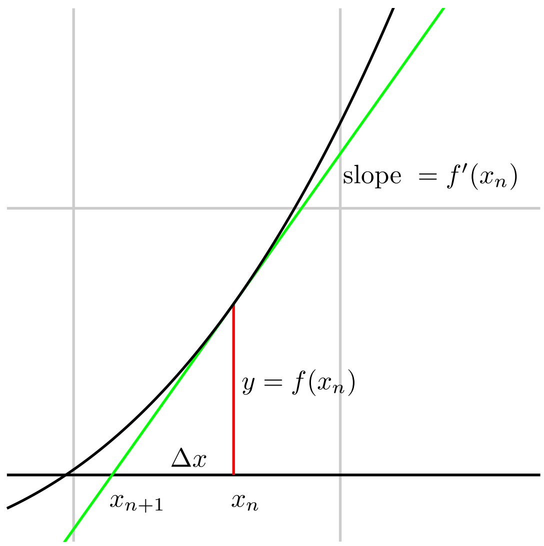
Mathematics 102 — Fall 1999

Newton's method for finding roots of equations

The basic idea

It is reasonably common to find yourself in the situation of wanting to find a number x such that $f(x) = 0$ for some function $f(x)$. One simple special case is to find the square root of a number c, which means solving $x^2 - c = 0$. One of the best and simplest procedures for doing this is called **Newton's method.** This method starts with a rough approximation to the root wanted, and then finds better and better successive approximations, until you get an approximation which is close enough for your purposes. The way to find better approximations is based on the following picture:



What the picture suggests is that if we have one approximation x_n to a root of $f(x) = 0$, then a better one can be found as the intersection of the tangent line to the graph of $y = f(x)$, over the point x_n, with the x-axis. This is because near the point x_n the tangent line and the graph of $y = f(x)$ lie very close to each other. This is the principle of using the tangent line as a **linear approximation** to the graph.

This intersection can be found from the definition of the derivative as slope:

$$\begin{aligned}
\frac{f(x_n)}{\Delta x} &= \frac{f(x_n)}{x_n - x_{n+1}} \\
&= f'(x_n) \\
\Delta x &= \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}
\end{aligned}$$

An example

Here is a run of a program to find roots of

$$f(x) = x^3 - 3x + 1 = 0$$

Here $f'(x) = 3x^2 - 3$. We know that there exists at least one real rot since $f(x) \to \pm\infty$ as $x \to \pm\infty$. We can locate the roots very roughly by plotting

$$\begin{array}{cccccccc} x & -\infty & -2 & -1 & 0 & 1 & 2 & \infty \\ f(x) & -\infty & -1 & 3 & 1 & -1 & 3 & \infty \end{array}$$

So there are roots between -2 and -1, 0 and 1, 1 and 2. The program prompts for an initial guess and then runs until $|f(x)| < 0.00000000001$.

```
initial guess?  1.5
  1.533333333333   0.005037037037
  1.532090643275   0.000007101762
  1.532088886241   0.000000000014
  1.532088886238  -0.000000000000

initial guess?  .5
  0.333333333333   0.037037037037
  0.347222222222   0.000195580418
  0.347296353164   0.000000005725
  0.347296355334  -0.000000000000

initial guess?  -0.5
  0.555555555556  -0.495198902606
  0.316798941799   0.081397613626
  0.346958322453   0.000891902395
  0.347296310248   0.000000118944
  0.347296355334   0.000000000000

initial guess?  -2.0
 -1.888888888889  -0.072702331962
 -1.879451566952  -0.000503850074
 -1.879385244837  -0.000000024801
 -1.879385241572   0.000000000000
```

Convergence is very rapid in Newton's method, if the initial approximation is at all close. It usually happens as it does here that the number of decimals of accuracy doubles at every step. Therefore the only serious difficulty is obtaining a rough idea ahead of time where the roots will lie. To do this automatically is tricky, but often a quick sketch of the graph of $f(x)$ will help.

One drawback of Newton's method is that it requires knowing how to calculate the derivative. Calculators don't do this; they approximate the derivative by the formula

$$f'(x) \sim \frac{f(x+h) - f(x)}{h}$$

for h small. Another problem is that it will never find **multiple roots,** those where $f'(x) = 0$ and the graph of $f(x)$ is tangent to the x-axis.