Mathematics 102 — Fall 1999

Sequences

In this chapter we look at functions whose independent variable ranges over a domain of discrete values.

Arithmetic progressions

The odd positive integers are

$$1,\ 3,\ 5,\ 7,\ \ldots$$

In other words, we get them all by starting with 1 and then successively adding an increment of 2. This is one of the simplest examples of an **arithmetic progression**. More generally, an arithmetic progression is a sequence we get by starting with an **initial value**, say b, and then successively adding on an **increment**, say a:

$$b,\quad a+b,\quad 2a+b,\quad 3a+b,\ \ldots$$

We will assign to each term in this sequence, and indeed in any sequence, an **index**. *The first term gets index* 0*, the second gets index* 1*, etc. Thus the* n*-th term gets index* $n-1$, and we write it as t_{n-1}. This may be slightly confusing, but is extraordinarily convenient, since now we have a simple formula: *The term* t_n *with index* n *is* $an+b$*.*

Exercise 1. *What is a formula for the* n *term in the sequence of odd numbers?*

From the sequence of odd numbers we can build a second sequence by adding the first several odd numbers together. We get

$$0,\ 1,\ 1+3=4,\ 1+3+5=9,\ \ldots$$

which we have already seen to be the same as the sequence of non-negative square integers. Note that we start with 0, so this sequence is defined by this rule: the term with index n is the sum of the first n odd positive integers.

Exercise 2. *What is a formula for the* n*-th term in the sequence* $0, 1, 4, \ldots$ *?*

Exercise 3. *In the sequence of odd numbers, what index does* 1001 *have?* 1617*?*

In general, **a sequence is just a function whose independent variable** n **takes values among the non-negative integers** $0, 1, 2, 3, \ldots$ The index of a term is often written as a subscript. Thus the odd positive integers form a sequence t_n with $t_n = 2n+1$. A sequence is an example of a function where the independent variable is discrete, which means that it is taken from a set of isolated numbers. Another example would be a function whose domain contains all integers, both positive and negative–a two-sided sequence—but we shall not need this notion right now.

If we are given one sequence

$$f_0,\ f_1,\ f_2, \ldots$$

then we can calculate a second sequence, called its **sum sequence**, by the rule that the term s_n with index n in the new sequence is the sum of the first n terms in the original sequence. Thus

$$s_0 = 0,\ s_1 = f_0,\ s_2 = f_0 + f_1, \ldots$$

The Greek letter for capital S is Σ (Sigma). So if f is the original sequence, we write Σf for its sum sequence:

$$(\Sigma f)_n = f_0 + \cdots + f_{n-1}\ .$$

The symbol Σ can usually be read “sum of”.

We can also construct from any sequence f_n a new sequence called its **difference sequence** d_n by the rule that $d_n = f_{n+1} - f_n$. Thus from

$$0,\ 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ \ldots$$

we get the difference sequence

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \ldots$$

The Greek capital D is Δ (Delta) so if f is the original sequence we write Δf for its difference sequence:

$$(\Delta f)_n = f_{n+1} - f_n\ .$$

The symbol Δ can usually be read as "change in".

It is often convenient to represent a sequence and these related sequences in one table. We can display a familiar example here in this way:

Δf	2	2	2	2	2	...		
f	1	3	5	7	9	11	...	
Σf	0	1	4	9	16	25	36	...

Note that if we are given a sequence f_n, then we can calculate any given element of its difference sequence just by looking at two successive terms f_{n+1} and f_n, but that unless we are lucky, if we want to calculate the term indexed by n in its sum sequence we will have to calculate all the terms f_0, $f_0+f_1, \ldots, f_0+f_1+\cdots+f_{n-1}$, or in effect all the terms of the sum sequence up to the one we want. Inefficient, perhaps a lot of work, but nonetheless possible. And, furthermore, this summing process is indeed a rule, and not a very complicated one. It does illustrate that a rule that defines a function is just some well determined process for calculating something, not necessarily an efficient one. In general, there may or may not turn out to be some simple rule short-circuiting the sum sequence definition.

Also, it is commonly the case that the difference sequence is simpler than the original sequence, and that we can often figure out a rule for a sequence by looking at its difference sequence. For example, here are the first few terms of a 'mystery sequence'.

$$1,\ 4,\ 10,\ 19,\ 31,\ \ldots$$

What is the next term? If we calculate the difference sequence we get

$$3,\ 6,\ 9,\ 12,\ \ldots$$

It is natural to guess that this continues as

$$3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ \ldots$$

which means that we can extend the original sequence:

3	6	9	12	15	18	21	...
1	4	10	19	31	46	64	...

Exercise 4. *Find a rule for the sum sequence of $f_n = n$. Of $f_n = an + b$.*

Exercise 5. *Find an explicit formula for the difference sequences of n^2, n^3, n^4. Hint: Let's do the first one. We have*

$$\Delta f)_n = f_{n+1} - f_n = (n+1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1\ .$$

Exercise 6. *Write down the first 6 terms of the difference sequence for*

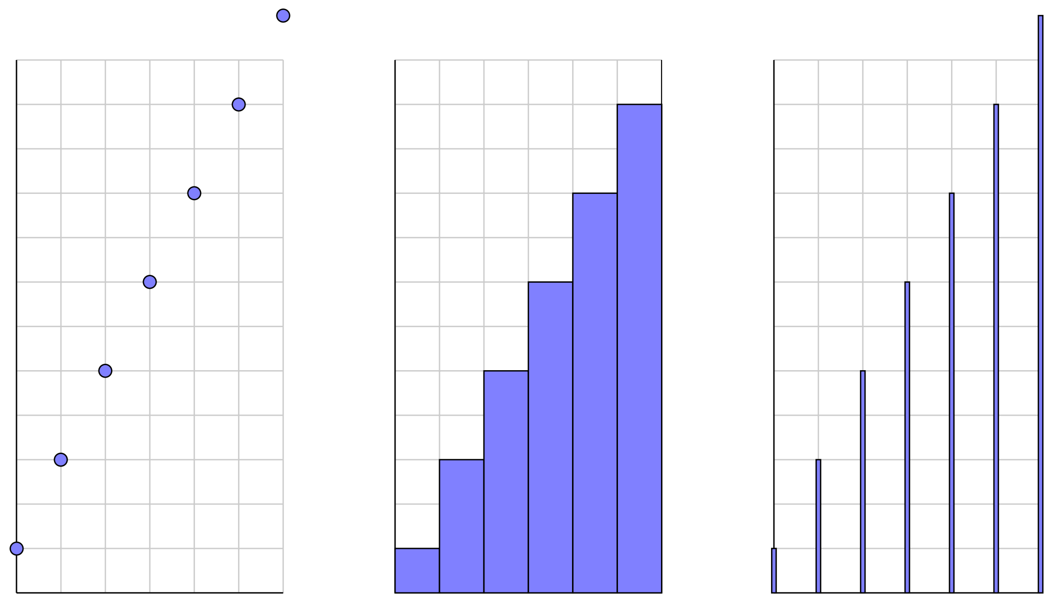
$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ \ldots$$

Of the sum sequence.

Exercise 7. *Find a formula for the term with index n for the sequence $1, 4, 10, 19, \ldots$.*

Exercise 8. *True or false? A sequence whose difference sequence takes constant values is an arithmetiuc progression.*

We can picture a sequence in any of several ways: (1) a simple plot of data points; (2) a **bar graph**; (3) a **comb**. The difference between a bar graph and a comb is largely one of width, but notice that (a) the comb teeth are centered on the x-values, while (b) the bars have their left side at these values, and run all the way to the next one.



Exercise 9. *What sequence are we plotting here?*

In all the examples we have looked at, the sequences have integer terms. But a sequence can have real values as well, for example:

$$1,\ 1/2,\ 1/4,\ 1/8,\ \ldots$$

or

$$3.2,\ 6.4,\ 9.6,\ 12.8,\ \ldots$$

Exercise 10. *What are the next 3 terms in these sequences? Plot the first 7 terms by bar graphs.*