

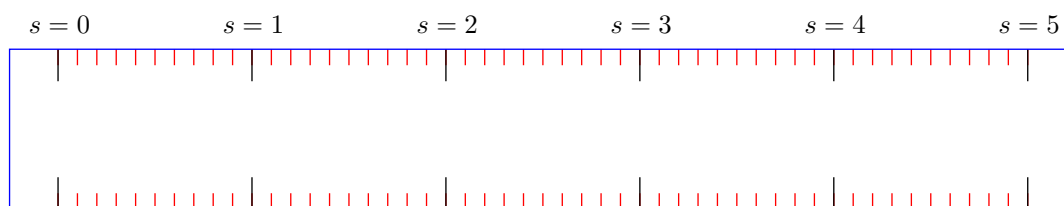
Mathematics 102 — Fall 1999

Speed

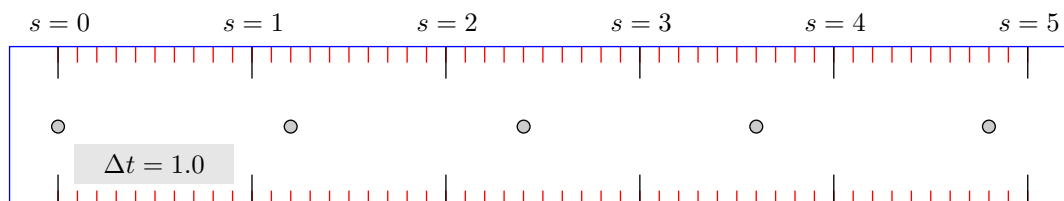
In this chapter and the next we shall pursue two themes that arose in Galileo's experiments. In this one, we shall examine more carefully the concept of speed.

Uniform motion

What was notable about Galileo's experiments on rolling balls descending under the influence of gravity was that the speed of the object increased as it descended. We saw certain features of this behaviour, but in order to understand it better we will now look at a much simpler experiment. We shall (in our thought experiment) roll a heavy ball along a flat surface, assuming as before that we can ignore friction. Just to fix the image in your mind, we shall take the table to be roughly 5 metres in length, and we shall mark off a coordinate s along it, which represents the distance in metres from the left hand mark, so the view from above is like this:



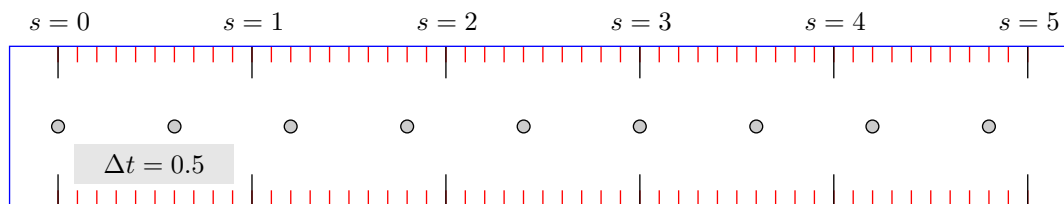
Now we start the ball rolling from the left, so that at $t = 0$ seconds it crosses the left hand edge where $s = 0$, and then freeze its location at successive intervals of 1 second. We get this picture:



Here is a record of the positions:

t	1	2	3	4
s	1.2	2.4	3.6	4.8

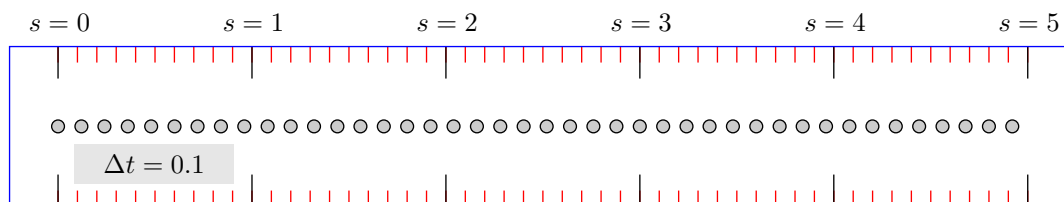
The pattern is very even, rather dull in fact compared to what we saw in Galileo's experiment. In every one of the time intervals of one second, the ball travels 1.2 metres, so using these data we say the speed is 1.2 metres per second. If we cut the length of time between observations, say to a half-second, we get this picture instead:



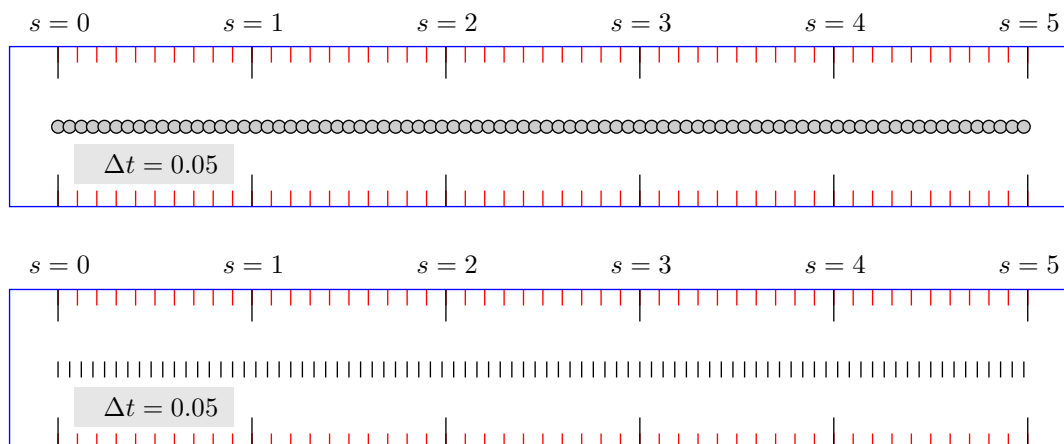
We see the ball twice as often. Here, it travels 0.6 metres in every half-second, so we say its speed is

$$\frac{0.6}{0.5}$$

metres per second. But of course $0.6/0.5 = 1.2/1$, so the speed is the same calculation we made before. And so on. No matter how often we take observations, the speed we calculate will be the same, since the distances traversed in one time interval shrink in the same proportion as the time interval. Here is what we get if the time interval is 0.1, where the ball rolls 0.12 metres in every interval:



A bit later on, we are going to want to allow ourselves to look at very very small intervals of time. If we continue to draw the same kind of figure we have been drawing, they will begin to look confusing, since the balls will overlap. So we will just record the location by a line. The next two figures demonstrate both ways of dealing with the time interval $\Delta t = 0.05$:



We can summarize what we know in this way:

- As long as we know that the speed is uniform, any two observations of position at different times will tell us what the speed is. If the ball rolls a distance Δs (change in s) in a time interval of length Δt (change in t), then we have

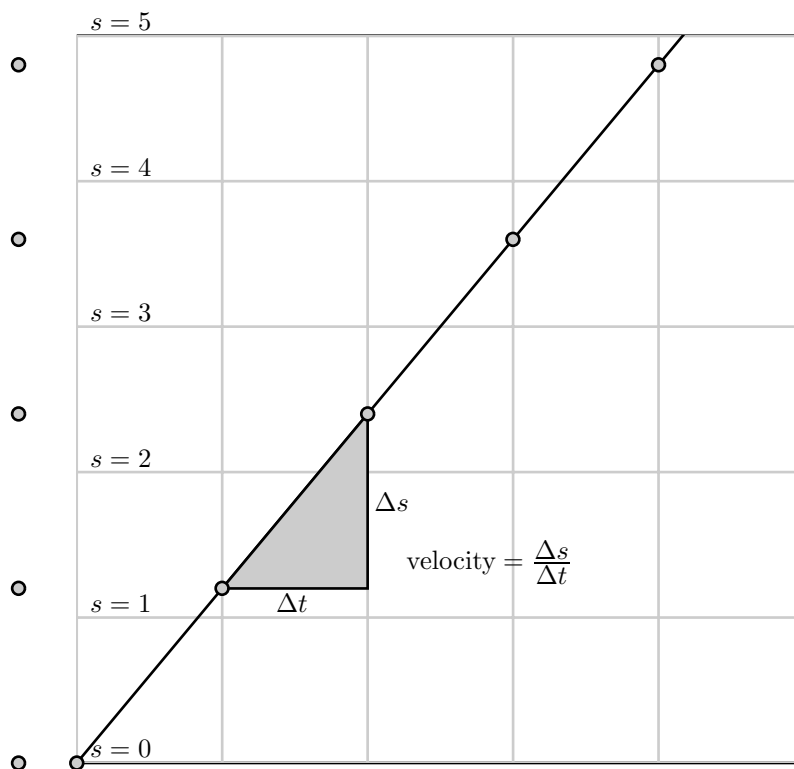
$$v = \frac{\Delta s}{\Delta t}.$$

We use v here because while we have written of **speed** so far, from now on we shall work with **velocity**, which can be positive or negative, telling us something about the direction of motion as well. Incidentally, it may seem like a trivial remark, but it is extremely important to realize that *it really does take two observations of position; one observation of position can tell us absolutely nothing about velocity.*

There is one further step to take in order to understand thoroughly motion at constant velocity. We must now look at the graph of position s versus time t . Since the ball rolls with velocity 1.2 metres per second, at time t the position s is given by the formula:

$$s = 1.2t.$$

So the graph of s against t is a straight line with **slope** 1.2 that passes through the origin.



We have also indicated in this figure several observations, as well as some extra information. At time $t = 1$ the position is $s = 1.2$, and at $t = 2$ it is 2.4. The velocity of motion is therefore equal to

$$\frac{\Delta s}{\Delta t} = \frac{1.2}{1}.$$

It is not an accident that this is the same as the slope of the graph, as the figure also indicates.

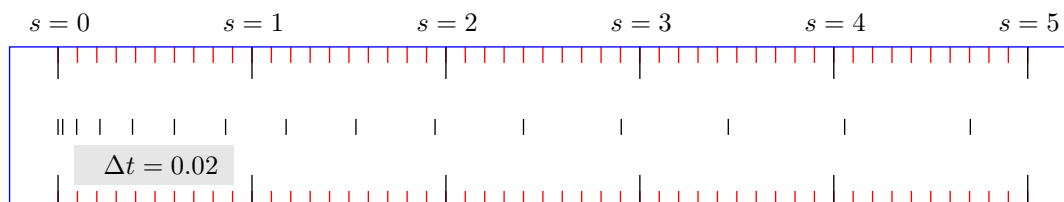
We'll write this again, because it is always true, and one of the most important things you have to understand in this course.

- *The velocity of a uniform motion (i.e. one at constant speed) is the slope of the graph of position versus time.*

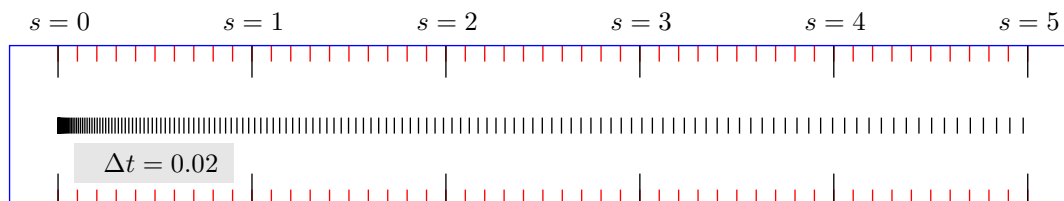
We shall come back to review linear graphs and slopes a bit later on. In the next section, however, we go on to discuss Galileo's experiment again.

Accelerated motion

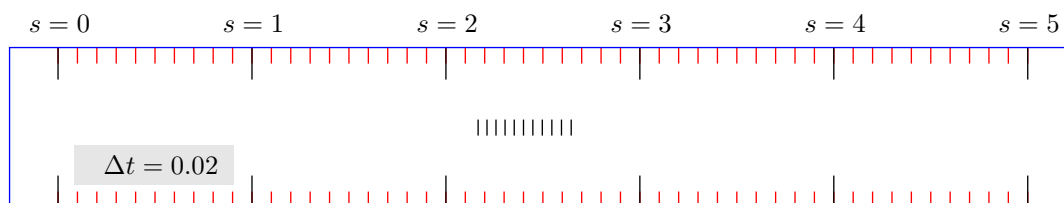
Let's look again at a ball rolling without friction down a slightly inclined ramp. Marking our ramp as we marked the flat table we get this figure for a time interval of $\Delta t = 0.02$:



Sure enough, we can see that the velocity is increasing, since the distance traveled between two observations is increasing. We now want to ask the more precise question: *How can we actually get a good estimate for the velocity at any given moment t ?* For example, at $t = 2$? In doing this, we want to explain a simple basic idea which will work in any similar situation. This idea is suggested by the following figure, where we have taken $\Delta t = 0.02$.



It is a bit harder to see here that the velocity is increasing, because from one interval to another the velocity grows only a small amount. We can see this even more strikingly if we throw away all the observations except several around $t = 2$ (to be precise, in the range $t = 1.9$ to $t = 2.1$):



Here is the basic idea we mentioned, which this figure illustrates:

- *If we take observations at a very small time interval around any given moment, then even though the velocity may be changing, it will change very little, and the record of the motion looks very much like that of uniform motion.*

The consequence is that if we want to estimate velocity at that moment, we just calculate $\Delta s / \Delta t$ for one of those small time intervals around the moment. In this example, the formula for the motion is (according to Galileo's Law)

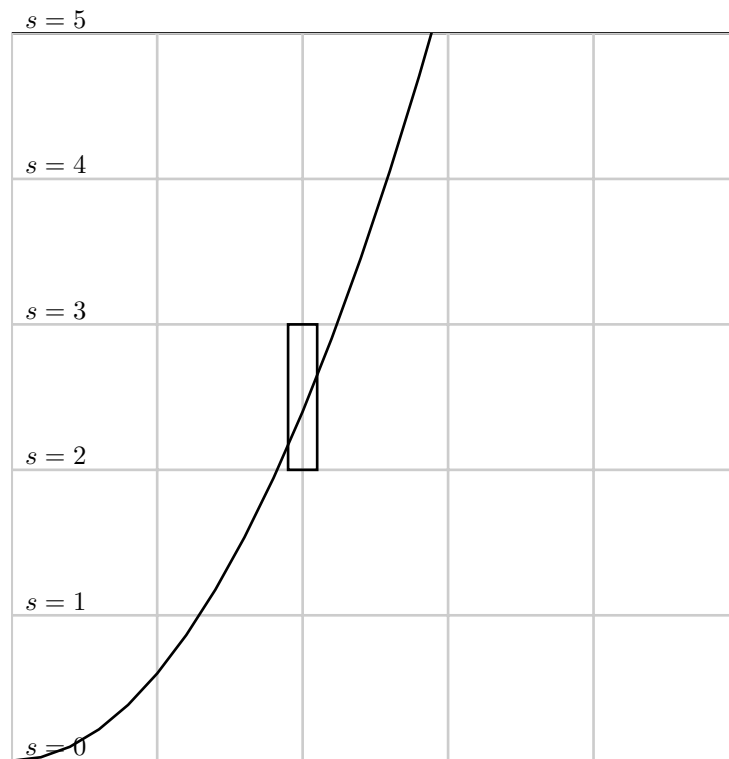
$$s = ct^2$$

where c happens to be 1.2. Here is the record of three positions:

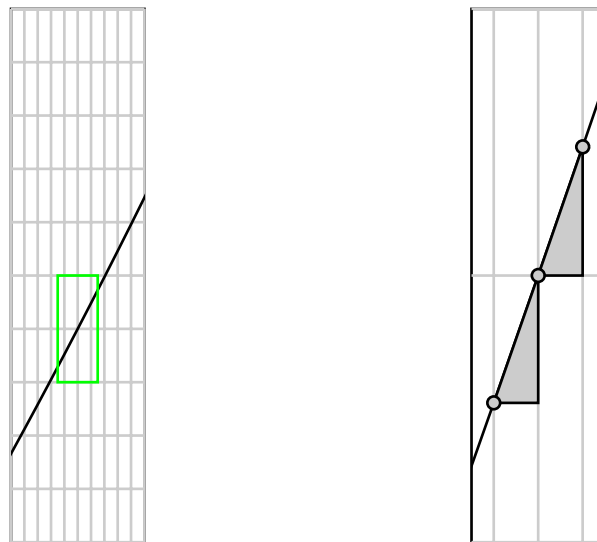
t	s	Δs	$\Delta s / \Delta t$
1.98	3.9204	0.0796	3.98
2.00	4.0000	0.0804	4.02
2.02	4.0804		

In the third column we have recorded the change in position Δs between one moment and the next; and in the fourth we have recorded $\Delta s / \Delta t$, our estimate of the velocity.

We can interpret our basic idea very neatly in terms of the graph of position against time. Here is the graph covering the same range as our figures:



And here is what we see when we zoom in to the region around $t = 2$, covering the range $t = 1.9$ to $t = 2.1$ (look at the figure on the left):



Of course this is a rather dull graph, and *that's the point!* It's dull because it looks just like a straight line, whereas the shape of the position/time graph overall is curved. It looks like a straight line precisely because in this small range of times around $t = 2$ the velocity isn't changing much. Of course we can now estimate the velocity of the ball at $t = 2$ to be the 'slope' of this almost-straight line, which we measure by looking at the time intervals just around $t = 2$. Look now at the figure on the right above, where we have zoomed in once more. This second figure matches the calculation we did earlier.

To sum up:

- *If we want to estimate the velocity of a moving object at time t , we observe its position at two times close to t separated by a small interval Δt . Suppose the change in position is Δs . Then we estimate*

$$v = \frac{\Delta s}{\Delta t}.$$

We can expect that our estimate will become more accurate if we use a smaller time interval. We can think of them as being extremely small. But not zero. We repeat the same remark here that we made earlier for uniform motion: *It takes two observations to say anything at all about velocity. From a single observation we can learn nothing at all about it.*

What is different about non-uniform motion is that two observations can only tell us something about velocity at times close to those when the observations were made.