## Mathematics 102 — Fall 1999

## Differential equations

A **differential equation** is an equation relating the derivative of a function to the function itself, as well as the independent variable. We have seen a small number of differential equations so far in this course, for example the equation

$$y' = y$$

satisfied by the function $y = e^x$. In this chapter we shall look at several more examples, and explore what a differential equation means.

Differential equations are important because the laws of nature are often most directly expressed in terms of them. The laws they express are often very simple, and the equations themselves relatively simple once you get used to the basic idea of a differential equation, but understanding how to derive from them predictions about the behaviour of natural systems is not usually so simple. This is the principal problem mathematics has to deal with regarding differential equations.

The main technical problem turns out to be that of finding all functions satisfying the equation. This is called **solving** the differential equation. In general, a differential equation will have lots of solutions. For example, suppose the equation to be $y' = y$. We know that $y = e^x$ is a solution. But if we let $y = ce^x$ then this $y$ is also a solution. It is not too difficult to see that these functions, as $c$ varies, make up all solutions of the differential equation. This is generally true—the solutions to a differential equation form a family with a 'constant that can be varied' to give different solutions.

### Radioactivity

One of the simplest places where a differential equation occurs is in the theory of radioactivity. Let's recall first what radioactivity involves. The atoms of some elements, such as radium or uranium, occasionally spit out from their nucleus an $\alpha$ particle (2 protons and 2 neutrons) or a $\beta$ particle (an electron). In this way the charge on their nucleus changes (down by 2 units in case an $\alpha$ particle is emitted, up by 1 if a $\beta$ particle), and the element changes into some other element. For example, radium has atomic number $88$ and atomic weight $226$, and it decays into a form of the gas radon with atomic number $86$ and weight $222$. There is an amazing fact about this process, which can be summarized in the single odd assertion that **atoms don't grow old**. People, by contrast, grow old. What this means is that their chances of dying increase after a while, and it becomes a near certainty in time. This does not happen to atoms. *Whether an atom of radium decays radioactively in any given time interval is completely independent of when that time interval starts.* To tell the truth, although we possess in quantum mechanics some very elaborate theories which describe what goes on very exactly, the underlying reasons for what we see are only imperfectly understood. Nonetheless, what we say about the probability of decay is true, and we accept it as given.

Radioactive decay is a statistical process. If we could look at only a very small number of radium atoms, then how many of them if any decay in the next ten minutes—or the next ten years, or the next ten thousand years—is totally unpredictable. But if we assemble a very large number of them, as is in fact present in any measurable quantity of radium, statistical fluctuations cancel out, and the process looks very regular. If we were to graph the exact number of radium atoms present in a sample of radium at time $t$ the true graph would have a very large number of small steps in it, but the size of those steps would be insignificant compared to the total numbers involved. The graph would look smooth from a distance, and it is not a serious error to assume that it is actually smooth.

Now the radioactivity of a given sample of radium is proportional to the quantity we are looking at. If one sample is twice the size of another, then twice as many atoms will decay in any given interval of time. Another way to say this:

- *At any given moment, the rate of radioactive decay of a radioactive substance is proportional to the amount present.*

This is the fundamental law of radioactivity. Taken together with what we said earlier, it can be translated directly (as a matter of definition, really) into the mathematical equation

$$\frac{dr}{dt} = -kr$$

where $k$ is a positive constant (no aging, decrease of substance), uniquely determined for every radioactive substance.

Think about it: $r = r(t)$ is a function of time whose derivative is equal to some constant times itself. It is easy to come up with a function with this property—$r(t) = e^{-kt}$. And it is also easy to see that any constant multiple of this satisfies the same differential equation. In fact, all solutions of the differential equation are of this form. So if $r(t)$ is the amount of radioactive substance present at any time $t$, then

$$r(t) = ce^{-kt}$$

for some constants $c$ and $k$. The physical meaning of $c$ is simple—if we set $t = 0$ we get $r(0) = c$, so $c$ is the initial amount of substance present. What about $k$? It controls the rate of radioactivity. This can also be measured by something called the **half-life** of the substance, the time interval $h$ it takes for the amount present to reduce by $1/2$. If $t = h$ we therefore get from the formula for $r$

$$1/2 = e^{-kh}, \quad -kh = \log(1/2), \quad k = \log 2/h \ .$$

So the constant $k$ is a simple multiple of $1/h$.

The half life is one simple measure of how fast something decays. Antoher is the **relaxation time** $\tau$, the amount of time it takes to decay by a factor equal to $1/e$, where $e = 2.718\ldots$ Here we have

$$1/e = e^{-k\tau}, \quad -k\tau = -1, \quad k = 1/\tau \ .$$

So $k$ is exactly equal to the inverse of the relaxation time.

**Exercise 1.** *What is $k$ for radium? For radon (half life $3.825$ days); $U_{238}$ (half life $4.498 \times 10^9$ years)? Strontium$_{90}$ ($25$ years)? Relaxation time? How long for each of them to reduce by a factor of $100$?*

### The basic theoretical fact about a first order differential equation

Given a differential equation

$$y' = f(x, y)$$

and a point $(x_0, y_0)$ in the plane, there exists a unique curve through this point representing the graph of a solution. Finding the unique solution with this property is called **solving the differential equation with the given initial condition**. The most common value for $x_0$ is $0$. Each possible $y_0$ then gives a unique solution, and this can be considered the 'constant that varies'. In the case of $y' = y$ we have said that the solutions are all of the form $y = ce^x$. If we set $x = 0$ we get $y(0) = c$, so $c = y_0$.

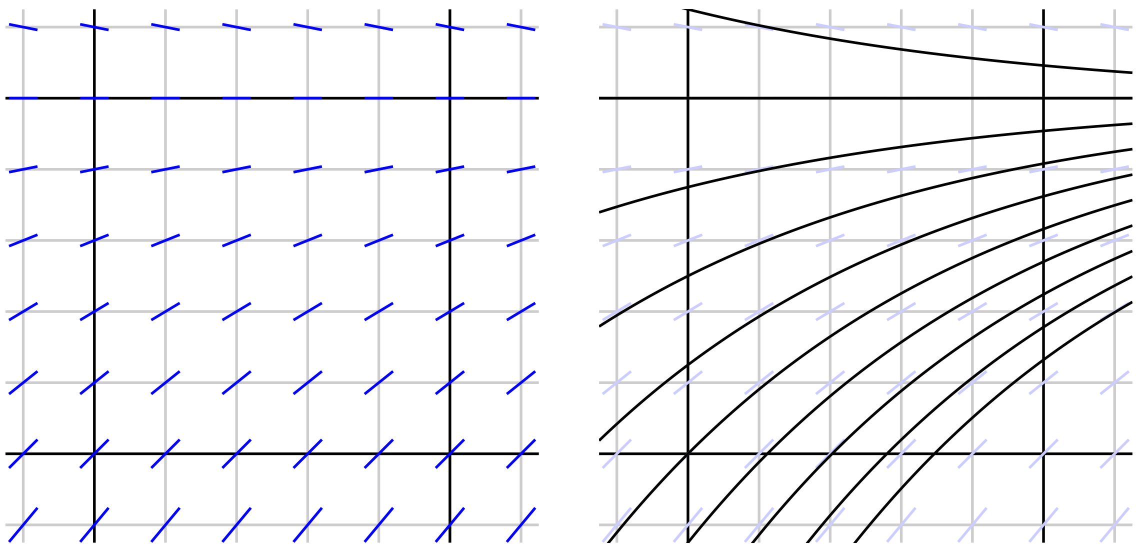
## Slope fields

You can picture a differential equation geometrically through a **slope field**. You should imagine at each point in the $(x, y)$ plane a very small line segment whose slope is equal to $f(x, y)$. The relationship between this and the differential equation is that

- *The graph of any solution has the property that at each point of the graph $y = y(x)$ the slope of the graph, that is to say $y'(x)$, is equal to the slope of one of the segments in the slope field, which is $f(x, y)$.*

In practice, to draw a slope field, (1) you should fix some region of the plane and a grid covering it; (2) at each point of the grid draw a small segment with the right slope. There several reasons why you might want to do this. The principal one is that almost always the picture you see will suggest what the graphs of solutions look like.

On the left is the slope field for $y' = -y + 1$, and on the right a few solution graphs superimposed on it.



## Finding an approximate solution

The most important thing to know about solving a differential equation is that *there is no formula that works all the time.* The best we can do if we are confronted with an arbitrary equation and initial condition

$$y' = f(x, y), \quad y(x_0) = y_0$$

is to find good approximations. There are many ways to do this. Unfortunately the ones which are very accurate are also very complicated, and in this chapter we just look at a simple one which is not too accurate, but accurate enough to be useful. It is called **Euler's method**. All methods have one thing in common. They start with $(x_0, y_0)$ and then produce a sequence of points $(x_1, y_1)$, $(x_2, y_2)$ , . . . which will generally lie close to the graph of the true solution we are looking for. The values of $x$ are easy to calculate—we choose an interval $\Delta x$, and set $x_1 = x_0 + \Delta x$, $x_2 = x_1 + \Delta x$, . . . In other words, the values of $x$ just march across uniformly. The values for $y$ are a bit trickier. The basic idea is not too complicated, however. If we have just calculated a point $(x, y)$, we know that the slope of the graph through that point is $s = f(x, y)$. This represents, essentially, the ratio of the rate of change in the value of $y$ to that in $x$. If we change $x$ by $\Delta x$, then the change in the value of $y$ will be approximately $s\Delta x$. So we set the next value of $y$ to be $y + f(x, y)\Delta x$. Laid out in detail, here is Euler's method:

(1) *Choose a step size $\Delta x$ by which $x$ changes in each step.*
(2) *Decide how far you are going to go, or equivalently how many steps you are going to take.*
(3) *If you have just calculated $(x_n, y_n)$, you do the next step by calculating*

$$x_{n+1} = x_n + \Delta x$$
$$y_{n+1} = y_n + f(x_n, y_n)\,\Delta x$$

(4) *Stop when you have gone as far as you wanted to.*

If you are drawing the graph of the approximation, the best thing to do is plot the points $(x_n, y_n)$ you get and connect them by line segments ('connect the dots').

Let's do one example completely. Suppose the equation and initial condition are

$$y' = -y + 1, \quad y(0) = 0\ .$$

Let's choose a step size of $\Delta x = 0.2$, and decide to to $5$ steps, so we shall have an approximation to the solution across the $x$-range $[0, 1]$.

We have already seen the slope field.

The calculations we have to make are not too difficult, but it is best to lay it out in a table, which starts out like this:

| $n$ | $x$ | $y$ | $f(x,y) = -y + 1$ | $\Delta y = f(x,y)\,\Delta x$ |
|---|---|---|---|---|
| 0 | 0 | 1 | | |

We can fill in the first row easily enough:

| $n$ | $x$ | $y$ | $f(x,y)$ | $\Delta y$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0.2 |

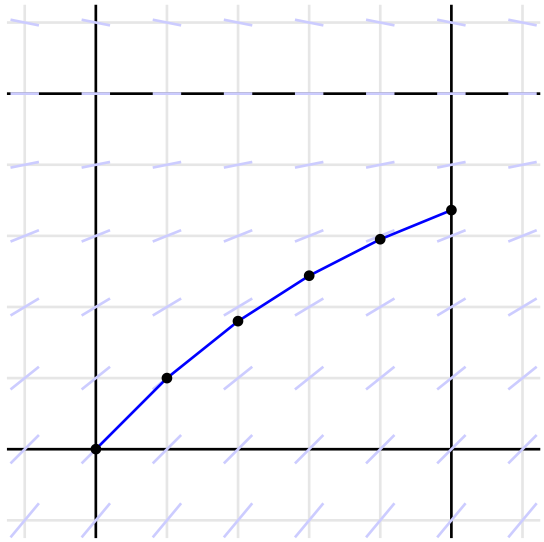
But then we can do the second row—$y_1 = y_0 + f(x_0, y_0)\,\Delta x$:

| $n$ | $x$ | $y$ | $f(x,y)$ | $\Delta y$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0.2 |
| 1 | 0.2 | 0.2 | | |

And finally the full table of calculations:

| | | | | |
|---|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 1.0000 | 0.2000 |
| 1 | 0.2000 | 0.2000 | 0.8000 | 0.1600 |
| 2 | 0.4000 | 0.3600 | 0.6400 | 0.1280 |
| 3 | 0.6000 | 0.4880 | 0.5120 | 0.1024 |
| 4 | 0.8000 | 0.5904 | 0.4096 | 0.0819 |
| 5 | 1.0000 | 0.6723 | 0.3277 | 0.0655 |

Here is a picture of the approximating polygonal path:



A final remark. The polygon we get is certainly not the graph of a true solution. In this case, for example, as the line segments march across at a constant slope, the true slope indicated by the slope field decreases. So the polygon will lie too high. We can improve the accuracy of the approximation by decreasing the step size.

#### Newton’s law of cooling

Suppose you pour hot coffee into a cup and set it down in a room at a normal temperature. The coffee will of course start to cool off. One interesting feature of the way it cools is that its rate of cooling will be greater the hotter the coffee is. In other words, the rate at which it cools down will be greatest when you first pour the coffee, and as the coffee cools off the rate at which it cools down also decreases. Very near room temperature, for example, it will cool very slowly. This situation is reasonably well modeled by **Newton’s law of cooling**, which says that

- *Any small object in a large environment, changes temperature at a rate proportional to the difference in temperature between the object and its environment,.*

This was first suggested rather casually by Isaac Newton. To a reasonable extent, the constant of proportionality depends on physical properties of the object and the environment (such as the size of the cup, and what the cup itself is made of), but does *not* depend on the temperatures themselves. It is important that the object be small in order that it make sense to speak of it as having a single temperature. If it is large, the temperature will vary throughout the interior of the object in a possibly very complicated way. For example, a coffee cup might be considered small in this sense, but a human body not. It is important that the environment be large so that we do not have to worry about the interaction between the object and its environment.

Newton’s law can be directly translated into mathematics. Suppose that $\theta(t)$ is the temperature of the object, $\theta_E$ that of the environment. The law asserts that

$$\frac{d\theta}{dt} = -k(\theta(t) - \theta_E)$$

where $k$ is a positive constant. The larger $k$ is, the more rapidly the object will change temperature. Because of the minus sign, if the difference $\theta(t) - \theta_E$ is positive the object will cool off, and if it is negative the object will heat up. One thing that is useful to realize is that $\theta_E$ here may itself vary with time. For example, the room might be heating up because someone has just turned the heater on, or cooling off because someone left the door to the outside open. But even if $\theta_E$ does vary, Newton’s law will still remain essentially correct. It is not an exact law, but as long as the temperature difference is not too large it will be a good approximation to reality.

It is simple to solve the differential equation

$$\frac{d\theta}{dt} = -k(\theta(t) - \theta_E)$$

when the room temperature $\theta_E$ is fixed. Since $\theta_E$ is a constant, the derivative of $\theta - \theta_E$ is the same as the derivative of $\theta$. Thus we can also write the differential equation as

$$\frac{d(\theta - \theta_E)}{dt} = -k(\theta - \theta_E) \ .$$

But we know that the differential equation

$$\frac{dy}{dt} = -ky$$

has solutions $y = ce^{-kt}$, so we deduce

$$\theta - \theta_E = ce^{-kt}, \quad \theta = \theta_E + ce^{-kt}$$

for some constant $c$. If we set $t = 0$, we get

$$\theta_0 = \theta_E + c, \quad c = \theta_0 - \theta_E$$

where $\theta_0$ is the initial temperature. All in all

$$\theta(t) = \theta_E + (\theta_0 - \theta_E)e^{-kt} \ .$$

is a formula for the temperature at time $t$ if it is initially $\theta_0$ and the room temperature is a constant $\theta_E$.

**Exercise 2.** *Find a similar formula for the solution of*

$$y' = ay + b \ .$$

**Exercise 3.** *The coffee cup takes $20$ minutes to cool from $90°$ to $50°$ in a room at $20°$. What temperature is it after $20$ minutes more? How long from the beginning does it take to cool to $25°$?*

**Exercise 4.** *Same coffee cup, same room. Coffee poured at $90°$. Right at the beginning you throw away half the coffee and replace it by an equal amount of milk at $5°$. Doing this has the effect of reducing the temperature to the average temperature of coffee and milk, but does not otherwise change anything. What temperature is the mixture after $20$ minutes? Suppose that instead you first let the coffee cool for $20$ minutes and then put in the milk (straight out of the refrigerator). What is the temperature of the mixture then?*

**Exercise 5.** *The differential equation*

$$y' = -y + x$$

*ahs a solution of the form $y = Ax + B$. Find $A$ and $B$.*

**Exercise 6.** *The differential equation*

$$y' = -y + \cos x$$

*has a solution of the form $y = A\cos x + B\sin x$. Find $A$ and $B$.*