Mathematics 102 — Fall 1999

Exponential functions

> *The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide about every twenty minutes. This is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically; one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth.*

From page 247 of *The Andromeda Strain*, by Michael Crichton. Published by Dell in 1969.

Well, let's see how it goes. We start with one cell at midnight. At 12:20 a.m. it divides into two. At 12:40 each of these divides in turn, giving four in all. By 1:00 there are eight. So each cell becomes eight in one hour. By 2:00 there are 64, at 3:00 there are $8^3 = 512$, and at the following midnight there are 8^{24}, which is about 4.7×10^{21}. Now a single bacterium weighs about 10^{-12} grams, so all together these 8^{24} bacteria weigh about 4.7×10^9 grams; or about $4,700,000$ kilograms; or equivalently $4,700$ metric tonnes. Doesn't really seem enough, does it?

Exercise 1. *The Earth is about* $6,000$ *kilometers in radius. Assume that the density of the Earth is about the same as iron, about* 8 *times that of water. What is the Earth's mass? (One cubic centimetre of water is one gram.)*

Exercise 2. *Assuming the density of bacteria to be the same as that of water, what would be the volume of* 8^{24} *bacteria? If they were all packed in a cube, what would be the side of the cube?*

Exercise 3. *If the bacteria keep on doubling at the same rate, how long would it be before they had the same volume as the Earth? (And where on Earth would their food come from?)*

Geometrical progressions

A sequence of numbers such as

$$1,\ 2,\ 4,\ 8,\ 16, \ldots$$

is called a **geometric progression**. The idea is that the ratio of one term to the previous one is a constant. Thus a general geometric progression is of the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

in which the ratio of successive terms is r. If the absolute value of r is greater than 1 then the terms grow in size, while if it is less than 1 they shrink. Geometric progressions all have one characteristic property which you can see most easily from the original one. Suppose we calculate its difference sequence, that is to say form the sequence d_n where

$$d_n = 2^{n+1} - 2^n = 2^n(2-1) = 2^n\ .$$

Here are the two for comparison:

1	2	4	8	16	…
1	2	4	8	16	…

They are just the same!

Exercise 4. *Here is another one, with ratio* 3.

2	6	18	54	162	…	
1	3	9	27	81	243	…

What is the relationship between the two sequences here?

If we do the same calculation for an arbitrary geometric progression we get

$$\begin{array}{ccccccccc} a(r-1) & ar(r-1) & ar^2(r-1) & ar^3(r-1) & \dots & & & & \\ a & ar & ar^2 & ar^3 & ar^4 & \dots & ar^n & \dots \end{array}$$

In this case we don't get the same sequence, but we do get a geometric progression with initial term $a(r-1)$ and the same ratio r. In brief, *the difference sequence for a geometric progression is another geometric progression with the same geometric ratio*. Of course this is quite unlike an arithmetic progression where the differences are constant.

It might not be completely obvious, but Crichton is correct on one basic point. Any geometric progression with ratio greater than 1 grows pretty fast. In particular, it will grow faster than any polynomial sequence. If the ratio is near one it might take a while to get going, but sooner or later it will be ahead in the race.

Exercise 5. *Suppose $r = 1.01$. How many terms in the geometric progression r^n before you get a term greater than 10? Greater than 100? Greater than 1000? If $r = 1.001$? If $r = 1.0001$; $r = 1.00001$?*

Exercise 6. *What is the smallest value of n such that 1.01^n is greater than n? n^2? n^3? (Trial and error may be the only good method.)*

Summing terms in a geometric progression

Suppose you calculate the sums of one of the simplest geometric progressions:

$$1,\ 1+2=3,\ 1+2+4=7,\ 1+2+4+8=15,\ \dots$$

There is a simple pattern, which leads you to guess that

$$1+2+4+\cdots+2^{n-1} = 2^n - 1\ .$$

A little algebra will confirm this. More generally, let

$$s = 1 + r + r^2 + r^3 + \cdots + r^{n-1}$$

be the sum of the first n terms in any simple geometric progression. If we multiply s by r and subtract it from itself we get

$$\begin{array}{ccccccccccc} 1 & + & r & + & r^2 & + & \cdots & + & r^{n-1} & & \\ & & r & + & r^2 & + & \cdots & + & r^{n-1} & + & r^n \\ 1 & + & 0 & + & 0 & + & \cdots & & 0 & - & r^n \end{array}$$

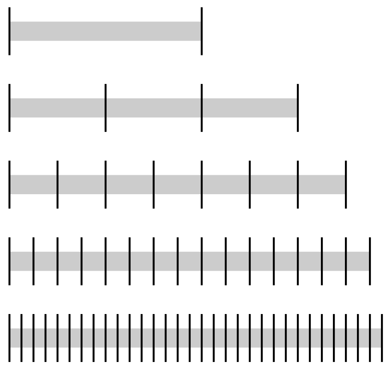
which means that

$$s - sr = s(1-r) = 1 - r^n, \quad s = \frac{1-r^n}{1-r}\ .$$

The most interesting things happen when we take $|r| < 1$. For example, look at the sums

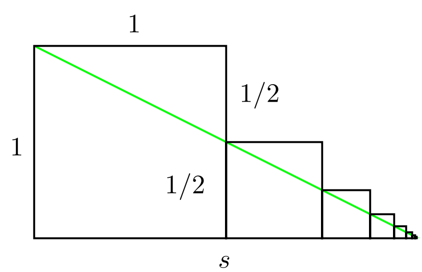
$$\begin{aligned} &1 \\ &1 + 1/2 = 3/2 \\ &1 + 1/2 + 1/4 = 7/4 \\ &1 + 1/2 + 1/4 + 1/8 = 15/8 \\ &1 + 1/2 + 1/4 + 1/8 + 1/16 = 31/16 \end{aligned}$$

We can picture these nicely:



In other words, these sums can also be written as $2-1$, $2-1/2$, $2-1/4$, $2-1/8$, $2-1/16$, … and as we keep on going we get the sum to be closer and closer to 2. The point that is visible from it is that at each point the next term we add on is half the difference between the current sum and 2.

But we can draw a different picture which, we hope, will explain a more general idea.



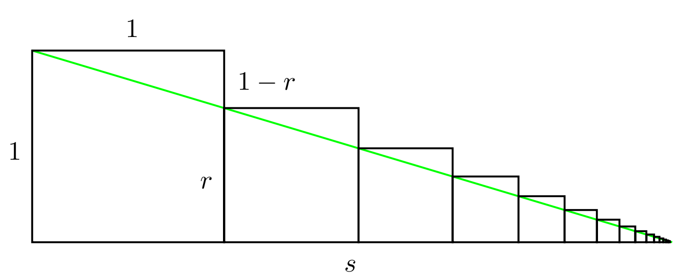
First of all, we have changed the picture from one to two dimensions. The squares have sides 1, 1/2, 1/4, etc. Therefore the distance s is the (infinite) sum

$$s = 1 + 1/2 + 1/4 + 1/8 + \cdots$$

But the squares are all similar, and this implies that the diagonal line connects their upper left corners, and that it strikes the bottom line at $(s, 0)$. Furthermore, an observation about similar triangles tells us that the slope of the diagonal is $-1/2$. We deduce that equation

$$\frac{s}{1} = \frac{1}{1/2}$$

which implies that $s = 2$. If we take a ratio r with $0 \le r < 1$ instead of $1/2$, we get this picture:



Now we see that

$$\frac{s}{1} = \frac{1}{1-r}, \quad s = \frac{1}{1-r} \; .$$

We have deduced from geometrical reasoning, in other words, the following fact:

- *For $0 \le r < 1$ the infinite sum*

  $$1 + r + r^2 + r^3 + r^4 + \cdots$$

  can be evaluated in the sense that as we take larger and larger values of n the finite sum

  $$1 + r + r^2 + \cdots + r^{n-1}$$

  approaches more and more closely the number

  $$\frac{1}{1-r} \; .$$

We say that the infinite sum

$$1 + r + r^2 + r^3 + r^4 + \cdots$$

converges to $1/(1-r)$. Actually, this holds whenever $|r| < 1$ —i.e. even if $r < 0$. We could give another geometric argument, but we can also show this algebraically. We have seen that the finite sum

$$1 + r + r^2 + \cdots + r^{n-1}$$

is equal to

$$\frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r} \; .$$

But if $|r| < 1$ the second term on the right becomes smaller and smaller as n becomes larger and larger.

Exercise 7. *Let $r = 0.9$. How large does n have to be to get the sum*

$$1 + r + r^2 + \cdots + r^{n-1}$$

to be within $1/1000$ of $1/(1-r)$? Within $1/1,000,000$?

Negative powers

It is easy to see how to extend a geometric progression

$$1, \;\; r, \;\; r^2,, \;\; r^3,, \;\; \ldots$$

backwards. In order to keep the ratio constant we must make it

$$\ldots \;\; 1/r^3, \;\; 1/r^2, \;\; 1/r, \;\; 1, \;\; r, \;\; r^2, \;\; r^3, \;\; \ldots$$

and this suggests that we can keep the rule of exponents only if we set

$$r^{-n} = 1/r^n \; .$$

Fractional powers

Crichton's colony doubles every 20 minutes, because *on the average* each bacterium splits in two every 20 minutes. If we start out with 1, then the population size will be 1 for a while, and then double. But the two bacteria now present might not grow in step, so maybe one splits after 19 minutes and the other at 21. As time goes on, more and more of the little guys will fall out of synch, and the population increase will look more like a continuous growth. So it is not a trivial question to ask: *If there are a million bacteria at one moment, how many are there 10 minutes later?*

We can generalize this question to ask how to estimate the number every ten minutes, assuming a continuous geometric growth. So if the initial number is A, we have to fill in the blanks in this sequence:

$$A,\ [\ \ ],\ 2A,\ [\ \ ],\ 4A,\ [\ \ ],\ 8A,\ [\ \ ],\ 16A,\ \ldots$$

How can we do this in a reasonable way? Continuous growth is not really tied to doubling. Maybe on some other planet a cell splits into triplets, every time. The characteristic property is that *in a given interval of time the population multiplies by a constant, independent of when that interval is.* In the first 10 minutes the size of the colony will multiply by some ratio r, and then so will it in the second 10 minutes. Since in those 20 minutes it doubles, the effect of multiplying by r twice must be the same as that of multiplying by 2 once. Hence $r^2 = 2$, $r = \sqrt{2}$. Every 10 minutes, the size of the colony grows by a factor of $\sqrt{2}$.

Exercise 8. *What happens to the colony in 5 minutes? In 1 minute?*

Look at the sequence

$$1,\ r,\ r^2,\ r^3,\ r^4,\ \ldots$$

again. The terms are in geometric progression, but the exponents are in arithmetic progression. This is just another way of expressing the fact that $r^{m+n} = r^m r^n$. This is at first only valid, essentially by definition, if m and n are positive integers. But now we want to extend the notion of powers to fractional exponents. We must preserve the identity (the basic law of exponents)

$$2^{x+y} = 2^x 2^y$$

because it is hard to see how we could claim to be making the proper definition otherwise. How do we interpret $2^{1/2}$, for example? We must have

$$2^{1/2}2^{1/2} = 2^{1/2+1/2} = 2^1 = 2$$

which implies that $2^{1/2}$ has to be $\sqrt{2}$ if the basic law of exponents is to hold. Similarly

$$2^{1/3}2^{1/3}2^{1/3} = 2^{1/3+1/3+1/3} = 2^1 = 2$$

which means that $2^{1/3} = \sqrt[3]{2}$. And more generally for any $a > 0$

$$a^{1/n} = \sqrt[n]{a}\ .$$

In addition the basic law of exponents gives us

$$a^{m/n} = a^{1/n+1/n+\cdots+1/n} = (a^{1/n})^m$$

which tells us how to interpret a^x for any fraction x. Let's say it again: we are forced to this formula if we want to maintain the basic law of exponents

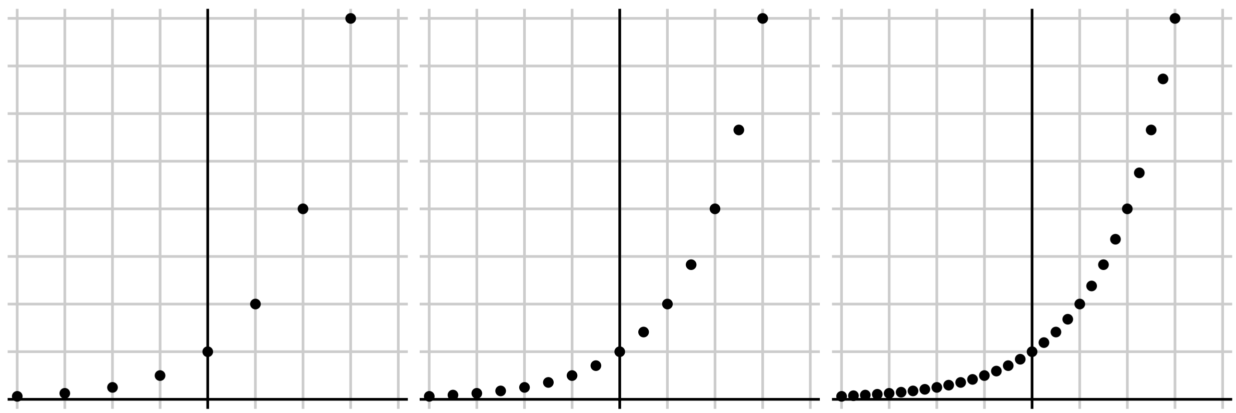
$$a^{x+y} = a^x a^y\ .$$

Note that if n is an integer, then a^n is defined for any a, while if x is a fraction then a^x is in general only defined when $a > 0$.

But there are lots of numbers x which are not fractions. For example, $\sqrt{2}$ and π are known not to be fractions. We may come back to explain later why this is, and various ways to understand it. But we would like now to extend our definition of 2^x to all numbers x, whether a fraction or not. This is not at all a trivial question. We shall discuss the problem later on in detail. But before we say even a little bit about it here, let's point out that even the definition of $\sqrt{2}$ is rather suspicious. We shall explain what we mean in a long digression in the next section, where we will also explain what 2^x means for arbitrary values of x.

An approximation

At any rate, it is this idea we use to explain how to evaluate a^x for any x. First of all we show a few pictures. In the first we plot 2^n for integer values of n, and in the next we allow n to be a fraction with denominator of 2, then of 4.



We can see that as the denominator is allowed to be bigger the graph fills in, and it is suggested taht in the limit we get a continuous grpah. If this is so, then even though we can't even think about calculating a=n arbitrary value of 2^x exactly we can think about calculating better and better approximations to it, after approximating x by numbers n whose denominators are powers of 2, which is all we can ask for.

There is one way in which we are fudging. In order for this to work, we have to know that if x and y are two fractions which are close together then a^x and a^y will be close, too. But say $y = x + h$. Then

$$a^y = a^{x+h} = a^x a^h \ .$$

So to know that a^y is close to a^x, all we really have to know is that a^h is close to 1 if h is close to 0. Suppose for example that $h = 1/N$ for an integer N, and suppose that $a = 2$. Why is the N-th root of 2 close to 1 if N is large? We just argue plausibly—the N-th root of 2 is a number which when multiplied by itself N times gives 2 . . .

Details: if $x > 0$ then

$$(1 + x/N)^N = 1 + x + \cdots$$

so

$$1 < 1 + x \le (1 + x/N)^N, \quad (1 + x)^{1/N} \le 1 + x/N, \quad 1 < a^{1/N} \le 1 + (a - 1)/N$$

for $a > 1$, and hence $a^{1/N} \to 1$ as $N \to \infty$.

Money

It costs money to borrow money. The cost is called **interest**. It is assessed as a proportion of the amount borrowed, called the interest rate. It also depends on the length of time the loan is made for.

For example, if you want to borrow \$100 for one year and the interest rate is 5%, you would have to pay back \$105 at the end of that year. You are in effect paying rent on the money you borrow. How much would it cost to borrow \$200 for one year? 5% of \$200, or \$10. To borrow x dollars? ix dollars, where $i = 5/100$. Let's say that again: if the interest rate is i (now expressed as a fraction) and you borrow an amount of money x, then at the end of the year you owe $(1 + i)x$. At the end of n years it would be $(1 + i)^n x$.

Unless you are taking out a **mortgage** on a house. Mortgages usually work a little differently, because accounting is done on a monthly basis. That is to say, if you borrow an amount of money x at interest i you are required to pay interest each month. You are required to make 12 interest payments during the year in such a way taht the total paid during the year is xi. Thus, each monthly payment must be $xi/12$. What we are next going to say seems to make more sense if we reverse the situation, and think now not of taking out a loan but of investing money into a savings account. So we shall assume we put an amount x into the bank and are paid at interest rate i on the amount. At the end of each month, the amount of interest paid is $xi/12$, making an annual total of xi. But we have a choice here—suppose we just leave the interest paid in the bank as well. This extra amount put in will accumulate interest as well. So at the end of a month we have $(1+i/12)x$, and the end of two months we have $(1+i/12)^2x$, etc. At the end of a year we shall have $(1+i/12)^{12}x$. Now interest is the amount of money earned over and above the original invested, so the interest we have accumulated will be $(1+i/12)^{12}x = x$. As we shall see in a moment, this will be more than just ix, what we would have if simple annual interest were involved. This is what we expect, since we are earning interest on more money, on the average.

Example. Let's do an explicit case. Sup[pose $x = \$1.00$, $i = 0.05$. Then here is the schedule of money in the account:

Month		Amount
1	$(1+0.05/12)$	$= 1.00417$
2	$(1+0.05/12)^2$	$= 1.00835$
3	$(1+0.05/12)^3$	$= 1.01255$
4	$(1+0.05/12)^4$	$= 1.01677$
5	$(1+0.05/12)^5$	$= 1.02101$
6	$(1+0.05/12)^6$	$= 1.02526$
7	$(1+0.05/12)^7$	$= 1.02953$
8	$(1+0.05/12)^8$	$= 1.03382$
9	$(1+0.05/12)^9$	$= 1.03813$
10	$(1+0.05/12)^{10}$	$= 1.04246$
11	$(1+0.05/12)^{11}$	$= 1.04680$
12	$(1+0.05/12)^{12}$	$= 1.05116$

In other words, under this scheme we see that the **true interest** is 0.05116 as opposed to the **nominal interest** of 0.05.

We can also do things algebraically. The amount held after one year can be calculated according to the binomial theorem to be

$$\begin{aligned}(1+i/12)^{12} &= 1 + 12\left(\frac{i}{12}\right) + \frac{12\cdot 11}{2}\left(\frac{i}{12}\right)^2 + \frac{12\cdot 11\cdot 10}{3!}\left(\frac{i}{12}\right)^3 + \cdots \\ &= 1 + i + \frac{11}{12}\left(\frac{i^2}{2}\right) + \frac{11\cdot 10}{12\cdot 12}\left(\frac{i^3}{3!}\right) + \cdots\end{aligned}$$

One thing we can see from this is that the amount is more than $1+i$.

Of course the bank could adjust the interest lower to take this extra bit of interest, but this might very well be a bit complicated to explain. Besides, although they just make a few extra pennies on each loan, they deal in a large number of loans. Banks' income is based on such pennies. This question is related clearly to what we have said previously about fractional powers. The amount due after n years is $(1+i)^n x$, and we would expect this to hold true even if n is a fraction. This would suggest that the money you owe after one month is then $(1+i)^{1/12}x$ where $i = 5/100$ and $x = 100$. This would imply a monthly interest rate of $(1+i)^{1/12} - 1 = 0.00407$. However, if you took out a mortgage you would find that this is not the case!

Let's summarize: The interest rate associated to a mortgage is only a **nominal interest**. *By convention, if the annual interest rate on a mortgage is i then the true monthly rate is $i/12$*, and the true annual interest is $(1+i/12)^{12}$. For a mortgage, the annual interest is said to be **compounded monthly**, or twelve times a year. If it were compounded N times a year the true annual rate would be $(1+i/N)^N - 1$.

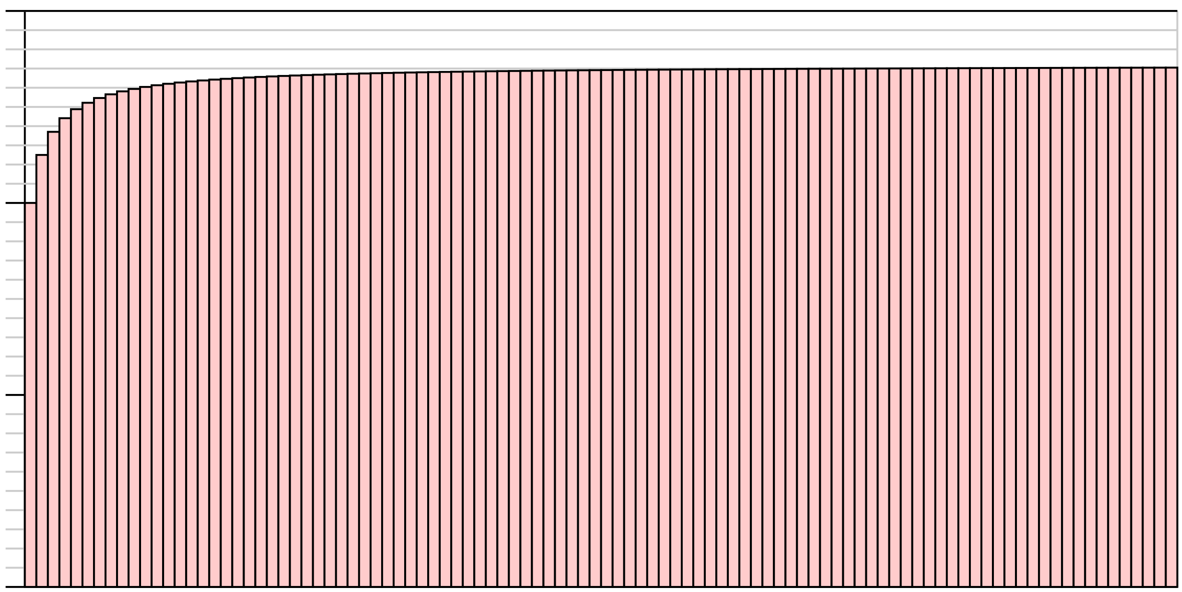
Other kinds of compounded interest are in fact used. If you borrow money on a **line of credit**, the interest is compounded daily. Here you are normally allowed to borrow or pay back at any time, and costs are based on the day on which you pay things back. If i is the nominal interest then the true annual interest is $(1+i/365)^{365}-1$.

Exercise 9. *If $i=5/100$, what is the true annual interest if compounded daily? If compounded every hour? Every minute? (Give all answers correct to 8 figures.)*

We are not sure exactly how the conventions of compounded interest came about, but from a mathematical standpoint it raises very interesting questions. Suppose we have a nominal interest rate of i which is compounded N times a year. Then if one unit of money is borrowed for one year, the amount to be paid back is $(1+i/N)^N$. We are going to presen a few new sample calculations, but since we are mathematicians we don't have to be very realistic. We are going to take i to be 1, which means a nominal annual interest rate of 100%! Here are a few sample values of $(1+1/N)^N$ for small values of N:

$N=1$	2	3	4	5	6	7
1	2.25	2.3704	2.4414	2.4883	2.5216	2.5465

This doesn't tell us much, but we can understand what we are looking at much better if we plot these numbers, and a few more as well:



Now we can see that we have a **trend**. It looks very much as if these numbers were tending towards some particular number, which we can estimate by drawing a rough line across the top at a bit above $y=2.7$.

Exercise 10. *Calculate $(1+1/N)^N$ for $N=100$; for 1000; for $10,000$; for $1,000,000$.*

We can understand better what is going on by using a little algebra.

According to the binomial theorem

$$\begin{aligned}(1+i/N)^N &= 1+N\frac{i}{N}+\frac{N(N-1)}{2}\,\frac{i}{N}^{\,2}+\frac{N(N-1)(N-2)}{1\cdot 2\cdot 3}\,\frac{i}{N}^{\,3}+\cdots \\ &= 1+i+\left(1-\frac{1}{N}\right)i^2+\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\frac{i^3}{6}+\cdots\end{aligned}$$

Now we can notice two things. First of all, the terms on the right are one by one less than those of the expression

$$E(i)=1+i+\frac{i^2}{2}+\frac{i^3}{1\cdot 2\cdot 3}+\frac{i^4}{1\cdot 2\cdot 3\cdot 4}+\cdots\ .$$

We shall see in a moment why we might expect that for any value of i the expression for $E(i)$ makes good sense and can be evaluated without too much trouble. Assuming that this is so, then we see that no matter how large N is, the quantity $(1+i/N)^N$ remains bounded by $E(i)$. Second, as N gets larger and larger the quantity $1/N$ gets smaller and smaller, so that in fact we would expect $(1+i/N)^N$ to approach $E(i)$ as N gets larger and larger.

For any $i \geq 0$, as N gets larger and larger, the quantity $(1+i/N)^N$ approaches

$$E(i) = 1 + i + \frac{i^2}{1\cdot 2} + \frac{i^3}{1\cdot 2\cdot 3} + \frac{i^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

which is finite.

Now we want to look at the expression for $E(i)$ in more detail. It has an infinite number of terms, so we cannot regard it as defining $E(i)$ exactly, and in fact we might not expect that $E(i)$ can be defined exactly by any finite expression. But we would like to check that this expression does describe $E(i)$ *in the sense that it provides a method by which $E(i)$ can be approximated arbitrarily closely.* We will not prove it, but just let you convince yourself by some calculations that you can do this in at least a few cases.

Example. Take $i = 1$. Then by calculation we get successive terms

$$\begin{aligned}
&1\\
&1+1=2\\
&1+1+1/2=2.5\\
&1+1+1/2+1/6=2.66666\ldots\\
&1+1+1/2+1/6+1/24=2.7083333\ldots\\
&1+1+1/2+1/6+1/24+1/120=2.7166666\ldots
\end{aligned}$$

So we can guess that the sum will converge to a number about equal to 2.72. We can do a bit better, however. The terms $1/m!$ with $m > 5$ we are adding on from now on are terms no larger than that in a certain geometric series:

$$\begin{aligned}
&1/6! \leq 1/720\\
&1/7! \leq (1/720)(1/7)\\
&1/8! < (1/720)(1/7)^2\\
&\ldots
\end{aligned}$$

so that the sum of what's left is less than

$$(1/720)(1+1/7+1/49+\cdot\,) = \frac{1}{720}\frac{1}{1-1/7} = \frac{1}{720}\frac{7}{6} = 0.00162037\ldots$$

and therefore the number $E(1)$ we are calculating satisfies

$$2.7166666\ldots + 1/6! = 2.7180555\ldots < E(1) < 2.7166666\ldots + 0.00162037\ldots = 2.718287037\ldots$$

so that we can guarantee that it is equal to 2.718 correct to 3 digits after the decimal point.

Exercise 11. *Calculate a similar 'sandwich'*

$$1 + 1/2 + 1/6 + \cdots + 1/8! < E(1) < \,?$$

by telling what ? is. How many figures of accuracy does this guarantee?

Exercise 12. *How many terms would you need to calculate it to 10 decimal places? 20?*

The surprising thing is that this works even for very large numbers. Let's look at

$$\begin{aligned} E(1) &= 1 + 10 + \frac{10^2}{2} + \frac{10^3}{6} + \frac{10^4}{4!} + \cdots \\ &= 1 + 10 + 50 + 166.6666\ldots + 416.6666\ldots + \cdots \end{aligned}$$

The terms we are adding are pretty big, and it isn't clear at all that this sum is finite. But if you keep on going, you come to the terms in the series

$$\cdots + \frac{10^{10}}{10!} + \frac{10^{11}}{11!} + \cdots$$

where the second term is smaller than the first, since $10/11 < 1$. From now on, the series is term by term smaller than some geometric series

$$\left(\frac{10^{10}}{10!}\right)(1 + 10/11 + (10/11)^2 + \cdots)$$

and so it must in fact converge in spite of first appearances.

Exercise 13. *Calculate the sum of the series for $i = 0.1$, $i = -1$, and $i = 2$, correct to 4 figures after the decimal point. Without doing any serious calculation, tell roughly how many terms of the series for $E(10)$ would be necessary to calculate it to 10 decimals (after the point).*

We now want to forget about the relation with interest, so we set

$$E(x) = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \cdots$$

We know, more or less, that no matter what x is this series sooner or later looks like a converging geometric series and therefore itself converges. We define the important constant e to be $E(1)$, which we know to be about 2.718. The amazing fact is that

- *We have*

  $$e^x = E(x)$$

  for all x.

We shall just make this plausible. First of all, in $E(x)$ to be the limit of

$$\left(1 + \frac{x}{N}\right)^N$$

as the integer N gets larger and larger, it is not necessary to restrict N to be an integer, because if $N \le Y < N+1$ then

$$\left(1 + \frac{x}{N}\right)^N \le \left(1 + \frac{x}{Y}\right)^Y < \left(1 + \frac{x}{N+1}\right)^{N+1} .$$

But we can rewrite

$$\left(1 + \frac{x}{Y}\right)^Y = \left(1 + \frac{1}{Y/x}\right)^{(Y/x)x}$$

and for any fixed value of x as Y gets larger and larger so does Y/x. So the limit of

$$\left(1 + \frac{1}{Y/x}\right)^{(Y/x)}$$

is just e, and the limit of

$$\left(1 + \frac{1}{Y/x}\right)^{(Y/x)x}$$

is e^x.

The function $f(x) = e^x$ has one remarkable property which explains why it is important in Calculus. Suppose we attempt to calculate its derivative. We have

$$\begin{aligned} f(x) &= e^x \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots . \end{aligned}$$

and we can try to take the derivative term by term in this sum, which is not obviously justified, but nonetheless a good idea.

$$\begin{aligned} f(x) &= 1 + x + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots \\ f'(x) &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= f(x) . \end{aligned}$$

In other words, it looks like—and it is in fact true—that the function e^x is equal to its derivative. We add this function to the list of 'elementary' functions whose derivatives you are responsible for, which currently contains just three entries:

$f(x)$	$f'(x)$
constant	0
x^r	rx^{r-1}
e^x	e^x

More about the derivative of an exponential function

Suppose that

$$f(x) = a^x$$

for $a > 0$. Then

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \left(\frac{a^{\Delta x} - 1}{\Delta x}\right) .$$

If $x = 0$ this becomes just

$$= \frac{a^{\Delta x} - 1}{\Delta x} .$$

As Δx becomes smaller and smaller, this has as limit the derivative of $f(x)$ at $x = 0$. So we can write

$$f'(x) = a^x f'(0) .$$

In other words, the derivative of an exponential function is just some constant times f itself. The number e is distinguished by the property that this constant is 1.

What about for other numbers a? We can write

$$a = e^\alpha$$

where $\alpha = \log_e a$. Then

$$a^x = e^{\alpha x}$$

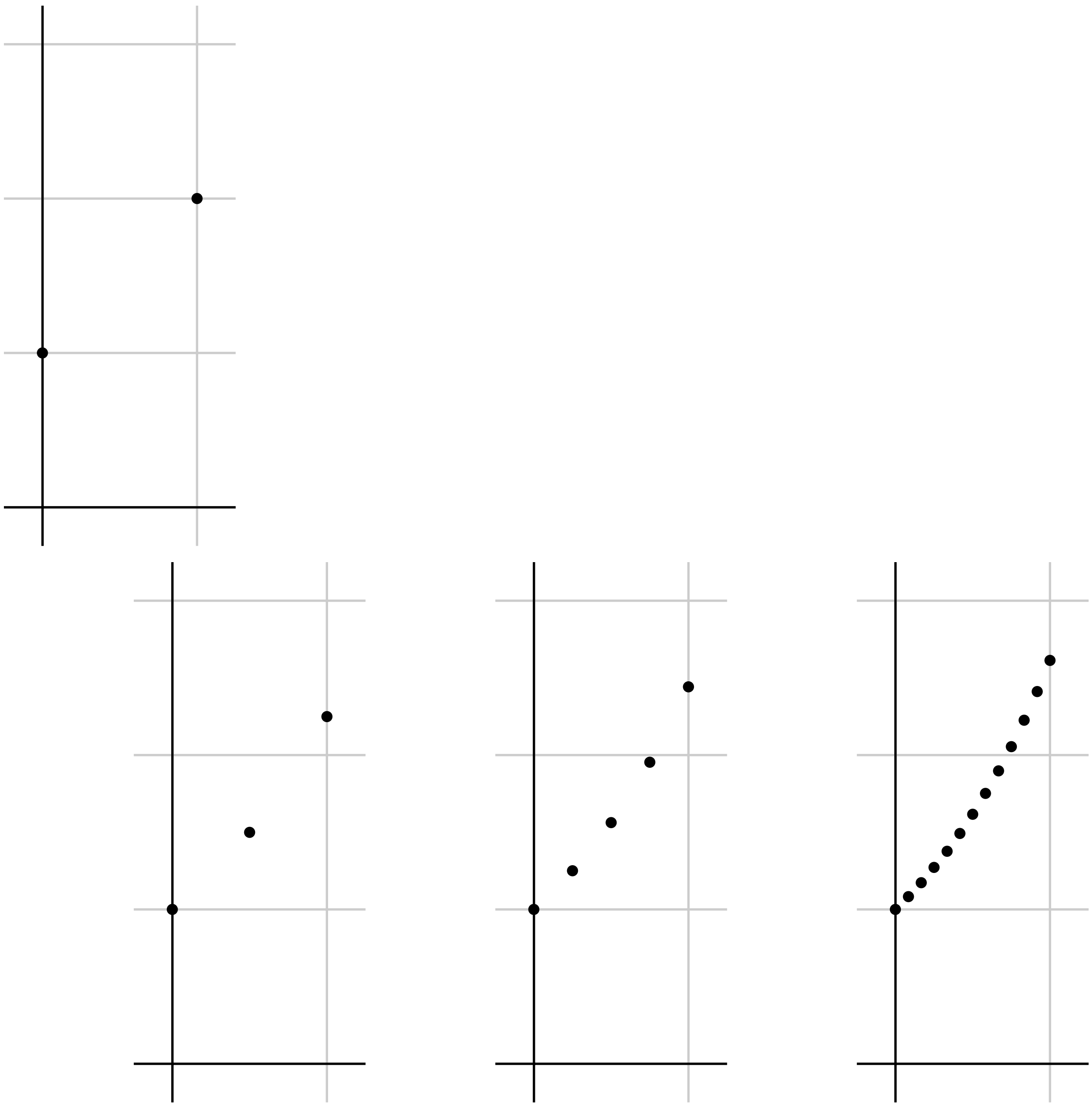
and if $f(x) = a^x$ we can calculate its derivative by the chain rule:

$$f'(x) = \alpha e^{\alpha x} = (\log_e a)\, a^x .$$

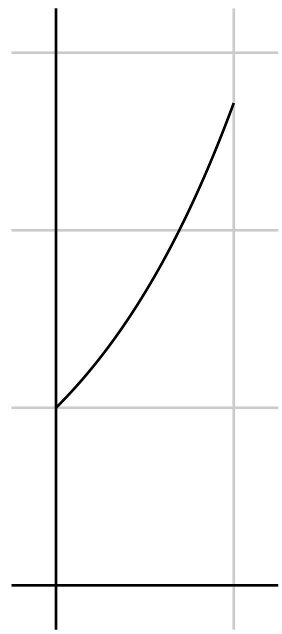
A graphical approach

We are going to make one more pass at trying to explain about compound interest and the number e, this time with a sequence of graphs.

Suppose we calculate at 100% compounded every N months, starting with \$1.00. We can plot the amount owed in this way, where N is equal to 12, 6, 3, 1:



It looks very much as though these sequences of points were converging to a single curve. Here, for example, with a slightly different format, is what we get for $N = 1000$:



Here is the main result:

- *These sequences of curves do in fact converge to asingle ultimate graph.*
- *Call e the point on this graph where $x = 1$. It is equal to the limiting value of*

$$\left(1 + \frac{i}{N}\right)^N$$

  as N gets larger and larger.
- *The graph is the curve $y = e^x$.*

A full demonstration of all of these would use an argument like the one we gave above to derive the infinite series

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots$$

But instead of giving one, let's see if we can understand why the third point is true. The k-th amount in the sequence of amounts calculated for compounding at 100% N times a year is

$$\left(1 + \frac{1}{N}\right)^k \ .$$

The x-value for the point we plot is then $x = k/N$. Suppose we fix this ratio, or in other words this x-value, and let k and N get larger and larger. In other words, we look at values kM and NM as M gets larger and larger. The value of y at our plotted points is

$$\left(1 + \frac{1}{NM}\right)^{kM} \ .$$

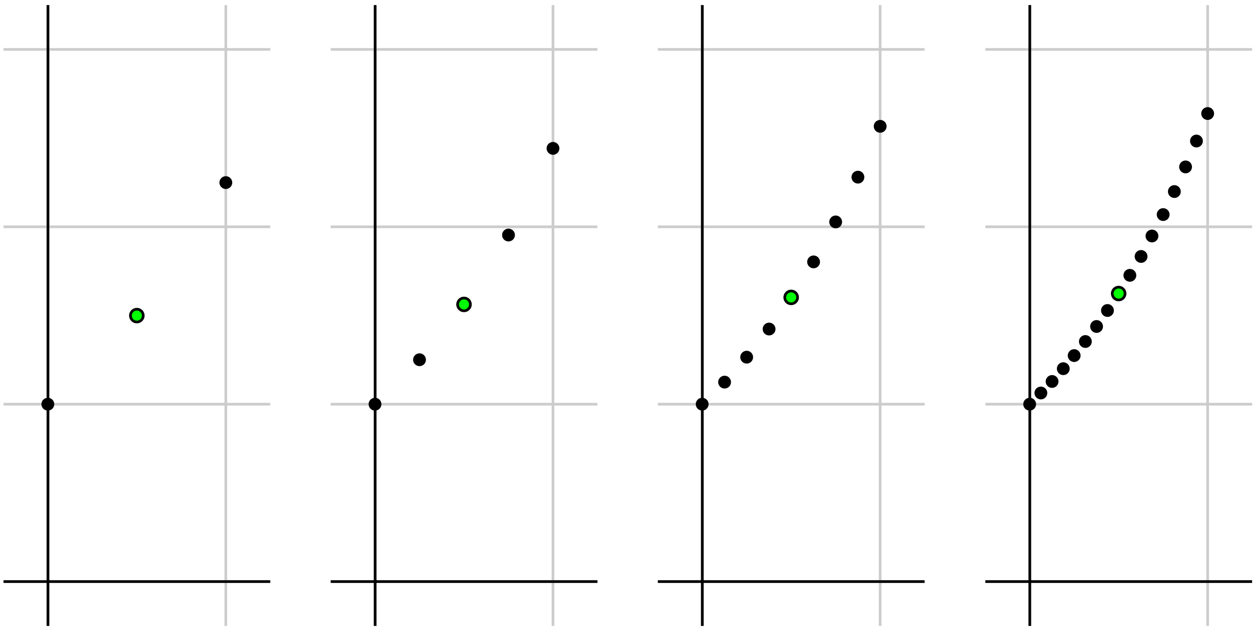
But this is the same as the

$$\left(1 + \frac{1}{NM}\right)^{NM(k/N)} = \left(1 + \frac{1}{NM}\right)^{NMx} \ .$$

And as M gets larger and larger this has limiting value e^x since e is the limit as M gets large of

$$\left(1 + \frac{1}{NM}\right)^{NM}$$

For example, the point in the middle in the figures below converges to $e^{1/2}$.



There is one last point to be established, that the function e is equal to its own derivative. We have seen that this will be OK if we know that the slope of $y = e^x$ at the special point $x = 0$ is 1. But that follows immediately from what we have said in this section. No matter what N is the first point in our plotted sequence is $(0, 1)$. The second is $(1/N, 1 + 1/N)$. But the slope of the line segment from the first to the second is always

$$\frac{\Delta y}{\Delta x} = \frac{1/N}{1/N} = 1$$

and this property remains true in the limit for the tangent line as well.