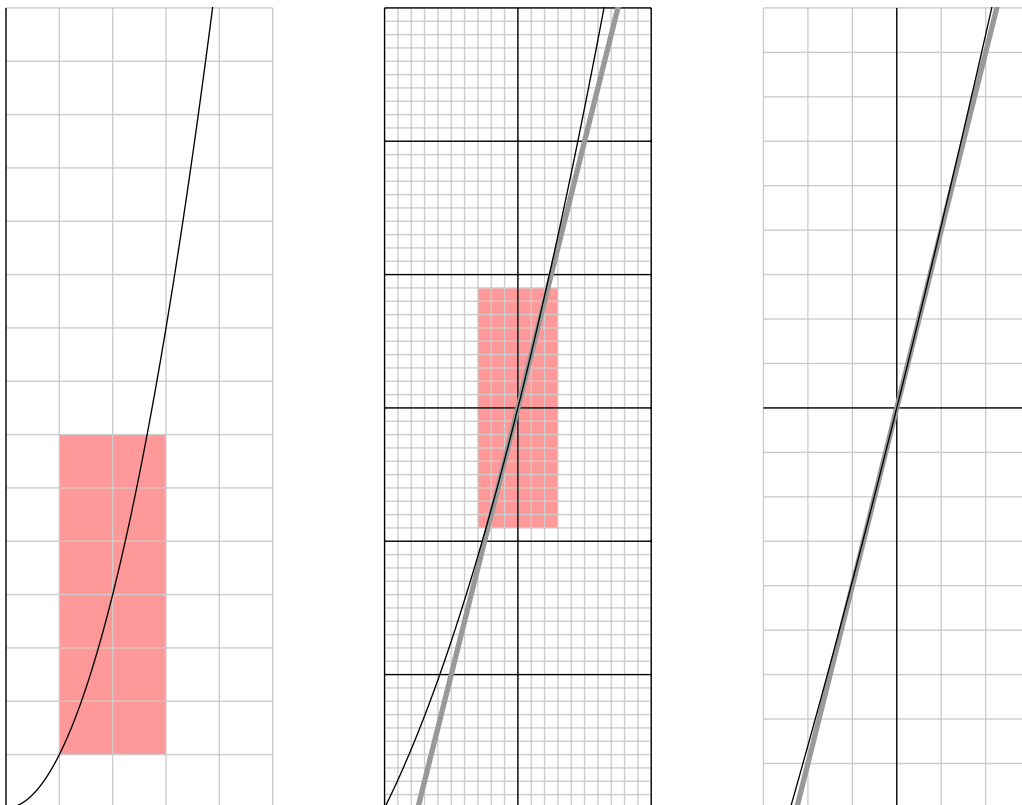**Mathematics 102 — Fall 1999**

Tangents

In this chapter we shall finish off our analysis of Galileo's experiments.

The principal idea of differential calculus

Let's arrange things so we are dealing with a ball rolling down an inclined ramp at such an angle that the position coordinate s at time t is t^2. We have therefore the graph of s versus t (shown at the far left):

We are going to try to figure out what the exact velocity at some specific time is, say at $t = 2$. Our principal motivation for doing what we are going to do is that if we want to find out what the velocity is at $t = 2$, then we don't really have to look at the whole graph, but just that part of it near where $t = 2$. So we **zoom in** to that part, as indicated in the two figures at the right above. First we zoom in to the range from $t = 1$ to 3, and then to the range 1.7 to 2.3. What we can see from these pictures is that

- *If we look only at a small part of the graph, then it is almost indistinguishable from a straight line.*

You can see this very roughly without any help, but the pictures make this clearer by adding something. Recall that the **tangent line** to the graph at a popint (x, y) is a straight line which just grazes the graph at (x, y). In this case, a tangent line will touch the graph at (x, y) without crossing it. In the picture above we have drawn the **tangent line** to the graph at $(2, 4)$ for comparison with the graph itself. You can see that in the middle figure the graph looks still a bit curved at this scale, but that in the third figure the curvature is just about invisible. Of course the curvature does not vanish completely, but it does vanish almost up to the dimensions of the thickness of the lines we are drawing. This discussion also suggests a way to make the principle more precise:

- *If we look only at a small part of the graph around a point (x, y), then it is almost indistinguishable from the tangent line at that point.*

This simple idea, believe it or not, is what Calculus is all about.

We would like to point out that there is at least something subtle about this idea, because it is not true of all graphs. Here, for example, is the graph of the function $y = |x|$, the **absolute value** of x. For example, $|4| = 4$ but $|-4| = 4$ also. Explicitly we have

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$
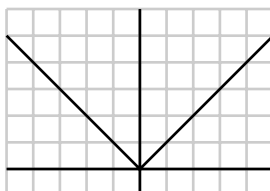
This graph does *not* look like a straight line at $(0, 0)$, and in fact has *no* tangent line there.

Velocity and tangent slopes

We zoom in so that the graph of position versus time looks like a straight line. We want to know how to calculate the velocity from what we see. Now what we have seen already in class is that if the graph were a straight line, then the velocity would be equal to its slope. It is not a straight line, but it is very very close to a straight line, namely the tangent line. So we see that

- *The velocity at any point on the graph is equal to the slope of the tangent line at that point.*

Keep in mind that this is an entirely theoretical idea. We have no way at the moment to calculate the slope of the tangent line; all we know how to do is to calculate the slope of a line through points close together on the graph according to the rule

$$\text{slope} = \frac{\Delta s}{\Delta t} \ .$$

Nonetheless a bit of logic can get what we want. So learning how to calculate the slope of the tangent line is our next order of business.

Calculating the slope of the tangent line

Let's continue to keep the point $t = 2$ in mind, and let's look at the graph in the middle above, with a few modifications.
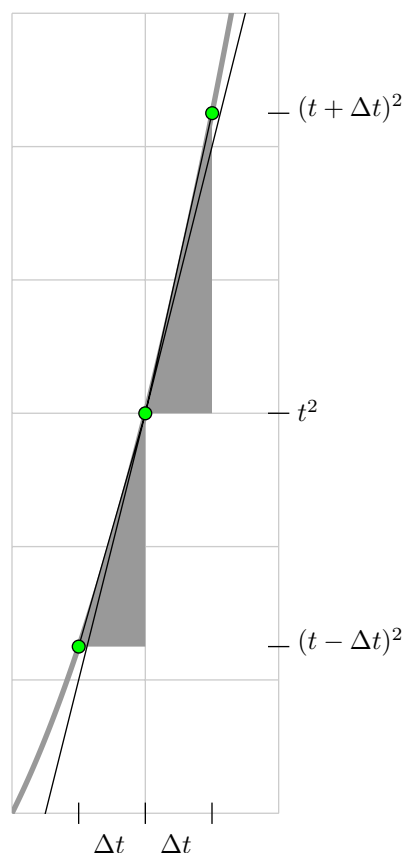
First of all, we choose some time interval Δt, not too big. Say $\Delta t = 0.5$. Then we plot on the graph the points at t-distance Δt away from $t = 2$. These will have t-values $2 + \Delta t = 1.5$ and $2 - \Delta t = 2.5$, and the s-values will be

$$(2 + \Delta t)^2 = 6.25$$
$$(2 - \Delta t)^2 = 2.25$$

We also plot what are called the **secant lines** from the lower point to the middle one, and from the middle one to the upper one, and the triangles representing their slopes. We calculate the lower slope to be

$$\frac{4 - 2.25}{0.5} = \frac{1.75}{0.5} = 3.5$$

and that of the upper one to be

$$\frac{6.25 - 4}{0.5} = \frac{2.25}{0.5} = 4.5 \ .$$

Now what we can see directly from the picture is that the lower secant line is not as steep as the tangent line, and the upper secant line is steeper. Therefore the slope of the tangent line is bracketed between the slopes of the upper and lower secants:

$$3.5 < \text{ slope of tangent line } < 4.5 \ .$$

Now do this over again with a smaller value of Δt, say $\Delta t = 0.1$. Then the upper point has height $2.1^2 = 4.41$ and the lower one has height $1.9^2 = 3.61$. The lower secant slope is

$$\frac{4 - 3.61}{0.1} = 3.9$$

while that of the upper is

$$\frac{4.41 - 4}{0.1} = 4.1 \ .$$

Again we have an inequality, this time

$$3.9 < \text{ slope of tangent line } < 4.1 \ .$$

We leave it to you to verify that if $\Delta t = 0.01$ then we get

$$3.99 < \text{ slope of tangent line } < 4.01 \ .$$

To understand exactly what is going on, we use a bit of high school algebra. We recall that

$$(a + b)^2 = a^2 + 2ab + b^2 \ .$$

Therefore

$$(2 + \Delta t)^2 = 4 + 4\Delta t + (\Delta t)^2$$
$$(2 - \Delta t)^2 = 4 - 4\Delta t + (\Delta t)^2$$

This means that the slope of the higher secant is

$$\frac{(2 + \Delta t)^2 - 2^2}{\Delta t} = \frac{4\Delta t + (\Delta t)^2}{\Delta t} = 4 + \Delta t$$

and that of the lower secant is

$$\frac{2^2 - (2 + \Delta t)^2}{\Delta t} = \frac{4\Delta t - (\Delta t)^2}{\Delta t} = 4 - \Delta t$$

This means that for every possible value of Δt, no matter how small, we have a 'sandwich'

$$4 - \Delta t < \text{ slope of tangent line } < 4 + \Delta t \ .$$

What else can the slope of the tangent line be except 4 exactly?

Now instead of 2 we look at an arbitrary value of t and inquire what the slope of the tangent line is there. We get for each Δt a sandwich

$$2t - \Delta t < \text{ slope of tangent line } < 2t + \Delta t \ .$$

and we deduce that the slope of the tangent line must be $2t$.

Exercise 1. *Graph the function $y = x^2 + x + 1$ for $x = -5$ to $x = 5$. Find by the same method the slope of the tangent line to the graph of at (x, y).*

Back to Galileo

If we set $s = ct^2$ a similar calculation will tell us that $v = 2ct$. In particular the velocity of any falling object is proportional to time. Galileo understood this, but his argument was arguably much clumsier than ours. Nonetheless, he understood also that the **rate of increase of velocity** was constant. This is called **acceleration**. To us it is hard to imagine the satisfaction that Galileo derived from the simple statement that *the acceleration of any falling body is constant during its fall.* It was in fact the first time in all of history, as far as we can tell, that a natural law was formulated so precisely and simply. A few years later Isaac Newton put together his Laws of Motion. Analyzing Galileo's work, he saw that the best way to understand it was to introduce the notion of force, which Galileo probably did not arrive at. Newton's second law asserts that force and acceleration are proportional. This was important because force is something that people develop an intuitive perception of, and also because it often has a simple mathematical expression—for example, in any gravitational field, even one far from the Earth. Newton's Laws of motion reduced the analysis of motion to two steps: (1) the physical one of understanding forces; (2) the mathematical one of deducing the motion of objects from the description of the forces on them. In this he was he broke the apparently intractable problem of understanding motion, even of complicated systems like the Solar system, into two relatively simpler ones. Even the man in the street was impressed.

Other slopes

We want now to forget about falling objects and look at the mathematical problem of finding the slopes of the tangent lines for a wide range of graphs. In a later chapter we shall look at arbitrary polynomial functions, but to finish off this one and to motivate the later discussion we shall look at the graph of $y = x^4$.
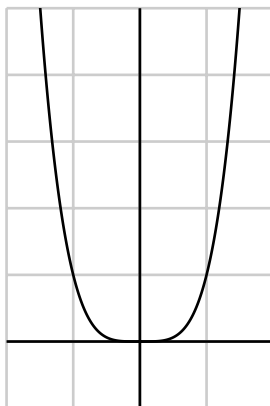
The pictures are essentially the same here. Only the algebra is different. We look at a point (x, x^4) and then the points $(x - \Delta x, (x - \Delta x)^4)$, $(x + \Delta x, (x + \Delta x)^4)$ before it and after it on the curve. We then sandwich the slope of the tangent line

$$\frac{x^4 - (x - \Delta x)^4}{\Delta x} < \text{ slope of tangent line } < \frac{(x - \Delta x)^4 - x^4}{\Delta x} \ .$$

Here we have to calculate

$$(a + b)^4 \ .$$

We shall look at similar calculations later on in more detail, but here we just do it directly:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = (a + b)(a + b)^2$$
$$= (a + b)(a^2 + 2ab + b^2)$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = (a + b)(a + b)^3$$
$$= (a + b)(a^3 + 3a^2b + 3ab^2 + b^3)$$
$$= a^4 + 4a^3b + 6a^2b^2 + 6ab^3 + b^4 \ .$$

Therefore we get a new sandwich

$$\frac{4x^3\Delta x - 6x^2(\Delta x)^2 + 4x(\Delta x)^3 - (\Delta x)^4}{\Delta x} < \text{ slope of tangent line}$$

$$< \frac{4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4}{\Delta x}$$

or

$$4x^3 - 6x^2\Delta x + 4x(\Delta x)^2 - (\Delta x)^3 < \text{ slope of tangent line } < 4x^3 + 6x^2\Delta x + 4x(\Delta x)^2 + (\Delta x)^3$$

This is much more complicated than what we had for the parabola, but not a great deal. The point here is that all the terms with a Δx in them become very small as Δx does, and we conclude that the slope has to be $4x^3$.

Exercise 2. *Let $\Delta x = 0.1, 0.01, 0.001, 0.0001$ in succession. Write down explicitly the 'sandwich' equations for the slope of $y = x^4$ for $x = 1$.*

Exercise 3. *Let $f(x) = x^3$. Calculate the algebraic expressions*

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}, \qquad \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

What do these become as Δx becomes smaller and smaller?