Mathematics 102 — Fall 1999

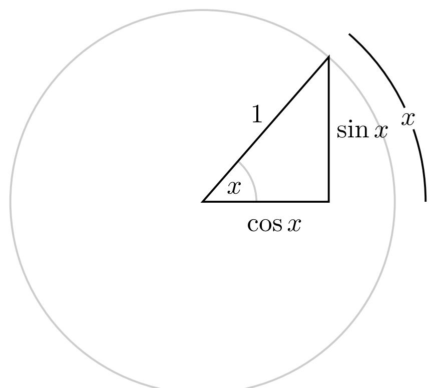
Trigonometry and calculus

Trigonometrical functions

In calculus, *angles are always measured in* **radians**. I recall that the measure of an angle in radians is the ratio of an arc of a circle covered by the angle to its radius. Thus, for example, what you were taught in grammar school to be a $360°$ angle is now 2π radians.

Exercise 1. *How many radians is* $180°$*?* $90°$*?*

For any number x, $\cos(x)$ and $\sin(x)$ are thus defined to be the horizontal and vertical coordinates of the point on the unit circle at a distance x measured along the arc of the circle from the point $(1, 0)$.



By Pythagoras' Theorem

$$\cos^2 x + \sin^2 x = 1$$

always. Two more formulas taht remain valid are the rule for the cosine and sum of angles:

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$
$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

Exercise 2. *Find a formula for* $\tan(u + v)$.

Exercise 3. *Find a simple formula for* $1 + \tan^2(x)$.

Exercise 4. *What is* $\cos(\pi/2)$*?* $\cos(\pi/4)$*?* $\sin(\pi/4)$*?* $\sin(-\pi/2)$*?*

Exercise 5. *An angle* a *has* $\cos(a) = -1/\sqrt{2}$ *and* $\sin(a) = -1/\sqrt{2}$*. What is* a*? Are there any other possibilities?*

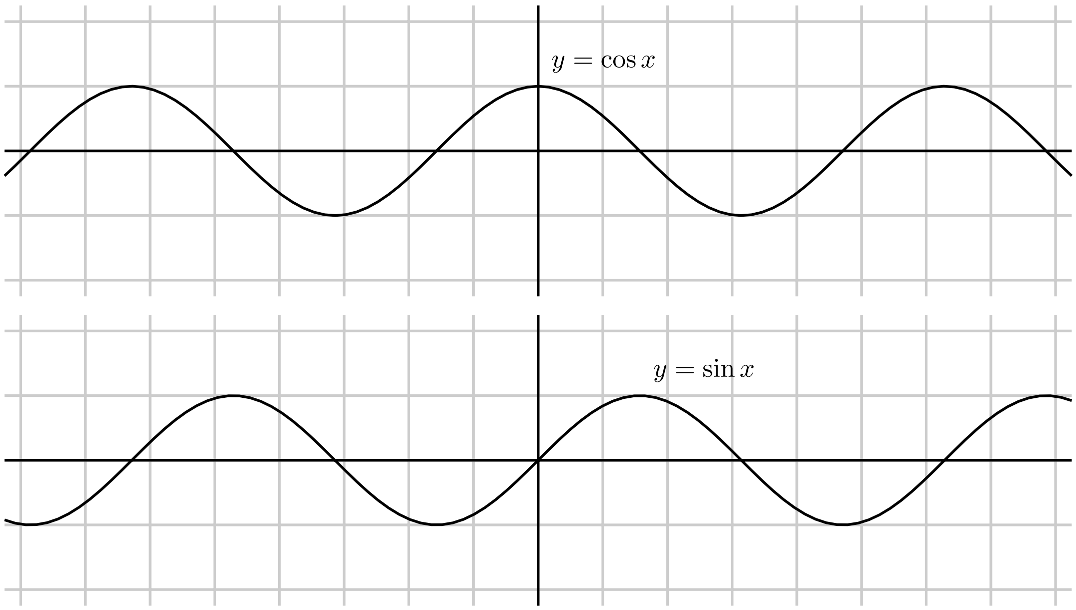
Exercise 6. *What is* $\sin(\pi/6)$ *(exactly)? Why?* $\cos(\pi/6)$*?* $\cos(\pi/3)$*?* $\sin(\pi/3)$*?*

Exercise 7. *Describe all angles such that* $\cos(a) = -1/2$.

Their derivatives

In order to calculate the derivative of $\sin(x)$ we have to apply the original idea of taking a limit of secant slopes. We will first find $\sin'(0)$ and $\cos'(0)$. The second we can just read off from the graph.

Here are the graphs of both functions.



We can see immediately from the graph, if we did not know it already, that $\cos(0) = 1$ is a maximum value of $\cos(x)$, so that $\cos'(0)$ has to be 0. The value of $\sin'(0)$ is more difficult. Basically, the idea is that if an angle is very small, then the arc length along that angle is just about the same as the vertical coordinate of the corresponding point on the circle. In other words, as a first order approximation we have

$$\sin(\Delta x) \sim \Delta x$$

which means that the limiting ratio of secant slope

$$\frac{\sin(\Delta x) - \sin(0)}{\Delta x} = \frac{\sin(\Delta x)}{\Delta x}$$

is 1. So we have

$$\begin{aligned}
\cos'(0) &= \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - \cos(0)}{\Delta x} \\
&= 0 \\
\sin'(0) &= \lim_{\Delta x \to 0} \frac{\sin(\Delta x) - \sin(0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} \\
&= 1
\end{aligned}$$

But now for any angle x at all we have

$$\begin{aligned}
\sin'(x) &= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x} \\
&= \sin(x) \lim_{\Delta x \to 0} \left(\frac{\cos(\Delta x) - 1}{\Delta x} \right) + \cos(x) \lim_{\Delta x \to 0} \left(\frac{\sin(\Delta x)}{\Delta x} \right) \\
&= \sin(x)\cos'(0) + \cos(x)\sin'(0) \\
&= \cos(x) \ .
\end{aligned}$$

The two limit expressions are the derivatives at 0 of $\sin(x)$ and $\cos(x)$. The derivative of $\cos(x)$ at 0 is 0 since $\cos(0) = 1$ is the maximum possible value. And for Δx very small we can see that $\sin(\Delta x)$ is just about the same as Δx, which means that the derivative of $\sin(x)$ at $x = 0$ is 1. Thus

$$\sin'(x) = \cos(x) \ .$$

And since $\cos(x) = \sin(\pi/2 - x)$, the derivative of $\cos(x)$ is

$$\cos'(x) = -\sin'(\pi/2 - x) = -\cos(\pi/2 - x) = -\sin(x) \ .$$

With this convention, the derivative of $\sin(x)$ turns out to be $\cos(x)$ and that of $\cos(x)$ is $-\sin(x)$.

Simply periodic functions

The functions $\cos x$ and $\sin x$ repeat every interval of 2π (because angles do). They rae said to be **periodic**. In fact they are called **simply periodic**, because they are among the simplest of periodic functions, even in a technical sense which I can't explain here. The other simply periodic functions are the ones obtained from these by scaling and shifting either vertically or horizonatlly. It turns out that all of them are of the form $C\cos(\omega x - \alpha)$ for various constants $C > 0$, $\omega > 0$, and α. This function oscillates between C and $-C$, and for this reason C is called its **amplitude**. The constant ω is called its **frequency**; the **period** of the function is $2\pi/\omega$.

What is not at all obvious is this: *any combination $A\cos\omega x + B\sin\omega x$ (two terms of the same frequency) can be expressed as a simply periodic function of this form*. It is just a matter of finding C and α. I will not explain this completely here, but just show how the calculation works. We want to write

$$A\cos\omega x + B\sin\omega x = C\cos(\omega x - \alpha)\ l.$$

To do this we apply the cosine sum formula on the righ to get

$$C\cos(\omega x)\cos(\alpha) + C\cos(\omega x)\sin(\alpha) = A\cos\omega x + B\sin\omega x,$$

which after equating corresponding coefficients leads to equations

$$\begin{aligned}
A &= C\cos(\alpha) \\
B &= C\sin(\alpha) \\
A^2 + B^2 &= C^2 \\
C &= \sqrt{A^2 + b^2} \\
\cos(\alpha) &= A/C \\
\sin(\alpha) &= B/C
\end{aligned}$$

This gives a direct formula for C, and exactly enough information to find the angle α.

Example. Look at

$$\cos x + \sin x \ .$$

We have

$$C = \sqrt{2}$$
$$\cos(\alpha) = 1/\sqrt{2}$$
$$\sin(\alpha) = 1/\sqrt{2}$$
$$\alpha = \pi/4$$

so

$$\cos x + \sin x = \sqrt{2}\cos(x - \pi/4) \ .$$

Functional inverses

If $f(x)$ is any function then another function $F(x)$ is called its **functional inverse** if $f(F(x)) = x$. In other words, if we first apply f and then F we get back where we started from. Equivalently, this means that for any number y we get $F(y)$ by solving for y in the equation $y = f(x)$.

For example, if $f(x) = e^x$ then we solve

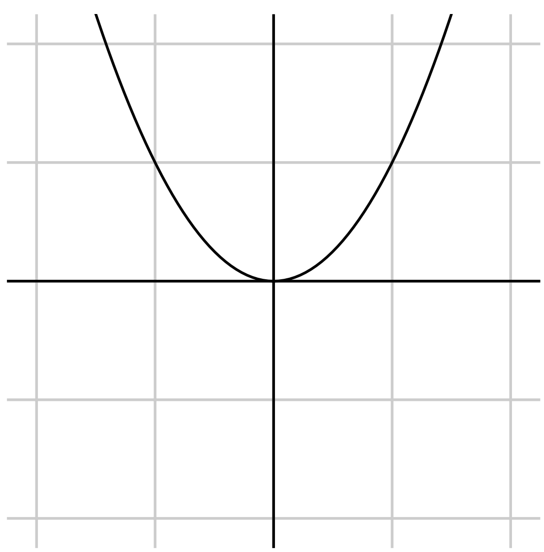
$$y = e^x, \quad x = \log(y)$$

so that log is the functional inverse of the exponential function. We disciussed this before, from a slightly different point of view—we saw that the graph of $y = \log(x)$ is obtained from the graph of $y = e^x$ by swapping x and y.

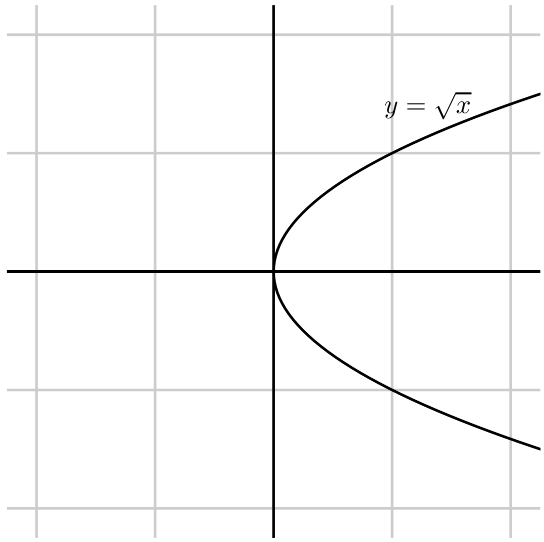
Similarly,

$$y = x^2, \quad x = \sqrt{y}$$

so the square root is the functional inverse of the square. Let's look at this example a bit more. Here is the graph of $y = x^2$:



And here is what we get when we swap x and y:



There are a few problems. The first is that not every number has a square root—the domain of $y = \sqrt{x}$, the set of numbers for which it is defined, is the range of non-negative numbers. A second problem is that even for $x > 0$ there isn't a unique square root of x, but two So we must pick one—by convention the non-negative one. So in order to define the functional inverse of $y = x^2$ we must add a bit to the description. For $x \geq 0$ the number $\sqrt{x}$ is the unique non-negative one such that $(\sqrt{x})^2 = x$.

In any event, we are interested in finding a formula for the derivative of a functional inverse. By the chain rule, we have

$$\begin{aligned} f(F(x)) &= x \\ f'(F(x))F'(x) &= 1 \\ F'(x) &= \frac{1}{f'(F(x))} \end{aligned}$$

For example, the functional inverse of e^x is $\log(x)$, so

$$\begin{aligned} \log'(x) &= \frac{1}{e^{\log(x)}} \\ &= \frac{1}{x} \end{aligned}$$

since the derivative of e^x is e^x.

The functional inverse of $\tan(x)$

Recall that

$$\tan(x) = \frac{\sin x}{\cos x} \ .$$

We can calculate the derivative by the quotient rule

$$\tan'(x) = \frac{1}{\cos^2 x} \ .$$

Exercise 8. *Sketch the graph of* $y = \tan(x)$ *for* $x = -\pi/2$ *to* $\pi/2$.

We define $\arctan(x)$ to be the unique number y such that y lies between $-\pi/2$ and $\pi/2$, and $\tan(y) = x$.

Exercise 9. *Sketch the graph of* $y = \tan(x)$ *for* $x = -10$ *to* 10.

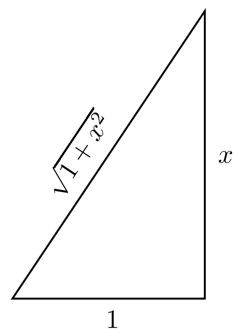
Exercise 10. *Sketch the graph of* $y = \arctan(x)$ *for* $x = -10$ *to* 10.

As for its derivative, if $F(x) = \arctan(x)$ then

$$F'(x) = \frac{1}{\cos^2(\arctan(x))}$$

since the derivative of $\tan(x)$ is $1/\cos^2(x)$. This formula can be simplified, since the following picture shows that

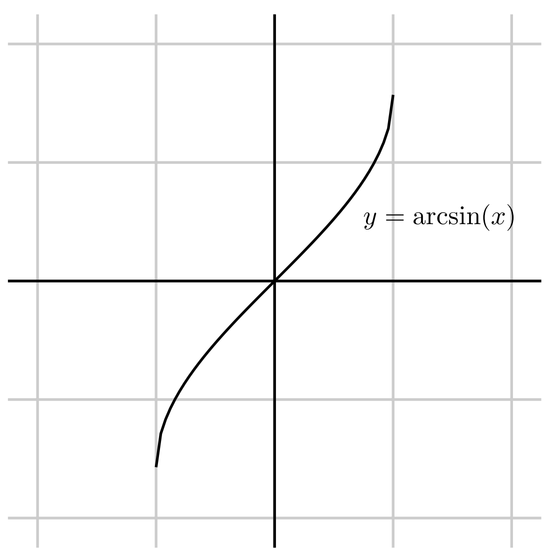
$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}} .$$



Exercise 11. *What is* $\arctan(1)$*?* $\arctan(0)$*?* $\arctan(-1)$*?*

Exercise 12. *What happens to* $\arctan(x)$ *as* $x \to \infty$ *(i.e.* x *gets larger and larger positive)?*

For any x with $-1 \le x \le 1$ $\arcsin(x)$ is defined to be the unique number y such that $\pi/2 \le y \le \pi/2$ and $\sin(y) = x$. It is a functional inverse of $\sin(x)$. A similar calculation will show that the derivative of $\arcsin(x)$ is $1/\sqrt{1-x^2}$.



Exercise 13. *Define* $\arccos(x)$ *and find its derivative.*

Exercise 14. *What is* $\arcsin'(1)$*?*

The table of derivatives so far

Recall that calculating derivatives is a recipe in two parts. The first part is memorizng the derivatives of a handful of functions that are in some sense basic. The second involves applying a set of four or five rules (sum rule, product rule, chain rule, quotient rule, and now in addition the functional inverse rule, even though it is a consequence of the chain rule) to the functions in this table. In this chapter we have in effect added some functions to the table. Here is what is it is now:

$f(x)$	$f'(x)$
constant	0
x^r	rx^{r-1}
e^x	e^x
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$

In addition, there are a few that you might want to memorize even though they can be derived from the rule for the derivatives of functional inverses:

$\log(x)$	$1/x$
$\arctan(x)$	$1/(1+x^2)$
$\arcsin(x)$	$1/\sqrt{1-x^2}$