Mathematics 103 — section 203 — Spring 2000

Eighth homework — due Wednesday, March 29

Exercise 1. *Graph the functions* $f(x) = e^{-x^2}$, e^{-2x^2}, e^{-2x^2-2}, $e^{-x^2/4}$, $e^{-x^2/4-2x}$, e^{-8x^2}, *each on a separate graph, along with the standard bell curve. Then find the integrals* $\int_2^3 f(x)\,dx$. *In each case, indicate on your graph the areas you are finding, and the corresponding area under the standard bell curve.*

Exercise 2. *Do Exercise 1 in the notes on bell curves.*

Exercise 3. *(a) Calculate the probabilities of getting between* k *heads* ($k = 0$ *to* 9) *from* $n = 9$ *coins, but when the probability of getting heads on each coin is only* $p = 0.46$. *(b) Draw the bar graph. (c) Calculate the probability of getting between 5 and 8 heads in two ways: (i) exact; (ii) using bell curve with mean* $m = np$, *spread* $s = \sqrt{np(1-p}$. *(d) Graph the bell curve on top of the bar graph.*

Exercise 4. *If* $p(x)$ *is a probability distribution on the interval* $[a, b]$, *then the mean value of a function* $G(x)$ *of* x *is the integral*

$$\overline{G(x)} = \int_a^b G(x)p(x)\,dx \ .$$

For example, the mean value of x *is just*

$$M_1 = \overline{x} = \int_a^b xp(x)\,dx \ .$$

A special case is the **second moment**

$$M_2 = \overline{x^2} = \int_a^b x^2 p(x)\,dx \ .$$

The **variance** *is the mean of* $(x - \overline{x})$, *or*

$$V = \int_a^b (x - M_1)^2 p(x)\,dx$$

It has something to do with measuring the spread of the distribution. For example, it will be small if all values are bunched up around the mean of x, *and large if there are a lot of values far away. This can also be expressed as* $M_2 - M_1^2$. *The* **standard deviation** σ, *sometimes called the* **spread** *is* $\sqrt{V}$.

Compute the mean, variance, and standard deviation of

(a) $p(x) = 1/b \quad (0 \le x \le b)$

(b) $p(x) = ke^{-kx} \quad (k > 0)$

(c) $p(x) = 2(1 - x) \quad (0 \le x \le 1)$.

Exercise 5. *For the differential equation*

$$y' = 2y(2 - y)$$

(a) Graph the slope field—grid 0.25×0.25*—in the region* $0 \le x \le 2$, $0 \le y \le 3$. *(b) Find explicitly the solutions with initial conditions* $y(0) = 0$; $y(0) = 1$; $y(0) = 3$. *(c) Sketch their graphs on the graph in (a). (d) Find approximations of the form* $3 + Ae^{-Bt}$ *(for large* t*) to the solutions in (b) that approach the line* $y = 3$.

Exercise 6. *Let* $K(t)$ *be the amount of knowledge you have. Of couse when you start studying* $K(t) = 0$. *From the moment you start, the rate at which you acquire knowledge is constant, while the rate at which you forget is proportional to what you know. (a) Write down a differential equation for* $K(t)$. *(b) What is the most knowledge you will every possess? (c) At what moment will you have learned half the total?*

Exercise 7. *Find the slope field of*

$$y' = y(y^2 - 1) \quad (0 \le t \le 3, -1 \le y \le 1)$$

Find as best you can the solution with $y(0) = 2$*, and graph it in the range* $0 \le x \le 3$*.*

Exercise 8. *A certain substance* Y *is involved in a chemical reaction in which it is produced and degraded. Its concentration satisfies* $y' = \alpha - \beta y$ *with* $\alpha > 0$*,* $\beta > 0$*. In the diagram below, a solution* $y(t)$ *is graphed. (a) What are* α *and* β*? (b) Write down the formula for* $y(t)$*.*