Mathematics 103 — Spring 2000

Elementary probability

Many of the principal applications of calculus are to questions of probability and statistics. We shall include here an introduction to elementary probability, and eventually some discussion of what calculus has to do with it.

The three basic rules

Probability is assigned to certain repeatable events. The probability $P(E)$ of an event E is a measure of the relative frequency with which those events occur. It is therefore a number lying between 0 and 1. If it is equal to 0, that means the event cannot occur at all. If it is 1, that means it is certain. For example, if you toss a coin in the air, it will presumably land 'heads' exactly as often as it lands 'tails', so the probability of either event is 1/2. If you throw a single die, the probability of any one of the 6 faces coming out on top is 1/6. Even elementary events like this can be combined to make up more complicated ones, as we shall see in a moment. For example, if you throw a single die, what is the probability that the score is 3 or less?

There are three basic rules which tell us how to compute the probability of more complicated events without a great deal of trouble.

- **Complementarity.** If E is an event with probability $P(E)$, then the probability that E doesn't occur is $1 - P(E)$.
- **Sum.** If E and F are two events which cannot occur simultaneously, then the probability that either E or F occurs is $P(E) + P(F)$.
- **Product.** If E and F are two independent events, then the probability that both occur is $P(E)P(F)$.

The meaning of these will become clear soon. Even now we can make a few remarks, however. If we combine the first and second rule, we have this consequence:

- If an event has possible distinct outcomes E_1, E_2, ..., E_n then the sum of the probabilities $P(E_i)$ is 1.

Informally, you can just say that, after all, *something* has to happen! And if you have listed all the possibilities, then it has to be one of them.

As for the product rule, independent events are ones which do not affect each other at all, and the product rule should be clear. For example, if I throw two coins simultaneously a zillion times, how one of them lands will not affect how the other does. The first coin will come up heads about half a zillion times, and among the half a zillion times it comes up heads the second coin will also come up heads. This means that about one-quarter of the time they will both come up heads, and since $1/4 = (1/2)(1/2)$ this illustrates the product rule.

These rules are pretty simple, but in order to apply them correctly you will have to be careful.

Tossing coins

Let's look in detail at what happens when we toss two coins simultaneously. Being careful here means we should keep in mind that although the two coins look alike, they are really different. We could, in principle, mark them in order to make them distinguishable. Suppose we mark them as 'A' and 'B'. If we toss them both, there are altogether four possible outcomes: (1) they both come up **H**; (2) A comes up **H** and B comes up **T**; (3) A comes up **T** and B comes up **H**; (4) they both come up **T**. On grounds of symmetry, we can see that each of these occurs with the same probability, and since the sum of the probabilities of all possible outcomes must equal 1, they must each have probability 1/4.

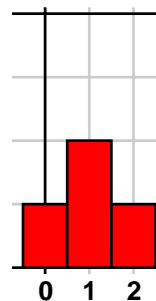| A	B	Probability
H	**H**	1/4
H	**T**	1/4
T	**H**	1/4
T	**T**	1/4

We could also have applied the product rule to see that the probability is 1/4, since the outcomes for the two coins are independent, and $1/4 = (1/2)(1/2)$.

Exercise 1. *Suppose one of the coins is* **biased**, *so that the probability of coming up heads is $p \neq 1/2$. What is the probability of getting tails? Make a table for this situation analogous to the one above.*

**Exercise 2.** *How many distinct outcomes if we throw two dice? Make a list of them all.*

In practice, when we examine the coins to see what the outcome of a toss is, we won't be able to tell the difference betwen them. The only thing that is easily observable is the total number which come up heads. Based on this observation, there are three possible **observable** outcomes: 0, 1, or 2 heads in total. What are the probabilities for these? We have to express these **compound** events in terms of the elementary ones. To get 0 heads, both coins must come up tails, so the probability is 1/4. To get 2 heads, both must be heads; probability 1/4. But to get 1 head, we can have either **HT** or **TH**. Each of these has probability 1/4, and they cannot happen at the same time, so we apply the sum rule. The probability is $1/4 + 1/4 = 1/2$. We get this table:

$n = 2$ Number of heads	Probability
0	1/4
1	1/2
2	1/4

For comparison, we can make up a similar table for the simpler case of a single coin:

$n = 1$ Number of heads	Probability
0	1/2
1	1/2
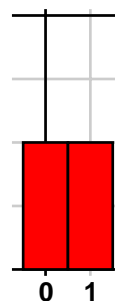
Exercise 3. *Make up a similar table and graph for a pair of coins for which the probability of heads is* 0.4.

What happens for three coins A, B, C? We can make up a list of all possible outcomes in the following way. *What happens to the third coin C is independent of what happens to the first two.* So to make a list of all possible outcomes for three coins we write down the list of outcomes for two, and then for each of these we put in a different outcome for the third. Since there were four cases for two coins, there are 8 for 3 coins. This is the list we get:

A	B	C
H	**H**	**H**
		T
H	**T**	**H**
		T
T	**H**	**H**
		T
T	**H**	**H**
		T

And as for probabilities, the product rule tells us each outcome has probability $(1/2)(1/2)(1/2) = 1/8$.

A	B	C	Probability	Number of heads
H	**H**	**H**	1/8	3
H	**H**	**T**	1/8	2
H	**T**	**H**	1/8	2
H	**T**	**T**	1/8	1
T	**H**	**H**	1/8	2
T	**H**	**T**	1/8	1
T	**T**	**H**	1/8	1
T	**T**	**T**	1/8	0

What about the probabilities here for the total number of heads? The sum rule applies again. For example, in order to get exactly two heads out of three coins, we can have **HHT**, **HTH**, or **THH**. So the probability is 3/8. Scanning the table of all outcomes, we build this one for 'observable' outcomes.

$n = 3$ Number of heads	Probability
0	1/8
1	3/8
2	3/8
3	1/8
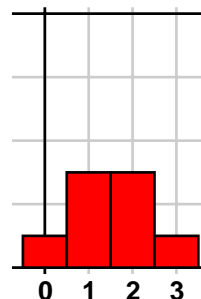
At this point, we want to understand the general rules for what happens. A few rules are very simple.

- *When we add a coin, the number of outcomes doubles.*

This is since for every way in which n coins can come up, there are 2 ways for $n + 1$ coins to come up.

- *The probability of each of these 2^n outcomes is $1/2^n$.*

In particular, for n coins the probability of 0 heads or n heads is $1/2^n$. So now what we wnat to know is how to calculate the probability of k heads where $0 < k < n$?

Let's look at $n = 4$ to see the pattern of calculation, and even more specifically let's figure out the probability of 2 heads among a toss of 4 coins. The point is this: we can break the case of 2 heads from 4 coins into two separate cases. Because one of two thing smust happen—either the fourth (new) coin comes up heads, or it comes up tails. Let's look at these cases.

(1) Suppose the fourth coin D to come up heads. In this case, if there are 2 heads in all, there must be exactly one among the first three. But we know how to count those, because we have just done it. Here is the list for this case.

A	B	C	D
H	**T**	**T**	**H**
T	**H**	**T**	**H**

      T   **T**   **H**   **H**

(2) Or, suppose coin D to come up tails. There must be 2 heads among the first three.

| A	B	C	D
T	**H**	**H**	**T**
H	**T**	**H**	**T**
H	**H**	**T**	**T**

The probability that there is 1 head among the first three is, as we have already calculated, $3/8$. The probability that the last coin is **H** is $1/2$. So the probability of the first case is $(3/8)(1/2) = 3/16$.

The probability that there are 2 head among the first three is, as we have already calculated, $3/8$. The probability that the last coin is **T** is $1/2$. So the probability of the second case is $(3/8)(1/2) = 3/16$.

By the sum rule, the probability of 2 heads from 4 coins is therefore $3/16 + 3/16 = 6/16$. The whole table for $n = 4$ is this:

$n = 4$ Number of heads	Probability
0	1/16
1	4/16
2	6/16
3	4/16
4	1/16
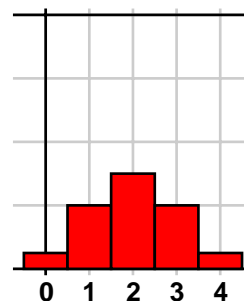
More generally, the probability of getting k heads from $n + 1$ coins, with $0 < k < n + 1$ is the sum of two other probabilities: (1) that of getting $k - 1$ heads from n coins tiomes that of getting 1 head from 1 coin; (2) that of getting k heads from n coins and 0 heads from 1 coin. This lets us generate one table from the previous one. For example, from the row for $n = 3$ we can generate that for $n = 4$:

$n = 3$	1/8	3/8	3/8	1/8	
$n = 4$	1/16	4/16	6/16	4/16	1/16

We get one entry, say $6/16$, by adding the two just above it and to the left, and multiplying by $1/2$.

$n = 4$	1/16	4/16	6/16	4/16	1/16	
$n = 5$	1/32	5/32	10/32	10/32	5/32	1/32

Exercise 4. *Build the next row, the one for $n = 6$.*

Tossing n coins

The probability of getting k heads from n coins will be of the form $m/2^n$ for some integer m. In order to figure out the formula for the probability, we need only figure out the formula for m. Here is a table of what we have calculated so far:

n	m					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

We even know the rule for generating this table, which is called Pascal's triangle. (1) The first and last numbers in each row are 1. (2) Any other number in the table is the sum of the two numbers (a) just above and (b) above and to the left of it. Using these two rules, we can, with enough time, generate any row we want.

Exercise 5. *Write down the next two rows.*

What we would like, however, is a way to generate any row without necessarilty knowing all the rows above it. Here is that rule: To write down the n-th row, (a) Start with a 1. Call this the 0-number in that row. (b) To get from the k-th number to the $k+1$-st, multiply by $n-k$ and divide by $k+1$. Here, for example, is the fourth row with the calculation sdispalyed:

n	0	1	2	3	4
4	1	$4 = (4/1)\,1$	$6 = (3/2)\,4$	$4 = (2/3)\,6$	$1 = (1/4)\,4$

We can detect the pattern of calculation. In the n-th ro wwe start with 1. The next is $(n/1)\,1 = n$; then $(n-1)/2$ times n, or $n(n-1)/2$. In the k-th column we shall get

$$n(n-1)(n-2)\ldots(n-k+1)/1(2)(3)\ldots(k) \ .$$

Exercise 6. *Apply this rule to get the 5-th row.*

Exercise 7. *Apply this rule to get the 10-th row.*

Exercise 8. *What is the probability of getting 5 heads in a toss of 10 coins? 5 heads in a toss of 11 coins?*