## Mathematics 103 — Spring 2000

## Formulas for simple sequences

Let's suppose we want to find a formula for the sum of the first $n$ squares

$$1 + 4 + 9 + \cdots + n^2 \, .$$

The first few sums, for example, are given by this table:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $n^2$ | 0 | 1 | 4 | 9 | 16 | 25 | 36 |
| sum | 0 | 1 | 5 | 14 | 30 | 55 | 81 |

certainly hard to know even where to start to find a formula for the numbers in the bottom row in terms of $n$. In fact, there is a very simple technique for finding such a formual due, I believe, to Isaac Newton. It works in very general circumstances.

First, write down a number of terms in the sequence we want to find a formula for.

$$0 \quad 1 \quad 5 \quad 14 \quad 30 \quad 55 \quad 81$$

Then write down above it its **difference sequence**.

$$\begin{array}{ccccccccccccc} & 1 & & 4 & & 9 & & 16 & & 25 & & 36 & \\ 0 & & 1 & & 5 & & 14 & & 30 & & 55 & & 81 \end{array}$$

In the case we are looking at, this is just the original sequence of squares. Then take the differences of that in turn.

$$\begin{array}{ccccccccccccc} & & 3 & & 5 & & 7 & & 9 & & 11 & & \\ & 1 & & 4 & & 9 & & 16 & & 25 & & 36 & \\ 0 & & 1 & & 5 & & 14 & & 30 & & 55 & & 81 \end{array}$$

Keep on doing this, and if you are in good circumstances, you will find a row of all zeroes. Stop there.

$$\begin{array}{ccccccccccccccc} & & & & \delta_4 = 0 & & 0 & & 0 & & 0 & & & & \\ & & & \delta_3 = 2 & & 2 & & 2 & & 2 & & 2 & & & \\ & & \delta_2 = 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & \\ & \delta_1 = 1 & & 4 & & 9 & & 16 & & 25 & & 36 & & 49 & \\ \delta_0 = 0 & & 1 & & 5 & & 14 & & 30 & & 55 & & 81 & & 130 \end{array}$$

The important numbers in this table are the ones at the far left. I shall label them by Greek $\delta$'s, starting at the bottom. Then the formula we are looking for is

$$\delta_0 \cdot 1 + \delta_1 \cdot n + \delta_2 \cdot \frac{n(n-1)}{2!} + \delta_3 \cdot \frac{n(n-1)(n-2)}{3!} + \delta_4 \cdot \frac{n(n-1)(n-2)(n-3)}{4!} + \cdots$$

Recall that $n!$ is the product of all the first $n$ integers, so that

$$1! = 1, \; 2! = 1 \cdot 2 = 2, \; 3! = 1 \cdot 2 \cdot 3 = 6, \ldots, n! = 1 \cdot 2 \cdot 3 \cdots n \, .$$

The pattern here should be pretty clear. You might have aleady noticed that the same expressions in $n$ occur also in the binomial formula. In our case the formula gives us

$$\begin{aligned} 0 \cdot 1 + 1 \cdot n + 3 \cdot \frac{n(n-1)}{2!} + 2 \cdot \frac{n(n-1)(n-2)}{3!} &= \frac{6n + 9n(n-1) + 2n(n-1)(n-2)}{6} \\ &= \frac{6n + 9n^2 - 9n + 2n^3 - 6n^2 + 4n}{6} \\ &= \frac{2n^3 + 3n^2 + n}{6} \\ &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \, . \end{aligned}$$

It is not even usually necessary to simplify the original formula, which is quite convenient to calculate with on its own. And of course, when you are through you can check easily enough whether your formula is correct.

This works for finding the sum of the first $k$-powers, for any positive $k$, because as you check easily enough, its $k$-differences are constant.

**Exercise 1.** *Find the formula for* $a_n$*:*

| | | | | | | |
|---|---|---|---|---|---|---|
| $n$: | 0 | 1 | 2 | 3 | 4 | 5 |
| $a_n$: | 1 | 5 | 11 | 19 | 29 | 41 |

**Exercise 2.** *Find the formula for* $a_n$*:*

| | | | | | | |
|---|---|---|---|---|---|---|
| $n$: | 0 | 1 | 2 | 3 | 4 | 5 |
| $a_n$: | 2 | 7 | 22 | 53 | 106 | 187 |