Mathematics 103 — Spring 2000

Pascal's triangle—Part I

Recall that

$$(a + b)^2 = a^2 + 2ab + b^2 \ .$$

It turns out to be important to be able to write down similar formulas for arbitrary powers $(a + b)^n$ with $n \geq 0$. We start off by writing down the first three:

$$(a + b)^0 = 1$$
$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$

We can continue this list if we just keep on multiplying what we have by $a + b$. For example, we can do the next by starting out

$$(a + b)^3 = (a + b)(a + b)^2 = (a + b)(a^2 + 2ab + b^2)$$

and then carrying out the multiplication explicitly. We should do this in a systematic way. To multiply something by $a + b$, first multiply it out in detail by a, and then by b, and then add the two parts together.

$$(a + b)(a^2 + 2ab + b^2) = (a^3 + 2a^2b + ab^2) + (a^2b + 2ab^2 + b^3) = a^3 + 3a^2b + 3ab^2 + b^3 \ .$$

Let's be even more systematic. After we have made the two multiplications, align the results under each other:

$$
\begin{array}{ccccccc}
a^3 & + & 2a^2b & + & ab^2 & & \\
 & & a^2b & + & 2ab^2 & + & b^3 \\
a^3 & + & 3a^2b & + & 3ab^2 & + & b^3
\end{array}
$$

We can do the same thing to find $(a + b)^4$. We get

$$(a + b)^4 = (a + b)(a + b)^3 = (a + b)(a^3 + 3a^2b + 3ab^2 + b^3) \ldots$$

and then

$$
\begin{array}{ccccccccc}
a^4 & + & 3a^3b & + & 3a^2b^2 & + & ab^3 & & \\
 & & a^3b & + & 3a^2b^2 & + & 3ab^3 & + & b^4 \\
a^4 & + & 4a^3b & + & 6a^2b^2 & + & 4ab^3 & + & b^4
\end{array}
$$

We can summarize what we have so far:

$$(a + b)^0 = 1$$
$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

We can now see something of a pattern emerging. First of all, $(a + b)^n$ is going to be sum of terms of the form $C_{m,n}a^m b^{n-m}$, where m ranges over $n, n-1, \ldots, 0$, and the coefficients $C_{m,n}$ are all positive integers. This makes $n + 1$ terms altogether. We don't really have to write down anything except the coefficients in the right order. So the essential part of the table above can be distilled into this table, which is called **Pascal's triangle**:

n					
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

The way in which we computed the rows suggests how to get from one row to the next—we **shift it and add aligned columns**. Thus the next row comes from the calculation

$$
\begin{array}{ccccccc}
1 & 4 & 6 & 4 & 1 & & \\
& 1 & 4 & 6 & 4 & 1 & \\
1 & 5 & 10 & 10 & 5 & 1 &
\end{array}
$$

And we don't even have to write this down. This rule for shift and add translates into these rules to generate all of Pascal's triangle:

(1) The whole left hand column is made up of 1.

(2) All entries but the first number in the first row are 0. (These have been invisible so far.)

(3) The number $C_{m,n}$ in column m (with $m = 0$ for the first column) and row n is the sum of the numbers (a) above it and (b) above it and to the left. In a formula

$$
C_{m,n} = C_{m-1,n-1} + C_{m,n-1}
$$

for $m > 0$, $n > 0$. For example, here is the calculation of two entries in the row for $n = 6$:

$$
\begin{array}{clccccccc}
n & & & & & & & & \\
0 & 1 & & & & & & & \\
1 & 1 & 1 & & & & & & \\
2 & 1 & 2 & 1 & & & & & \\
3 & 1 & 3 & 3 & 1 & & & & \\
4 & 1 & 4 & 6 & 4 & 1 & & & \\
5 & 1 & 5 & 10 & 10 & 5 & 1 & & \\
6 & & & 15 & & & & 1 &
\end{array}
$$

Exercise 1. *Fill in the whole row for $n = 6$.*

Furthermore, we can see from this table that the first and last non-zero entries in any row are 1, and that the second and second last in row n are equal to n. At any rate, if we have calculated the n-th row of Pascal's triangle then we can write

$$
(a + b)^n = \sum_{m=n}^{0} C_{m,n} a^m b^{n-m} = a^n + n a^{n-1} b + \cdots + n a b^{n-1} + b^n \ .
$$

We shall find later a formula for the numbers $C_{m,n}$ which does not require generating all rows successively.