Mathematics 103 — Section 103 — Spring 2000

First homework — due Friday, January 12

Exercise 1. *Write the sum*

$$1 + 16 + \cdots + k^4$$

in $\sum$ *notation. Find a formula for it.*

Exercise 2. *Find formulas for (a)* $\sum_{k=0}^{59} k$;
(b) $\sum_{k=20}^{120} k$;
(c) $\sum_{k=0}^{n} k$;
(d) $\sum_{k=0}^{n-1} (k^2 + 1)$;
(e) $\sum_{k=1}^{n} (k^2 + 1)$;
(f) $\sum_{k=0}^{n} (k^3 + k^2 - k)$;
(g) $\sum_{n=0}^{k-1} n^2$.

Exercise 3. *Use the* $\sum$ *summation notation to set up the following problems, and then compute.*

(a) Find the sum of the first 50 even numbers.

(b) Find the sum of the first 50 odd numbers.

(c) Find the sum of the first 50 integers of the form $n(n + 1)$ *(i.e. where* $n = 1, 2, 3, \ldots$ *)*

Exercise 4. *The Great Pyramid of Giza, Egypt, built around 2,720-2,560 BC by Khufu (anglicized to Cheops) has a square base. We will assume that the base has side length* 200 *meters. (Actual measurement:* 230 *m) The pyramid is made out of blocks of stone whose size is roughly* $1 \cdot 1 \cdot 0.73\text{m}^3$ *(Actual:* $1.17 \cdot 1.17 \cdot 0.73\text{m}^3$*). There are 200 layers of blocks, so that the height of the pyramid is* $200 \cdot 0.73 = 146$ *m. Assume that the size of the pyramid steps (i.e. the horizontal distance between the end of one step and the beginning of another) is* 0.5 *m. We will also assume that the pyramid is solid, i.e. we will neglect the (relatively small) spaces that make up passages and burial chambers inside the structure.*

(a) How many blocks are there in the layer that makes up the base of the pyramid? How many blocks in the second layer ?

(b) How many blocks are there at the very top of the pyramid?

(c) Write down a summation formula for the total number of blocks in the pyramid and compute the total. (Hint: you may find it easiest to start the sum from the top layer and work your way down.)

Exercise 5. *Your local produce store has a special on oranges. Their display of fruit is a triangular pyramid with 100 layers, topped with a single orange (i.e. top layer: 1). The layer second from the top has three* $(3 = 1 + 2)$ *oranges, and the one directly under it has six* $(6 = 3 + 2 + 1)$*. The same pattern continues for all* 100 *layers. (This results in efficient hexagonal packing, with each orange sitting in a little depression created by three neighbors right under it.)*

(a) How many oranges are there in the fourth and fifth layers from the top ? How many in the N*-th layer from the top ? Draw a picture of the second, third, and fourth layers.*

(b) If the pyramid of oranges only has 3 *layers, how many oranges are used in total? What if the pyramid has* 4*, or* 5 *layers?*

(c) Write down a formula for the sum of the total number of oranges that would be needed to make a pyramid with N *layers. Simplify your result so that you can use the summation formulae for* $\sum n$ *and for* $\sum n^2$ *to determine the total number of oranges in such a pyramid.*

(d) Determine how many oranges are needed for the pyramid with 100 *layers.*

Exercise 6. (**Geometric series**) *A (finite) geometric series with k terms is a series of the form*

$$1 + r + r^2 + r^3 + \cdots + r^{k-1} = \sum_{n=0}^{k-1} r^n \ .$$

The sum of this series is $S = (1-r^k)/(1-r)$ as long as $r \neq 1$. (a) If $r = 1$, what is the sum? (b) Find the sum of the series $1+2+2^2+2^3+\cdots+2^{10}$. (c) Find the sum of the series $1+(0.5)+(0.5)^2+(0.5)^3+\cdots+(0.5)^{10}$.

Exercise 7. *A branching colony of fungus starts as a single spore. The spore puts out a single branch which grows for 0.1 mm, producing a segment of that length. The tip splits into two new tips, and each grows for 0.1 mm, producing two new segments. (The total length at this point would be $0.1 + 0.2 = 0.3$ mm. Each tip splits again, and then the four tips grow for 0.1 mm. Suppose that this pattern of growth has continued for 20 generations (a generation is the time between branching events). How many tips will there be? What will be the total length of all the filaments in the fungus?*

Exercise 8. *(a) Find a formula for the area in N uniform strips of width $1/N$ both under and over the curve $y = x^4$, using the formula for the sum $\sum_1^n k^4$. The strips cover the interval $[0, 1]$. (b) Draw carefully the region $0 \leq x \leq 1$, $0 \leq y \leq x^4$, and then find its area.*

Exercise 9. *A* **right circular cone** *is the shape of an upside-down ice cream cone, circular at the bottom, say of radius r, and with the bottom perpendicular to the axis along the centre of the cone, say of height h. In this exercise you will find a formula for the volume of such a cone. (1) Make N uniform slices of the cone, each one parallel to the bottom and of height (or thickness) h/N. Inside each slice put a cylinder of the same height, as wide as possible. The radii of the circular slices varies from 0 at the top to a bit less than r at the bottom. At the left is a sideways view, with $N = 10$. Use similar triangles to answer these questions: (a) What is the radius of the smallest cylinder other than the one of radius 0? (b) Of the bottom cylinder? (2) Express the total volume of the N cylinders as a sum, and then find a simple formula for this sum. (3) As N gets larger and larger, the volume of the cylinders has as limit the volume of the cone. Explain what happens to the formula in (2) as N gets large, and tell what the volume of the cone is.*