Asymptotic series

In order to evaluate the integral of the normal distribution

$$\frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\, dt$$

we have so far explained how to compile a table of values and then interpolate if necessary for intermediate values of x. But this will not work over the entire range $[0, \infty)$, since the table must be finite. For this particular integral there is an extremely useful trick to deal with this problem.

Start off by keeping in mind that

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2}\, d1/2$$

so that since

$$\int_0^x = \int_0^\infty - \int_x^\infty$$

it suffices to calculate

$$\int_x^\infty e^{-t^2/2}\, dt\ .$$

Integration by parts asserts that

$$\int u\, dv = uv - \int v\, du\ .$$

Rewrite

$$\begin{aligned}
\int_x^\infty e^{-t^2/2}\, dt &= \int_x^\infty \left(\frac{1}{-t}\right) - te^{-t^2/2}\, dt \\
&= \int_x^\infty \left(\frac{1}{-t}\right) de^{-t^2/2} \\
&= \left[\frac{e^{-t^2/2}}{-t}\right]_x^\infty - \int_x^\infty \left(\frac{1}{t^2}\right) e^{-t^2/2}\, dt \\
&= \frac{e^{-x^2/2}}{x} - \int_x^\infty \left(\frac{1}{t^2}\right) e^{-t^2/2}\, dt
\end{aligned}$$

We can do this again (and again):

$$\begin{aligned}
\int_x^\infty \left(\frac{1}{t^2}\right) e^{-t^2/2}\, dt &= \int_x^\infty \left(\frac{1}{-t^3}\right) de^{-t^2/2} \\
&= \frac{e^{-x^2/2}}{x^3} - \int_x^\infty \left(\frac{3}{t^4}\right) e^{-t^2/2}\, dt \\
\int_x^\infty \left(\frac{3}{t^4}\right) e^{-t^2/2}\, dt &= \int_x^\infty \left(\frac{3}{-t^5}\right) de^{-t^2/2} \\
&= \frac{3\cdot 5\, e^{-x^2/2}}{x^5} - \int_x^\infty \left(\frac{3\cdot 5}{t^6}\right) e^{-t^2/2}\, dt \\
\int_x^\infty \left(\frac{3\cdot 5}{t^6}\right) e^{-t^2/2}\, dt &= \int_x^\infty \left(\frac{3\cdot 5}{-t^7}\right) de^{-t^2/2} \\
&= \frac{3\cdot 5\cdot 7\, e^{-x^2/2}}{x^7} - \int_x^\infty \left(\frac{3\cdot 5\cdot 7}{t^8}\right) e^{-t^2/2}\, dt
\end{aligned}$$

which gives us all in all

$$\begin{aligned}
\int_x^\infty e^{-t^2/2}\,dt &= \frac{e^{-x^2/2}}{x} - \int_x^\infty \left(\frac{1}{t^2}\right) e^{-t^2/2}\,dt \\
&= \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + \int_x^\infty \left(\frac{3}{t^5}\right) e^{-t^2/2}\,dt \\
&= \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + \frac{3e^{-x^2/2}}{x^5} - \int_x^\infty \left(\frac{3\cdot 5}{t^7}\right) e^{-t^2/2}\,dt \\
&= \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + \frac{3\,e^{-x^2/2}}{x^5} - \frac{3\cdot 5e^{-x^2/2}}{x^7} + \int_x^\infty \left(\frac{3\cdot 5\cdot 7}{t^9}\right) e^{-t^2/2}\,dt \\
&= \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + \frac{3\,e^{-x^2/2}}{x^5} - \frac{3\cdot 5e^{-x^2/2}}{x^7} + \frac{3\cdot 5\cdot 7e^{-x^2/2}}{x^9} - \int_x^\infty \left(\frac{3\cdot 5\cdot 7\cdot 9}{t^{11}}\right) e^{-t^2/2}\,dt
\end{aligned}$$

At first sight this may seem like a useless enterprise, since the integrals are getting more and more complicated. But the integral in each case is at least positive, so we see that

$$\begin{aligned}
\int_x^\infty e^{-t^2/2}\,dt &< \frac{e^{-x^2/2}}{x} \\
&> \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} \\
&< \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + \frac{3e^{-x^2/2}}{x^5} \\
&> \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + \frac{3\,e^{-x^2/2}}{x^5} - \frac{3\cdot 5e^{-x^2/2}}{x^7} \\
&< \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + \frac{3\,e^{-x^2/2}}{x^5} - \frac{3\cdot 5e^{-x^2/2}}{x^7} + \frac{3\cdot 5\cdot 7e^{-x^2/2}}{x^9}
\end{aligned}$$

We can phrase this in a simple way. The integral

$$\int_x^\infty e^{-t^2/2}\,dt$$

is sandwiched between any two successive finite sums from the series

$$e^{-x^2/2}\left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{3\cdot 5}{x^7} + \frac{3\cdot 5\cdot 7}{x^9} - \cdots\right)$$

For example

$$e^{-x^2/2}\left(\frac{1}{x} - \frac{1}{x^3}\right) < \int_x^\infty e^{-t^2/2}\,dt < e^{-x^2/2}\left(\frac{1}{x}\right) .$$

If $x = 4$, for example, we see that

$$e^{-4^2/2}\left(\frac{1}{4} - \frac{1}{4^3}\right) < \int_4^\infty e^{-t^2/2}\,dt < e^{-4^2/2}\left(\frac{1}{4}\right) .$$

which tells us that

$$0.000078624 < \int_4^\infty e^{-t^2/2}\,dt < 0.000083866$$

and

$$\int_4^\infty e^{-t^2/2}\,d0.00008$$

to five decimals.

It is easy to generate the terms in the series. The first term is $1/x$. Subsequent terms are of the form c_k/x^k. In going from one term to the next, multiply the numerator by $-k$, the numerator by x^2. The total factor is $-k/x^2$. It may happen that the first few terms are small, but the numerator of this factor grows indefinitely, while the denominator remains constant. Soone or later, then, the terms must start to grow larger in magnitude. In particular, the series never converges. As we have seen, however, this does not mean it is useless for computation. It does mean, however, that it aonly allows limited accuracy in computation. The most accurate calculation is obtained by adding all th terms up to the point where the magnitude of the terms grows.

Let's see how this works with $x = 4$. The terms will decrease until $k > 16$. We get

term	sum
0.25	0.25
-0.015625	0.234375
0.0029296875	0.2373046875
$-9.1552734375E\text{-}4$	0.23638916015625
$4.00543212890625E\text{-}4$	0.23678970336914062
$-2.2530555725097656E\text{-}4$	0.23656439781188965
$1.548975706100464E\text{-}4$	0.2367192953824997
$-1.258542761206627E\text{-}4$	0.23659344110637903
$1.1798838386312127E\text{-}4$	0.23671142949024215
$-1.2536265785456635E\text{-}4$	0.2365860668323876

The terms have started to grow. We conclude that

$$\int_6^\infty e^{-t^2/2}\,dt \sim e^{-8} \cdot 0.2366\text{-}0.2367 = 0.0000794$$

correct to all visible places.