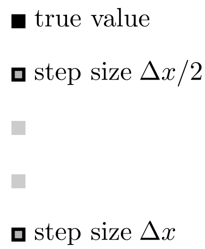
Accuracy in integrating

The trapezoid rule has an overall error of order Δx^2. This means that if you use the trapezoid rule to estimate

$$\int_a^b f(x)\,dx$$

with a reasonably small value of the step size Δx, and then you try again, say with a step size half as large, the error in the second estimate will be about 1/4 as large as that in the first. So we have the following simple mental picture:



As the picture shows, the error in the first estimate is 4/3 times the difference between the two estimates. So we have the following situation:

- *One estimate of the integral tells you nothing about how accurate it is.*
- *Two estimates with different small enough step sizes do give you a good estimate of the error in either of those estimates.*

We know roughly that the error is $C\Delta x^2$ for some constant C. Two estimates of the integral therefore enable us to estimate what C is. Let $A_{\Delta x}$ be the estimate with step size Δx. We have

$$C\Delta x^2 = (4/3)\big(A_{\Delta x/2} - A_{\Delta x}\big), \quad C = \frac{4}{3}\,\frac{A_{\Delta x/2} - A_{\Delta x}}{\Delta x^2}$$

Now suppose we would like to get an estimate of the integral to within a certain allowed error ε. To find the right step size to use we solve

$$C\Delta x^2 = \varepsilon$$

for Δx.

- *To obtain an estimate of an integral may require three different estimates in all.*

Similar reasoning applies to any of the methods of estimating an integral or solving a differential equation, except that the error will in general be proportional to some other power Δx^k.