Solving systems of differential equations

With a suitable normalization of constants, the law of gravity in the gravitational field of a planet asserts that for an object falling straight towards the planet

$$x'' = -\frac{1}{x^2}$$

where x is the distance from the centre of the planet. This is snot easily solved exactly, and the most convenient way to solve it is numerically. One can use exactly the same methods (Euler's Improved Euler's and Runge-Kutta of order 4) to solve as one used for ordinary differential equations $y' = f(t, y)$ where $y(t)$ is a function of the usual sort. The trick is to change'it into a first order differential equation in terms of a vector-valued function.

Introduce a new function $v(t)$, namely the velocity at time t. Then the equation $x'' = f(t, x)$ is equivalent to two other equations:

$$x' = v$$
$$v' = f(t, x)$$

which hold by definition of velocity and acceleration. If we consider a function $X = (t, x)$ with two coordinates then we can write these in the form

$$X' = (x', v') = F(t, X)$$

where now $F(t, X)$ has itself two coordinates, namely in this case

$$F(t, X) = (v, -1/x^2) .$$

All this means esentially just a change in notation. But this notation allows us to write down Euler's method for such first order differential equations for vector-valued functions.

We start with initial conditions $t_0 = 0$, $X_0 = X_0$, a given initial condition, in this case

$$X_0 = (x_0, v_0)$$

the initial position and velocity. We then choose a step size Δt and then find a succession of values for t and X:

$$t_{n+1} = t_n + \delta t$$
$$X_{n+1} = X_n + F(t_n, X_n)$$

until we decide to stop. In oure example, this becomes

$$t_{n+1} = t_n + \delta t$$
$$x_{n+1} = x_n + v_n \Delta t$$
$$v_{n+1} = v_n + (-1/x^2)\Delta t .$$

This is intutively clear. Position changes because the object has velocity, and velocity changes because it has acceleration.

This method has an overall error of order Δt. This means that if you want to get 6 decimals of accuracy, you have to take on the order of $1,000,000$ steps. Not very efficient. But we can also use an analogue of the Improved Euler's method.

$$t_{n+1} = t_n + \Delta t$$
$$S = F(t_n, X_n)$$
$$X_* = X_n + S\Delta t$$
$$S_* = F(t_{n+1}, X_*)$$
$$X_{n+1} = X_n + (1/2)(S + S_*)\Delta t$$

where all these are to be interpreted as vector formulas. This has overall error Δt^2.