Mathematics 220 — Fall 2000 — Bill Casselman's section

Second homework — due Friday, October 6

1. Use a variant of mathematical induction to prove that

$$1 + r + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

for all $n > 1$.

2. The Fibonacci numbers are

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 1, \quad \ldots, \quad f_{n+2} = f_{n+1} + f_n$$

Find numbers α and β, A and B such that $f_n = A\alpha^n + B\beta^n$ for all $n \geq 0$. Find α and β by subjecting them to the recursion relation $x^{n+2} = x^{n+1} + x^n$. Find A and B by considering f_0 and f_1. Use a variant of mathematical induction to prove the formula for f_n, for all $n \geq 0$.

3. Write a complete Java program `e` that has as input a single integer n and outputs the first n digits of e.

4. Write a Java program `sqrt` with two inputs a and n and outputs the first n digits of $\sqrt{a}$. Use Newton's method to do this. By experiment, answer this: roughly how many steps of Newton's method are required to do this?

5. Explain in your own words why $1.0000000\ldots$ and $0.99999999\ldots$ are the same number.

6. Explain in your own words why

$$1 + r + r^2 + \cdots = \frac{1}{1 - r}$$

if $|r| < 1$.

7. One method of solving equations is by iteration. This solves an equation $x = f(x)$ by starting with a avlue x_0 of x and setting $x_{n+1} = f(x_n)$. Explain by diagrams and words why this always works if $f(x) = x(a - x)$ where $1 < a < 4$, and x_0 is a positive number near 0. (The cases $a > 0$, $a < 0$ are different.) What is the explicit solution in this case? Roughly how many steps does it take to find n digits of the solution, if $x_0 = 1$?

8. The difference between

$$E_N = 1 + 1 + 1/2! + 1/3! + \cdots + 1/N!$$

and $(1 + 1/N)^N$ is roughly proportional to $1/N$. In fact it is equal to c_N/N where c_N converges to a limiting value as N gets large. Find an explicit formula for c_N, and an explicit formula for the limiting value. Similarly for the difference between e and $(1 + 1/N)^N$.

9. If you use your calculator to estimate e by calculating $(1 + 1/N)^N$ for large N, you will see that the estimate converges, but not to e. What does it converge to? Why is it not the same as e?