Real numbers and decimal representations

1. An informal introduction

It is likely that the reason real numbers were introduced was to make possible a numerical description of the ratios of the lengths of line segments, a task whose accomplishment seems to have escaped Greek mathematicians. It is, basically, representations of numbers with respect to a given base which makes this most feasible. The introduction of signed numbers allows further a complete one-dimensional coordinate system on a line.

Suppose then that we are given a line, with a particular pair of points on it to be designated as 0 and 1. We can introduce for each integer n a one-dimensional grid of points on the line with coordinates $10^n m$, where m is an integer. We will identify decimal fractions of this form with the points they correspond to. The closed intervals $[10^n m, 10^n (m+1)]$ then partition the line into segments overlapping only at their endpoints. The real numbers are now introduced in order to try to extend this partial coordinate system to all points of the line. We should think of the line with these finer and finer grids on it as being essentially a model for the real numbers, and try to translate our intuition about the real line into properties of the real numbers.

We can at any rate now assign to any real number (i.e. a point on our line) an infinite 'decimal expansion' in the following way. If x is negative, assign it a negative sign, and look at the point you get by reflecting it in the origin instead. So we may assume x non-negative. No matter how large x is, there will exist some power of ten, say 10^N, which is larger than x. Choose such an N, so that more precisely we now have

$$0 \le x < 10^N \ .$$

The range $[0, 10^N)$ can be partitioned into ten smaller ranges

$$[0, 10^{N-1}), \quad [10^{N-1}, 2 \cdot 10^{N-1}), \ \dots \ , \quad [9 \cdot 10^{N-1}, 10 \cdot 10^{N-1})$$

and there will then exist a unique integer a in the range $[0, 9]$ such that

$$a \, 10^{N-1} \le x < (a+1) \, 10^{N-1} \ .$$

Let this be a_{N-1}. We now have (subtracting $10^{N-1} a_{N-1}$ from both sides)

$$0 \le x_{N-1} = x - 10^{N-1} a_{N-1} < 10^{N-1}$$

and so we can apply the same idea all over again to x_{N-1} instead of x to find an integer a_{N-2} in the range $[0, 9]$ such that

$$a_{N-2} \cdot 10^{N-2} \le x_{N-1} < (a_{N-2}+1) \cdot 10^{N-2} \ .$$

Etc. In terms of the original number x we have at the i-th stage

$$0 \le x - \left(a_{N-1} 10^{N-1} + a_{N-2} 10^{N-2} + \cdots a_{N-i} 10^{N-i} \right) < 10^{N-i} \ .$$

Continuing in this way, we get a sequence of better and better approximations to x. We express this in a purely formal manner by writing

$$x = a_{N-1} 10^{N-1} + a_{N-2} 10^{N-2} + \cdots a_{N-i} 10^{N-i} + \cdots$$

or express x in decimal form as

$$x = a_N \dots a_0 . a_{-1} a_{-2} \dots a_{-n} \dots$$

even though this is to be interpreted only as a sequence of better and better approximations by finite sums. Of course unless we happen to know something very special about x, we cannot expect to describe the whole infinite sequence all at once.

We can, however, describe what we have done to find the decimal expansion of x more explicitly, and slightly more efficiently. To help in this, we first define $\lfloor x \rfloor$ to be the largest integer n with $n \leq x$, for any non-negative real number x. For example,

$$\lfloor 3.14159\ldots \rfloor = 3 \; .$$

Here is the description of the process. (1) We find N such that $x < 10^N$, and set $n = N - 1$. (2) We normalize x to by setting $x = x/10^{N-1}$.

And now we repeat this loop over and over: (3) Set $a_n = \lfloor x \rfloor$; (4) First set $x = x - a_n$, so that now $0 \leq x < 1$, and then and renormalize it by setting $x = 10\,x$; (5) Set $n = n - 1$. (6) Go back to (3).

For fractions, this procedure is quite practical. If $x = 1/3$, for example, then $N = 0$ since $10^{-1} < 1/3 < 1 = 10^0$. We start with $n = -1$, and the normalized value of x is $10/3$. Then in the first loop we set $a_{-1} = \lfloor 10/3 \rfloor = 3$ and x first equal to $10/3 - 3 = 1/3$, then to $10/3$. This is what we started with. So we just keep getting $a_i = 3$ for all $i \leq -1$. We could write this result in the conventional way as

$$1/3 = 0.333333\ldots$$

I remind you that this equation is an informal way of expressing the collection of inequalities

$$1/3 - 0.3\ldots \; n \text{ times } \ldots 3 < 1/10^n$$

for all $n \geq 0$. We shall see later on that all fractions have decimal expansions that repeat.

On the other hand, suppose we take x to be the ratio of the diagonal of a square to its side. Then by Pythagoras' Theorem we are looking for the decimal expansion of $x = \sqrt{2}$. Although the procedure described above is not directly applicable, we can figure out as many terms in the decimal expansion as we want to by the following straightforward procedure. Since we have $1^2 < 2 < 2^2$, we have

$$1 < x < 2 \; .$$

We can now calculate $1.0^2 = 1$, $1.1^2 = 1.21$, $\ldots$, $1.4^2 = 1.96$, $1.5^2 = 2.25$, $\ldots$ to see that

$$1.4 < x < 1.5 \; .$$

And so on, at each stage doing a simple search through a smaller and smaller range of numbers to get one more digit. At the next stage, for example, we see that $1.41 < x < 1.42$ since $1.41^2 = 1.9881 < 2 < 1.42^2 = 2.0164$. This procedure differs from that involved in finding the expansion of $1/3$ in several ways. First of all, there is no obvious pattern to the digits we get; in this sense we know the decimal expansion of $1/3$ in a much more explicit manner. Second, we are faced with an increasing amount of work at each step. Nonetheless, in principle we can figure out the decimal expansion of $\sqrt{2}$ to any length, given enough time to do the calculation and and space to write it down.

There is a technical matter to be dealt with now. We are going to extend the notion of decimal expansion to allow for digits of arbitrarily large order—with the extra condition that the ones of large enough order are simply zero. This is harmless, and amounts to padding a decimal expansion by an infinite number of zeroes at the left end.

I summarize: *if x is any real number, a decimal expansion of x is an infinite sequence of digits a_i such that (1) $a_i = 0$ for i large; (2) $0 \leq a_i < 10$ for all i; and (3)*

$$\left| x - \sum_{\infty}^{n} a_i 10^i \right| < 10^n$$

for all n. We have just seen how to construct, at least in principle, a decimal expansion for any real number, and indeed one with the slightly stronger property

$$0 \le x - \sum_\infty^n a_i 10^i < 10^n$$

If, conversely, we are given any sequence of decimal digits (a_i) satisfying (1) and (2) above, then intuition tells us that there should exist a real number x whose decimal expansion it is, the **limit** of the finite partial sums

$$\sum_\infty^n a_i 10^i$$

This, however, is something the ancient Greek mathematicians never understood, and might have even contested. It is a subtle point to which we shall return.

We are thus led to identify real numbers with infinite sequences of decimal digits. But this is not quite the correct thing to do. If x is a non-negative real number then at the i-th stage of the process we have described, to rephrase what we have already said, we get an inequality

$$a_{N-1}10^{N-1} + a_{N-2}10^{N-2} + \cdots a_{N-i}10^{N-i} \le x < a_{N-1}10^{N-1} + a_{N-2}10^{N-2} + \cdots a_{N-i}10^{N-i} + 10^{N-i}$$

The problem is that we don't get all sequences of digits (a_{N-i}) in this way. In partcular, we don't get those which end in an infinite string of nines. Why is that? Let's look at an example, the decimal expansion

$$0.999999\dots$$

If this were the representation found above for a number x, then we should have

$$0.9 + 0.09 + 0.009 + \cdots + 0.9 \cdot 10^{-}n \le x < 1$$

for all n. But the difference between the left and right sides is 0.1^n, which grows smaller and smaller as n gets large. Since $x < 1$, $1 - x > 0$ and hence on the one hand we can certainly find a negative power of ten less than $1 - x$, while on the other it has to be smaller than any power of ten. Contradiction.

We are thus led to ask: *Which real numbers can have more than one decimal representation?* The answer is suggested by our single example: *Any exact decimal fraction $m \, 10^n$ has two decimal expansions. Conversely, if x is a real number with more than one representation in a decimal expansion, then it has exactly two. One of these expansions is of the form*

$$x = \dots a \, 9 \, 9 \, 9 \dots$$

and the other is

$$x = \dots (a+1) \, 0 \, 0 \, 0 \dots$$

Rather than prove this, I shall adopt it as the cornerstone—a kind of axiom—of a theory of real numbers in the next section.

2. What is a real number?

Before doing this, I want to explore further the relationship between decimal expansions and numbers.

In other words, from now on a **decimal expansion** is to be an arbitrary double-ended sequence of decimal digits, i.e. a set (a_i) of decimal digits for each integer i with the property that for large i we have $a_i = 0$.

In essence, I **define** a non-negative real number to be a decimal expansion. This has to be modified by the extra stipulation that two expansions give rise to the same real number if they are of the form

$$\ldots a\,9\,9\,9 \ldots \quad \text{and} \quad \ldots (a+1)\,0\,0\,0 \ldots$$

Thus there is a slight distinction between a real number and its decimal expansion, but only in a very restricted way. So a real number amounts to either a single decimal expansion, or two of them in certain circumstances. Whether or not this definition captures excatly what is meant by a real number may not be obvious, It will in fact turn out to be in accord with both our experience and intuition.

For any decimal expansion a, let $\langle a \rangle$ be the corresponding real number we postulate to exist. We must now interpret the usual properties of real numbers in terms of properties of decimal expansions.

The first thing we must do is try to relate the notion of 'less than' in terms of decimal expansions.

The paradox of the equality

$$1.000000\ldots = 0.999999\ldots$$

is that normally we can tell the relative size of two real numbers by a simple comparison of their decimal expansions. For example, we can tell that $e < \pi$ because

$$e = 2.71828\ldots, \quad \pi = 3.14159\ldots$$

because the expansion for e starts with 2.7 and that for π starts with 3.1. But we cannot say that $0.9999\ldots < 1.0000\ldots$ even though $0 < 1$.

The point is that we have two notions of size for a decimal expansion. On the one hand, we have a notion of size for the decimal expansions themselves. Suppose a and b are two decimal expansions. We read the corresponding digits of a and b from the left until we find some place where the digits differ. This makes sense because of our convention that all digits are 0 if we go far enough to the left. At this point, suppose we are looking at the digits of order n. So we have $a_n \neq b_n$ but $a_i = b_i$ for all $i > n$. If $a_i < b_i$ we write $a \leq b$ and if $a_i > b_i$ we write $b \leq a$. Both equations $a \leq b$ and $b \leq a$ hold if in fact all their digits are the same. For example, consider $0.999999\ldots$ and $1.000000\ldots$. The first digits which differ are those of order 0. Then

$$0.999999\ldots \leq 1.000000\ldots$$

because $0 < 1$.

This is called the **lexicographic** ordering of decimal expansions, because it is the way things would be ordered in a dictionary for a language with alphabet 0123456789.

The relative size of integers, at least, is completely decided by the lexicographic order of their decimal representations.

There is another way to describe lexicographic ordering. If $a = (a_i)$ is a decimal expansion then we can consider the sequence of integers

$$A_{-n} = 10^n \sum_{\infty}^{-n} a_i 10^i$$

for all $n \geq 0$. For example if we are looking at the decimal expansion of $\pi = 3.14159\ldots$ we have

$$A_0 = 3$$
$$A_{-1} = 31$$
$$A_{-2} = 314$$
$$A_{-3} = 3141$$
$$A_{-4} = 31415$$
$$\ldots$$

The integer A_i is obtained by truncating from a all the digits of order lower than i, and then multiplying by 10^i to make the finite decimal expansion you get an integer. One infinite decimal expansion a is prior to another one, b, precisely when $A_{-n} \leq B_{-n}$ for some n.