Cauchy's criterion for convergence

1. **The definition of convergence**

The sequence x_n converges to X when this holds: for any $\epsilon > 0$ there exists K such that $|x_n - X| < \epsilon$ for all $n \geq K$.

Informally, this says that as n gets larger and larger the numbers x_n get closer and closer to X. But the definition is something you can work with precisely. In effect, the definition says that in order to show that a sequence x_n converges to X you have to explain how to get K from ϵ. It is important to realize that you do not have find the best possible value for K - just anything that works will enable you to verify convergence.

2. **Convergent subsequences**

Any bounded sequence of real numbers has a converging subsequence.

Let the sequence be x_n. We can assume all of one sign, because we can choose an infinite subseueqnce of one sign. Say non-negative. Because of boundedness, we can assume all $x_i \leq 10^k$ for some k. An infinite number of the x_i must have a first digit in common; throw away the rest. Let y_0 be the first in what is left. Among those left, an infinite number must have the same second digit. Let y_1 be the first among these. Throw away the rest. Etc. Then after the first n all the y_i will have the same first n digits. We can construct a particular real number to be that number Y whose first n digits agree with those of y_n. The difference between y_n and Y will be at most 10^{k-n}. This makes it easy to verify the convegence to Y.

3. **What this means for series**

A series is a formal expression

$$x_0 + x_1 + \cdots$$

and it corresponds to the sequence of **partial sums**

$$s_1 = x_0, \quad s_2 = x_0 + x_1, \quad s_3 = x_0 + x_1 + x_2, \ldots$$

The series is said to converge if the sequence of partial sums s_i converges. *If a series converges, then its individual terms must have limit* 0, *but this is not a sufficient condition for convergence.* Applying the definition literally, we see that the series converges to the number S if for any ϵ there exists K such that

$$|S_n - S| = |x_0 + x_1 + \cdots + x_{n-1} - S| < \epsilon$$

whenever $n > K$. We'll see how this works in practice in the next section.

4. **The geometric series**

Fix x with $|x| < 1$, and consider the series

$$1 + x + x^2 + \cdots .$$

Algebra tells us that the n-th partial sum is

$$S_n = 1 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

Intuitively, this should converge to $S = 1/(1-x)$ as n gets larger. Let's try to verify the definition in this case.

We have explicitly

$$S - S_n = \frac{1}{1-x} - \frac{1-x^n}{1-x} = \frac{x^n}{1-x}$$

So now we have to verify that for any $\epsilon > 0$ there exists K such that

$$\left|\frac{x^n}{1-x}\right| < \epsilon \text{ or } x^n < (1-x)\epsilon$$

if $n > K$. But we can practically take as given in this course that this is so, or in other words that if $|x| < 1$ then the sequence x^n converges to 0. Explicitly, we can solve

$$x^K = (1-x)\epsilon, \quad K = \frac{\ln(\epsilon(1-x))}{\ln(x)}$$

If $i \geq K$ then $|x|^i \leq |x|^K$, and hence $x^i \leq (1-x)\epsilon$.

So in this case we can use the definition to prove directly that the geometric series with $|x| < 1$ converges to $1/(1-x)$. It is rare to know exactly whjat a series converges to. The geometric series plays a crucial role in the subject for this and other reasons.

5. **Cauchy's criterion**

The definition of convergence refers to the number X to which the sequence converges. But it is rare to know explicitly what a series converges to. In fact, the whole point of series is often that they converge to something interesting which you might not know how to describe otherwise. For example, it is essentially the definition of e that it is the number to which the series

$$1 + 1 + 1/2 + 1/3! + \cdots$$

converges.

Therefore what is needed is a criterion for convergence which is internal to the sequence (as opposed to external).

Cauchy's criterion. *The sequence x_n converges to something if and only if this holds: for every $\epsilon > 0$ there exists K such that $|x_n - x_m| < \epsilon$ whenever n, $m > K$.*

This is necessary and sufficient.

To prove one implication: Suppose the sequence x_n converges, say to X. Then by definition, for every $\epsilon > 0$ we can find K such that $|X - x_n| < \epsilon$ whenever $n \geq K$. But then if we are given $\epsilon > 0$ we can find K such that $|X - x_n| < \epsilon/2$ for $n \geq K$, and then

$$|x_n - x_m| = |(x_n - X) - (x_m - X)| < |x_n - X| + |x_m - X| < \epsilon/2 + \epsilon/2 = \epsilon$$

for m, $n \geq K$.

To prove the other: Suppose the criterion holds. We know that we have a subsequence x_{n_i} which converges to some X. I claim that in fact the whole sequence converges to this same X. We know that for any $\epsilon > 0$ we can find K_1 such that $|x_{n_i} - X| < \epsilon$ for $i \geq K_1$. We also know that if we are given $\epsilon > 0$ we can find K_2 such that $|x_n - x_m| < \epsilon$ for n, $m \geq K_2$.

Now we want to prove that for any $\epsilon > 0$ we can find K such that $|x_n - X| < \epsilon$ for $n \geq K$.

First choose K_1 such that $|X - x_{n_i}| < \epsilon/2$ for $i \geq K_1$. Second, choose K_2 such that

$$|x_n - x_m| < \epsilon/2$$

for m, $n \geq K_2$. Suppose $n \geq K_2$. Choose some x_{n_i} with both $n_i \geq K_2$ and $i \geq K_1$. Then

$$|x_n - X| = |(x_n - x_{n_i}) + (x_{n_i} - X)| \leq |x_n - x_{n_i}| + |x_{n_i} - X| < \epsilon/2 + \epsilon/2 = \epsilon\ .$$

6. **Convergence by comparison**

Theorem. *If the series of non-negative terms*

$$x_0 + x_1 + x_2 + \cdots$$

converges and $|y_i| \leq x_i$ for each i, then the series

$$y_0 + y_1 + y_2 + \cdots$$

converges also.

Suppose we are given $\epsilon > 0$. By Cauchy's criterion, we know that we can find K such that

$$|x_m + x_{m+1} + \cdots + x_{n-1}| < \epsilon$$

for $K \leq m < n$. But then for the same K

$$|y_m + y_{m+1} + \cdots + y_{n-1}| \leq x_m + x_{m+1} + \cdots + x_{n-1} < \epsilon$$

Because of this

Lemma. (Cauchy's inequality) *We have*

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + |a_3| + \cdots + |a_n|$$

Prove this by induction, starting with 2. We have already used this inequality with two terms in a previous section.

Corollary. *If the series*

$$|x_0| + |x_1| + \cdots$$

converges, then so does

$$x_0 + x_1 + \cdots$$

The series is said to converge **absolutely** if the series

$$|x_0| + |x_1| + \cdots$$

converges.

Corollary. *If $|x_i| \leq Cq^i$ for all i, where $0 \leq q < 1$, then the series*

$$x_0 + x_1 + \cdots$$

converges.

7. **A new example**

Let's now look at the series

$$x + 2x^2 + 3x^3 + \cdots\ .$$

I claim that it converges for all x with $|x| < 1$.

None of the techniques mentioned so far apply directly to it. It looks something like the geometric series, but the coefficient of x^n grows with n instead of remaining bounded. We need a new idea to deal with it.

It is true that the coefficients grow with n, but they don't grow very fast. They form an arithmetic progression, and in particular

Theorem. *The sequence n grows less slowly than any geometric sequence r^n if $r > 1$.*

This is not immediately apparent. If r is close to 1 then the geometric sequence starts out slowly, perhaps very slowly. Nonetheless sooner or later it surpasses the arithmetic progression. This is not quite as easy to see as one might like, since it is not easy to specify the smallest n such that $r^n > n$.

Lemma. *If $r > 1$ then for some N we have $r^n > n$ whenever $n \geq N$.*

Let $r = 1 + x$, and choose M large enough that $Mx > N - 1$. Then by the binomial theorem, for $n \geq M$

$$r^n = (1+x)^n = 1 + nx + \text{ positive terms } > 1 + Mx > 1 + (N-1) = N \ .$$

We can now use this to see that the series

$$x + 2x^2 + 3x^3 + \cdots$$

converges for $|x| < 1$. Since $|x| < 1$, we can find $r > 1$ such that $r|x| < 1$ also (say $r = \sqrt{1/|x|}$). We can then find N such that $r^n > n$ for all $n \geq N$. This implies that for $n \geq N$ we have

$$n|x|^n < r^n|x|^n \ .$$

But then after the first N terms the series $\sum nx^n$ is dominated by the geometric series for $r|x|$, hence converges

8. **The ratio test**

The same argument used for one series in the previous section can be applied to prove this as well:

Theorem. (The ratio test) *Suppose*

$$x_0 + x_1 + x_2 + \ldots$$

is a series such that the limit of $|x_{n+1}/x_n|$ is less than 1. Then the series converges.

This will show, for example, that the series

$$x + 4x^2 + 9x^3 + \cdots$$

converges for $|x| < 1$.

To test yourself: *How many terms of this series are required to compute its limit to within 10^{-100}?*

9. **Power series**

Another related result is this:

Theorem. *Suppose the series*

$$\sum c_n y^n$$

to converge. Then so do all series

$$\sum c_n x^n$$

with $|x| < y$.