Mathematics 220 — Solutions to final examination for December 19, 2000

1. *(a) State precisely what it means for the sequence y_n to converge to a number y.*

(b) State precisely what it means for the series

$$y_1 + y_2 + y_3 + \cdots$$

to converge.

(c) Prove directly from these definitions that if the series

$$y_1 + y_2 + y_3 + \cdots$$

converges then the terms y_i converge to 0.

(a) The **sequence** y_i converges to y when for any $\epsilon > 0$ we can find N with the property that for all $n > N$ we have $|y_i - y| < \epsilon$.

(b) The **series** converges when the **sequence** of partial sums

$$s_1 = y_1, \quad s_2 = y_1 + y_2, \quad s_3 = y_1 + y_2 + y_3, \quad \ldots$$

converges.

(c) If the series converges to say y, then for any ϵ we can find N (depending on ϵ) with the property that $|s_i - y| < \epsilon$ for all $i > N$. But then by Cauchy's inequality (see later) for $i > N$

$$|y_i| = |s_i - s_{i-1}| \le |s_i - y| + |y - s_{i-1}| < 2\epsilon \ .$$

So we choose N for the original series suitable for $\epsilon/2$.

2. *Give a complete proof that if $|x| < 1$ then the series*

$$1 + x + x^2 + \cdots$$

converges to $1/(1-x)$.

Look at the course notes.

3. *(a) What is the output from the following program?*

```
public class t {
    static int y = 13;
    public static void main(String[] arg) {
        int[] a = {11, 9, 12};
        int[] b = {10, 2};
        int[] c = mystery(a, b);
        for (int i=0;i<c.length;i++) {
            System.out.print(c[i] + ":");
        }
    }

    public static int[] mystery(int a[], int[] b) {
        int[] c = new int[a.length+b.length];
        for (int i=0;i<b.length;i++) {
            int z = 0;
            int d = b[i];
```

```
            for (int j = 0;j<a.length;j++) {
                int x = c[i+j] + d*a[j] + z;
                c[i+j] = x % y;
                z = x/y;
            }
            c[i+a.length] = z;
        }
        return(c);
    }
}
```

(b) Explain in your own words what this program is really doing.

(c) There is something not quite right about the `mystery` *routine. What problem might occur?*

(a) `6:3:4:9:2`.

(b) It is multiplying numbers expressed in terms of 'digit' arrays in base 13, written as usual in low to high order.

(c) If `b` has length 0 then `c` will have length the same as `a`, and in the line `c[i+a.length] = z;` we will get a range error.

4. *State and prove Cauchy's inequality in the most general form we have seen in this course.*

The inequality states that

$$|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|$$

for all $n \ge 1$.

For $n = 1$ this is tautological. For $n = 2$ we prove it by cases according to the signs of a_1 and a_2. For $n > 2$ we prove it by mathematical induction, using the case of $n = 2$, since then

$$\begin{aligned} |a_1 + a_2 + \cdots + a_n| &\le |a_1 + a_2 + \cdots + a_{n-1}| + |a_n| \quad \text{(case } n = 2\text{)} \\ &\le |a_1| + |a_2| + \cdots + |a_n| \quad \text{(inductive assumption)} \end{aligned}$$

5. *State Cauchy's criterion for convergence. Use it to give a complete proof that if the series*

$$y_1 + y_2 + \cdots$$

converges and $|x_i| \le y_i$ for all i, then so does the series

$$x_1 + x_2 + \cdots$$

See the notes, from which this is taken directly.

6. *(a) Prove using Cauchy's criterion directly that the series*

$$x + 8x^2 + 27x^3 + 64x^4 + \cdots$$

converges for $|x| < 1$.

(b) Make a good estimate of how many terms of this series are required to compute its value to within an accuracy of 10^{-100}, when $x = 0.9$?

(a) If $y_n = n^3 x^n$ then

$$\rho_n = \frac{y_{n+1}}{y_n} = \left(\frac{n+1}{n}\right)^3 x$$

which has limit x as n grows large, and in particular we can find r with $x < r < 1$ such that $\rho_n < r$ for large n, say for $n \geq N$. Then for all $n \geq N$ we have

$$y_n < y_N r^{n-N} .$$

Comparison with the geometric series for r guarantees convergence.

(b) We want

$$n^3 x^n + (n+1)^3 x^{n+1} + \cdots < 10^{-100} .$$

We can rewrite and estimate this as

$$\begin{aligned} n^3 x^n &\left(1 + \left(\frac{n+1}{n}\right)^3 x + \left(\frac{n+1}{n}\right)^3 \left(\frac{n+2}{n+1}\right)^3 x^2 + \cdots \right) \\ &< n^3 x^n \left(1 + \left(\frac{n+1}{n}\right)^3 x + \left(\frac{n+1}{n}\right)^6 x^2 + \cdots \right) \\ &= n^3 x^n \left(\frac{1}{1-(1+1/n)^3 x}\right) \end{aligned}$$

with $x = 0.9$. We want to choose n so

$$n^3 x^n \left(\frac{1}{1-(1+1/n)^3 x}\right) < 10^{-100}$$

The integer n will be pretty large, so this term will be approximately

$$n^3 x^n \left(\frac{1}{1-x}\right)$$

and even more roughly

$$x^n \left(\frac{1}{1-x}\right)$$

so we solve

$$(0.9)^n = 10^{-101}, \quad n = -101 \ln(10)/\ln(0.9) = 2207 .$$

The true n must be a bit larger. Trial and error gives $n = 2430$.

7. *Suppose $y > 0$. Prove that if the series*

$$\sum c_i y^i$$

converges then so does every series

$$\sum_i c_i x^i$$

with $|x| < y$.

This is tricky, because although $y > 0$, the c_i could be negative. Therefore you cannot say that $|c_i x^i| \leq c_i y^i$, and cannot use a direct comparison. A really different idea is needed.

From an earlier question, we know that $c_i y^i$ converges to 0. In particular, it is bounded: we can find C such that $c_i y^i \leq C$ for all i. But then

$$c_i x^i = c_i y^i \left(\frac{x}{y}\right)^i \leq C \left(\frac{x}{y}\right)^i$$

so that convergence follows by comparison with the geometric series Cr^i with $r = x/y$.

8. *(a) Prove in as much detail as you can that if* $0 < r < 1$ *then for all large values of* n

$$r^n < \frac{1}{n}\ .$$

(b) Same, replacing $1/n$ *by* $1/n^2$.

(a) There are lots of ways to see this. (i) Let $x_n = nr^n$. Then

$$\rho_n = \frac{x_{n+1}}{x_n} = \left(\frac{n+1}{n}\right) r$$

and since $r < 1$, this ratio is less than 1 for large enough n, say less tahn $\rho < 1$ for $n \geq N$. So $nr^n = Nr^N \rho_N \rho_{N+1} \ldots \rho_{n-1} < Nr^N \rho^{n-N}$ for $n \geq N$. But these terms definitely converge to 0.

(ii) If the terms $1/r^n$ are not eventually less than $1/n$ then for arbitrarily large n we'll have $1/r^n \geq 1/n$. This implies that $1/r^n \geq 1/n$ for all large enough n. But the series $\sum r^n$ converges while the series $\sum 1/n$ does not, so this leads to a contradiction.

(b) But if you have proven the first part, the second is immediate. The inequality $r^n < 1/n^2$ is equivalent to $r^{n/2} < 1/n$, taking square roots. If $0 < r < 1$ then so is $0 < r^{1/2} < 1$, so the second case follows directly from the first.