Mathematics 220 — Fall 2000 — Bill Casselman's section

Study guide for the mid-term examination on Monday, October 23

The examination questions will be very similar to some of these, with the exception of one question which will exhibit a fragment of Java program and ask you to tell what it does.

1. Find constants a_3, a_2, a_1, a_0 such that

$$1 + 2^2 + 3^2 + \cdots + n^2 = a_3 n^3 + a_2 n^2 + a_1 n + a_0$$

for all integers $n >= 0$. Use mathematical induction to prove that your formula is correct.

2. Explain why the ratio of successive Fibonacci numbers f_{n+1}/f_n approaches a constant value as n gets large.

3. Find a series in powers of x for

$$\frac{1}{1+x} \; .$$

For which values of x does this series converge? Use it to find by a formal calculation a likely series expression for

$$\log_e(1+x) = \int_0^x \frac{1}{1+u}\, du \; .$$

For which values of x does this series converge?

4. One special case of this series asserts that

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

This series converges to a definite value. How many terms do you need to estimate the value of $\log_e 2$ correctly to 3 decimal places?

5. Use the integral

$$\int \frac{1}{1+x^2}\, dx$$

to find a convergent series for $\pi/6$. About how many terms of this series are required to calculate $\pi/6$ correctly to 100 figures?

6. Explain why the series

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} + \cdots$$

converges for $k \geq 2$.

7. Suppose you want to divide $a_{n-1}a_{n-2}\ldots a_0$ into $b_n b_{n-1}\ldots b_0$ (these are both meant to be decimal representations), which you may assume to give you a one-digit quotient q. Suppose that $\hat{q}$ is the maximum of 9 or the quotient $b_n b_{n-1}/a_n$. Explain why $q \leq \hat{q}$.

8. As we know from the example

$$1.00000\ldots = 0.9999999\ldots \; ,$$

some numbers may have more than one decimal representation. For any for any number $x \geq 0$, however, there will be exactly one representation of the form

$$x = \sum_n^{-\infty} a_i 10^i$$

with the property that for all m

$$\sum_n^{n-m} a_i 10^i \leq x < \sum_n^{n-m} a_i 10^i + 10^{n-m} \; .$$

Prove this (not hard to describe how to find it more or less explicitly) and show that no such representation will end with an infinite string $\ldots 9999999 \ldots$

9. Explain why the only numbers with more than one decimal representation are those with a representation ending in a string of nines.