Mathematics 220 — Fall 2000 — Bill Casselman's section

Study guide for the second mid-term examination on Wednesday, November 8

The examination questions will be very similar to some of these, with the exception of one question which will exhibit a fragment of Java program and ask you to tell what it does.

1. Find constants a_3, a_2, a_1, a_0 such that

$$1 + 2^2 + 3^2 + \cdots + n^2 = a_3 n^3 + a_2 n^2 + a_1 n + a_0$$

for all integers $n >= 0$. Use mathematical induction to prove that your formula is correct.

2. Explain in detail why the ratio of successive Fibonacci numbers f_{n+1}/f_n approaches a constant value as n gets large.

3. The derivative of $1/\ln(\ln(\ln(x)))$ is

$$\frac{1}{x \ln(x) \ln(\ln(x))(\ln(\ln(\ln(x))))^2} .$$

Explain why the series

$$\sum_N^\infty \frac{1}{n \ln(n) \ln(\ln(n))(\ln(\ln(\ln(n))))^2}$$

is convergent, where N is a suitably large fixed integer. Why does N have to be large? What is the least it can be? How many terms of this series are required to calculate its limit to with 10^3?

4. Suppose you want to divide $a_{n-1}a_{n-2}\ldots a_0$ into $b_n b_{n-1}\ldots b_0$ (these are both meant to be decimal representations), which you may assume to give you a one-digit quotient q. Suppose that $\hat{q}$ is the minimum of 9 or the quotient $b_n b_{n-1}/a_{n-1}$. Explain why $q \le \hat{q}$; why $\hat{q} - 2 \le q \le \hat{q}$ if $a_{n-1} \ge 5$.

5. Prove in detail, using only things that have been proven in class, that the series

$$f(x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$$

converges for all x with $|x| < 1$. Give a fairly accurate way of calculating how many terms are required to calculate the answer to within 10^{-k}. Prove that the sum of the series for two numbers x_1 and x_2 is the same as the series for the single number $x_3 = x_1 + x_2 + x_1 x_2$, as long as $|x_i| < 1$ for all i.

6. Explain why the expression

$$1 + 1/2 + 1/3 + 1/4 + \cdots + 1/(n-1) - \ln(n)$$

approaches a limit as n gets larger and larger.

7. We know that if we add two integers from left to right we cannot determine all the digits of the sum at a constant rate as we go; we must always expect to back-track. Show that this is not the case if we want to multiply an integer by 2, going from left to right. That is to say, describe a procedure that reads an integer n from left to right and calculates the digits of $2n$ regularly as it goes along.