Real numbers

This course is about real numbers. Real numbers measure things. They are essentially a modern invention, but thy have much in common with aspects of mathematics in two much earlier cultures. The Greeks had an arithmetic of geometric segments, but were somewhat shy of relating intervals to numbers in any direct way. They worked with ratios of objects which were in a more or less precise way comparable—two lengths or two areas or two weights, for example—but do not seem to have thought of a ratio itself as an independent entity, and it is not clear to me that they thought of ratios of, say, lengths as essentially the same kind of ratio as one of areas. They did have an extremely sophisticated theory of magnitudes, which their concern for rigour in mathematical reasoning apparently forced them to construct in the face of irrational ratios, such as that of the diagonal of a square to a side. What they had were operational criteria, due to the extremely clever mathematician Eudoxus, for comparing two ratios. The only ratios they somehow thought of explicitly seem been those which were **constructible** in a somewhat restricted fashion—those which result from ruler and compass constructions. They were very, very uneasy about dealing with infinite processes of any kind. We do not know exactly why this was, and in the modern mathematical environment it is difficult to recover their feelings.

On the other hand, the Babylonians, in astronomical calculations going back to about 500 B.C., had developed an extremely practical perfectly good system of floating point notation, using 60 as base. This was adopted by Greek astronomers ina bout 100 B.C., who were themselves very mathematical in what they did, but these practical techniques seem to have almost no effect on the theoretical mathematics of the period, although the two subjects overlapped significantly in the Greek beginnings of trigonometry.

It is likely that the development of decimal notation by the Hindus, that is to say the arithmetic of integers based on 10, was not entirely independent of the Babylonian and Greek astronomical calculation schemes. It was introduced to Europe at different times, every time as far as I know through Arabic intermediaries. In the late Middle Ages it slowly took over from older schemes of calculation based on the abacus. However, it wasn't until the late sixteenth century that decimal notation was extended to real numbers in general, wth the invention of the decimal point by the Dutch mathematician Simon Stevin. His notation, with a few modifications, was adopted in the course of the next half century. Usage was varied. Galileo is not known ever to have used the new notation; all known calculations by him are with integers, and he used Euclid as a model for his theoretical discussions of physics. His contemporary Kepler, on the other hand, was a master of calculation with the new techniques.

Thus mathematicians of the sevententh century were familiar with modern decimal notation, and developed extremely efficient ways to calculate with them, but real numbers played little role in their theory. It wasn't until the mid-nineteenth century that this step occurred, setting in motion the subject of modern analysis and setting also the style and language which is used most often nowadays to discuss numbers in theoretical mathematics. As the nineteength century passed, mathematicians became more and more confident about dealing with infinite processes, ideas introduced by d'Alembert and Cauchy ultimately arriving at Cantor's theory of sets and Dedekind's free use of axioms, Dedekind and others established conventions for rigourous argument which served to guide mathematical exploration through the tricky terrain that they encountered. This eventually led to the extremely abstract and beautiful theories of Lebesgue and other analysts of the twentieth century, and also to a certain amount of confusion about the real nature of mathematical reasoning when that reasoning itself became a focus for reserach.

In this course, modern conventions will be thrown out, to some extent. I want to reduce the level of abstraction usually required to understand analysis. In doing this I shall be motivated by considerations arising in the development of algorithms and the use of computers in the late twentieth century. It is hard to understand, but some mathematicians, will find this an unwelcome notion.

In this course, roughly speaking, *a real number will be identified with its decimal expansion*. That is to say, with one necessary proviso I shall explain in a moment, I shall in fact define a real number to be its decimal expansion - an infinite sequence of decimal digits $a_m a_{m-1} \dots$ for some set of indices starting at an integer m and going off to $-\infty$. Informally, this is to be identified with the infinite sum

$$a_m 10^m + a_{m-1} 10^{m-1} + \cdots$$

but in reality we do not assume this sum to have any meaning other than the sequence itself. Conventionally, I shall assign $a_i = 0$ for i large. Once again: a real number **is** its decimal expansion, nothing more and nothing less. Non-negative ones, anyway. A negative number with be represented as the negative of a positive one.

Well, almost. I have mentioned that we can't quite identify real numbers with their decimal expansions, and of course this is well known. The problem is that some numbers do not have a unique decimal expansion—we are all taught early that

$$1.00000\ldots = 0.9999999\ldots$$

so we have two distinct ways of representing the number 1. We must somehow allow for this non-uniqueness—every decimal expansion will certainly give rise to a real number, but we need some criterion for deciding when two decimal expansions correspond to the same real numbr. I could do this by fiat, simply declaring that decimal expansions ending in an infinite string of nines represent the same real number as some other one ending with an infinite string of zeroes. But I find this intellectually unsatisfying. It does not seem to me immediately apparent, although it is true, that these are the only examples of non-unique expansion. So I will deal with this equivalence in a slightly less direct fashion.

1. What is a real number?.

The non-negative real numbers are to be the collection of decimal expansions, up to some equivalence relation to be described shortly. We assume for convenience that the expansions go both ways, but that the high order digits eventually all vanish. We put an ordering $\le$ on decimal expansions, the lexicographic one (also called dictionary order). Thus we say that

$$(a_i) \le (b_i)$$

if for the largest index k where $a_k \ne b_k$ we have $a_k < b_k$. In other words, to tell if $a \le b$ we read their matching digits from left to right stopping when we arrive at digits which sre not the same. Suppose we stop at index i. Then $a \le b$ if $a_i < b_i$, otherwise $b \le a$.

Thus

$$0.99999\ldots 1.00000\ldots$$

because the first digit they disagree at has index 0, where $0 < 1$.

Of course if $a = b$ then there is no such largest index, and there is nothing to worry about. For example, the string of zeroes in both directions is the smallest possible non-negative of all these expansions.

We can define the sum of two decimal expansions according to the usual rules of arithmetic. We can also define their difference if one is less than or equal to the other. This is to be taken as a very formal definition, combining two sequences to obtain another sequence, but oif course it is motivated by the interpretation of real numbers in terms of magnitudes. The operations of addition and subtraction of decimal expansions requires adding and subtracting from the left, in some sense, contrary to what we usually do.

We say that if a and b are **equivalent decimal expansions**, and write $a \cong b$, if $a \le b + \epsilon$ and $b \le a + \epsilon$ for all positive ϵ, or equivalently for all ϵ of the form 10^{-n}.

• **Theorem.** The only equivalences generated in this way are from infinite strings of nines.

What is the exact statement? We asume Suppose we have two distinct decimal expansions a and b. Swapping if necessary, we may assume $a \le b$. The hypothesis in these circumstances means that $b \le a + 10^{-n}$ for all n. What is the conclusion? Say the first digit the two expansions disagree in is the i-th, so that $a_i < b_i$. I claim that $b_i = a_i + 1$; that $a_j = 9$ for $j < i$; and that $b_j = 0$ for $j < i$.