Mathematics 255—January 11, 1996

First homework solutions

Exercise 1. *Use a spread sheet and the numerical method suggested in the notes to calculate and graph the solution of the differential equation plus initial conditions in the range indicated:*

$$y' = xy + 1, \quad y(0) = 1, \quad 0 \le x \le 1$$

Use intervals $\Delta x = 0.5, 0.25, 0.125, 0.1$.

0.00000000	1.00000000
0.50000000	1.50000000
1.00000000	2.37500000

0.00000000	1.00000000
0.25000000	1.25000000
0.50000000	1.57812500
0.75000000	2.02539062
1.00000000	2.65515137

0.00000000	1.00000000
0.12500000	1.12500000
0.25000000	1.26757812
0.37500000	1.43218994
0.50000000	1.62432384
0.62500000	1.85084409
0.75000000	2.12044128
0.87500000	2.44423265
1.00000000	2.83657060

0.00000000	1.00000000
0.10000000	1.10000000
0.20000000	1.21100000
0.30000000	1.33522000
0.40000000	1.47527660
0.50000000	1.63428766
0.60000000	1.81600205
0.70000000	2.02496217
0.80000000	2.26670952
0.90000000	2.54804628
1.00000000	2.87737045

Exercise 2. *The same for the equation*

$$y' = x^2 + y^2, \quad y(0) = 1$$

in as much of the range $[0, 1]$ *as you can.*

0.00000000	1.00000000
0.50000000	1.50000000
1.00000000	2.75000000

0.00000000	1.00000000
0.25000000	1.25000000
0.50000000	1.65625000
0.75000000	2.40454102
1.00000000	3.99062039

0.00000000	1.00000000
0.12500000	1.12500000
0.25000000	1.28515625
0.37500000	1.49942207
0.50000000	1.79803352
0.62500000	2.23339908
0.75000000	2.90573614
0.87500000	4.03146146
1.00000000	6.15874977

0.00000000	1.00000000
0.10000000	1.10000000
0.20000000	1.22200000
0.30000000	1.37532840
0.40000000	1.57348122
0.50000000	1.83706554
0.60000000	2.19954651
0.70000000	2.71934700
0.80000000	3.50783181
0.90000000	4.80232022
1.00000000	7.18954816

Exercise 3. *Find the general solutions of the equations*

$$\begin{aligned} y' &= y + x \\ y' &= y + e^x \\ y' &= 2y + \sin x \\ y' &= y^2 \end{aligned}$$

(a) The formula gives

$$\begin{aligned} y &= Ce^x + e^x \int^x e^{-t} t \, dt \\ &= Ce^{-x} + \left[-te^{-t} - e^{-t}\right]^x \\ &= Ce^x + (-x - 1) \end{aligned}$$

(b) The formula gives

$$\begin{aligned} y &= Ce^x + e^x \int^x e^{-t} e^t \, dt \\ &= Ce^x + xe^x \end{aligned}$$

(c) The formula gives

$$\begin{aligned} y &= Ce^{2x} + e^{2x} \int^x e^{-2t} \sin t \, dt \\ &= Ce^{2x} + \frac{-\cos x - 2\sin x}{5} \end{aligned}$$

(Refer to the notes on complex exponentials for how to find

$$\int e^{ax}\cos bx\,dx, \quad \int e^{ax}\sin bx\,dx$$

easily.)

(d) This was done in class. It is separable.

$$\begin{aligned} \frac{dy}{y^2} &= dx \\ -\frac{dy}{y} &= x + c \\ y &= -\frac{1}{x+c} \end{aligned}$$

Exercise 4. *(a) Write down the differential equation for a tank of water, draining at the bottom through a hole of effective area* 1cm *and of varying of cross-sectional area* $h^2\,\mathrm{cm}^2$ *at height* h *(in other words, a tank the shape of a coffee filter). Assume other data as in the example discussed in class.*

(b) Find the equilibrium height if it exists, or if not tell why it doesn't exist.

(c) Write down the linear equation approximating yours near equilibrium height.

(d) Plot the slope field for heights from 0 *to equilibrium,* $t = 0$ *to* 20 *seconds.*

(e) Tell whether it is a separable equation or not. If it is, find an implicit equation for $h(t)$.

The basic equation is the same as for the example in class.

$$\frac{dV}{dt} = 200 - \sqrt{1960\,h}$$

What makes this different is that here the formula for V in terms of h is different:

$$V = \frac{h^3}{3}, \quad \frac{dV}{dt} = h^2\,\frac{dh}{dt}$$

Therefore we get

$$h' = \frac{200 - \sqrt{1960\,h}}{h^2}$$

(b) Of course the equilibrium is exactly what it was in the example in class.

(c) Let

$$F(h) = \frac{200 - \sqrt{1960\,h}}{h^2}$$

Then

$$\begin{aligned} F'(h) &= \frac{-\left(\dfrac{\sqrt{1960}}{2\sqrt{h}}\right)h^2 - 2h\left(200 - \sqrt{1960h}\right)}{h^4} \\ F'(h_{\mathrm{eq}}) &= -\left(\frac{\sqrt{1960}}{2h_{\mathrm{eq}}^2\sqrt{h_{\mathrm{eq}}}}\right) \\ &= \text{(say) } \Phi \end{aligned}$$

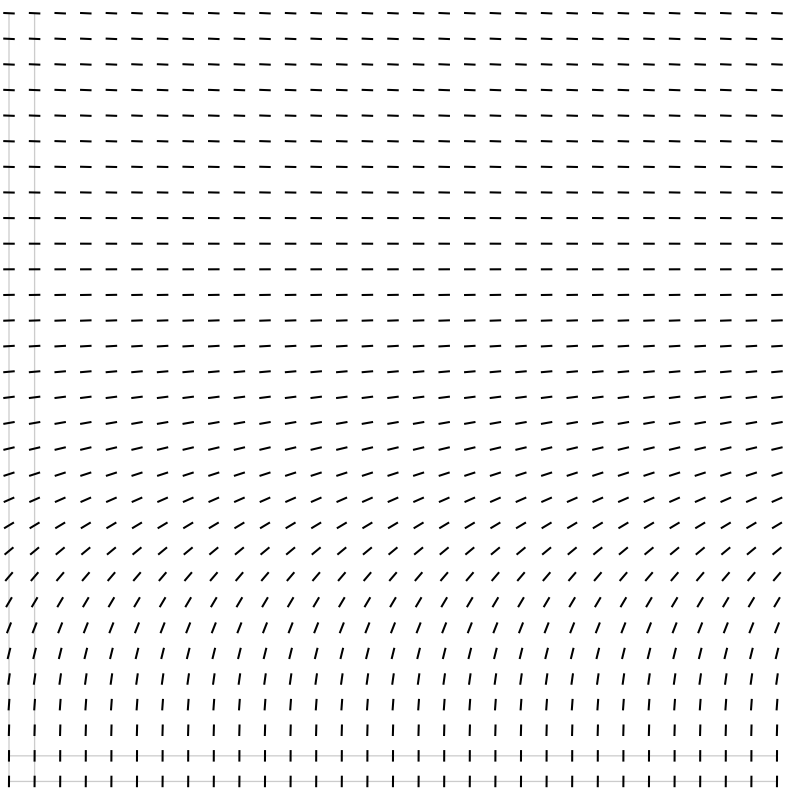
and the approximating equation is

$$h' = (h - h_{\text{eq}})\Phi = \Phi\, h - h_{\text{eq}}\,\Phi$$

with solution

$$h = h_{\text{eq}} + Ce^{\,\Phi t}$$

(d)



(e)

The equation is certainly separable. The equation becomes

$$\frac{h^2\,dh}{200 - \sqrt{1960\,h}} = dt$$

We again set $u = 200 - \sqrt{1960\,h}$ and get

$$\begin{aligned} h &= \frac{(2-u)^2}{0.196} \\ &= \frac{(u-2)^2}{0.196} \\ dh &= \frac{2(u-2)\,du}{0.196} \end{aligned}$$

so the left hand side becomes

$$\frac{2}{0.196^3}\int \frac{(u-2)^3\,du}{u}$$

which is simple to integrate.