Mathematics 255—February 13, 1996

Second order differential equations

A second order differential equation is of the form

$$y'' = F(t, y, y')$$

where $y = y(t)$. We shall often think of t as parametrizing time, and y position. In this case the differential equation asserts that at a given moment the acceleration is a function of time, position, and velocity. Such equations arise often in physical situations, as a rendering of Newton's Law $F = ma$.

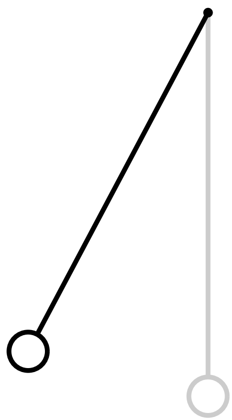
For first order equations, a solution is completely determined by its value $y(t_0)$ at any given moment. For second order equations we have this assertion: *given t_0, y_0, v_0, there exists a unique solution $y(t)$ with*

$$y(t_0) = y_0$$
$$y'(t_0) = v_0$$

In other words, we expect to specify two initial conditions to solve an equation completely.

The pendulum

Here is an example we shall consider often, sometimes with a few variations. *A pendulum of length ℓ is swinging on a fixed pivot. The mass of its bob is m. The bob is pulled away from the dead vertical position and released. Describe the subsequent motion.*



The first step is to write down what Newton's Second Law asserts in these circumstances. Gravity pulls downward on the bob, but if we assume that the shaft is rigid the component exerted along the shaft will have no effect. The only effect of gravity will therefore be to accelerate the bob in the direction tangential along the periphery of the swing. This component is $mg \sin\theta$ if θ is the angular displacement of the pendulum from equilibrium. If we let y be the linear distance along the arc of the pendulum then $y/\ell = \theta$. Since the force acts to decrease the angular displacement, the tangential force is precisely

$$-mg \sin\left(\frac{y}{\ell}\right)$$

There will be another force exerted on the bob, namely that of friction, but we will ignore this for the moment. Newton's Law now tells us that

$$my'' = -mg \sin\left(\frac{y}{\ell}\right)$$

or

$$y'' = -g\ \sin\left(\frac{y}{\ell}\right)$$

Thus we see that at least in this case we obtain a second order differential equation as a direct translation of Newton's Law into mathematics, no more and no less. This happens often. Neither Newton's Law nor the differential equation tell us directly how to describe the motion precisely. That can be, as it is here, a difficult problem.

The problem also tells us something else that is necessary to describe the motion explicitly: at time $t = 0$ we have

$$y(0) = \ell\theta_0, \quad y'(0) = 0$$

if θ_0 is the initial angle of displacement. The bob is released from rest at that displacement, so the initial velocity is 0.

Here, as for all second order differential equations, motion subsequent to any moment is completely determined by position and velocity at that moment.

A simple method of numerical solution

The physics of the example suggests a simple way to figure out approximately how to estimate the motion. Suppose at a certain time t the position is y and the velocity v. The acceleration at this moment will then be $a = -g\ \sin(y/\ell)$. After a short time interval Δt the position will be approximately $y + v\,\Delta t$, the new velocity $v + a\,\Delta t$. And so on for further intervals. If we start at time t_0 with position y_0 and velocity v_0 we therefore carry out the following procedure to estimate positions at subsequent times.

First we choose a small interval of time Δt. We will estimate the position and velocity at discrete times t_0, $t_0 + \Delta t$, etc.

Suppose that we have estimates y_n, v_n for position and velocity at time $t_n = t + n\,\Delta t$. The accleration at this moment will be $a_n = -g\ \sin(y_n/\ell)$. Then at time $t_{n+1} = t_n + \Delta t$ we obtain estimates

$$\begin{aligned} y_{n+1} &= y_n + v_n\ \Delta t \\ a_{n+1} &= v_n + a_n\ \Delta t \end{aligned}$$

This is the analogue of Euler's method for second order equations. All of the earlier methods we saw for approximating the solutions of first order differential equations can be applied to second order as well.

Linear equations

If x is small then $\sin x$ is approximately equal to x. Therefore if the displacement y is small the differential equation for the pendulum can be written approximately as

$$y'' = -g\ \sin(y/\ell)$$

This equation is a linear function of y, and such an equation is called a **linear** differential equation. More explicitly, a linear differential equation is one of the form

$$y'' = a(t)y' + b(t)y + f(t)$$

Linear differential equations play an important role in the general theory of differential equations because, as we have seen for the pendulum, arbitrary differential equations can be approximated in certain domains by linear ones. We saw this also in the case of the first order differential equation describing how height changed in a filling tank, where solutions to a linear equation gave a good approximation near the equilibrium height. For higher order equations this sort of thing becomes even more important, because they are far more difficult to solve explicitly, and we need all the help we can get.

We saw that for arbitrary first order linear equations

$$y' = a(t)y + f(t)$$

there exists an integral formula

$$y = Ce^{A(x)} + e^{A(x)} \int^x e^{-A(t)} f(t)\, dt$$

for the general solution. For second order equations we shall see a somewhat similar formula, although it will not turn out to be quite so explicit. The general structure of the formula will be the same. Recall that the first term in the first order formula is the general solution of the *homogeneous* equation where $f(t)$ vanishes, and the constant C is there because if $y(t)$ is a solution to a first order linear equation then so is $Cy(t)$. Something similar happens for arbitrary linear differential equations. They obey the **principle of linearity**, which comes in two parts:

(•) If y is a solution of the homogeneous linear differential equation

$$y'' = a(t)y' + b(t)y$$

then so is Cy for any constant C.

(•) If y_1 and y_2 are both solutions of the homogeneous linear differential equation above then so is $y_1 + y_2$.

These can be combined in the single statement:

(•) *Linear combinations of solutions of a linear homogeneous differential equation are again solutions.*

There is a practical effect of this result on the structure of solutions to such an equation. We have seen that solutions of an equation of second order depend on two parameters, the position and velocity at any given moment. This and linearity have the consequence that *the solutions of a second order linear homogeneous equation are in some sense a two dimensional vector space.* We can state this more precisely in terms of a possible coordinate system on the set of solutions.

Fix a time t_0. We can find a solution $y_{\#,0}$ of the differential equation satisfying the simple initial conditions

$$\begin{aligned} y_{\#,0}(t_0) &= 1 \\ y'_{\#,0}(t_0) &= 0 \end{aligned}$$

and another $y_{\#,1}$ such that

$$\begin{aligned} y_{\#,1}(t_0) &= 0 \\ y'_{\#,1}(t_0) &= 1 \end{aligned}$$

If y is any other solution of the differential equation, let $c_0 = y(t_0)$, $c_1 = y'(t_0)$. Then because of linearity $c_0 y_{\#,0} + c_1 y_{\#,1}$ will be a solution of the differential equation, and furthermore by assumption about the $y_{\#,i}$ it will satisfy the same conditions at t_0 that y does. Therefore it and y must be the same solution, which means that

$$y = c_0 y_{\#,0} + c_1 y_{\#,1}, \quad c_0 = y(t_0), \quad c_1 = y(t_1)$$

In other words, every solution y of the differential equation can be assigned coordinates in terms of its value and that of its derivative at t_0.

The coordinates of a solution will vary with the choice of t_0. No problem, but keep it in mind. The functions $y_{\#,0}$ and $y_{\#,1}$ are called the **fundamental solutions** determined by t_0. They will vary with the choice of t_0.

Linear equations with constant coefficients

I have said above that we can write down a formula for the general solution of any linear second differential equation, but that it would not be so explicit as the formula for first order equations. In fact, it is a formula that is almost useless unless we make some special assumption about the equation. The reason for this is that the formula involves the general solution to the homogeneous equation, and in general *there is no formula for solving second order homogeneous linear differential equations*. What the formula does is solve the inhomogeneous case in terms of the homogeneous case, but we are usually stopped right there. There is one large category of equations which can be solved, however — those linear homogeneous where the coefficients $a(t)$ and $b(t)$ are constants. Even in this case there is much to be said, and we shall spend a fair amount of time in this course on this topic.

Let's now consider such an equation, which I write in a form slightly different from earlier:

$$y'' + ay' + by = 0$$

To get some feel for what will happen, let's look at an explicit example

$$y'' - y = 0$$

This can be written in the old form

$$y'' = y$$

so that solutions of this differential equation are those functions $y(t)$ whose double derivatives are equal to themselves. There is certainly one function $y(t) = e^t$ which comes to mind quickly. Another and similar function is $y(t) = e^{-t}$, since for this function $y' = -y$, $y'' = -(-y) = y$. Now the solutions to any linear homogeneous equation satisfy the linearity principle, so we can obtain lots more solutions by taking linear combinations

$$ae^t + be^{-t}$$

of the ones we already have. The converse turns out also to be true.

Theorem. *Any solution of the differential equation*

$$y'' - y = 0$$

can be expressed as a linear combination

$$ae^t + be^{-t} \ .$$

This is another way of saying that we can assign coordinates, here the constants a and b, to every solution of the equation.

The proof involves a calculation we shall repeat often.

The point is to construct first the fundamental solutions associated to $t = 0$ from the solutions e^t, e^{-t}. The fundamental solution $y_{\#,0}$ for example satisfies

$$y_{\#,0}(0) = 1$$
$$y'_{\#,0}(0) = 0$$

WE sepeculate that $y_{\#,0}$ is a linear combination of e^t and e^{-t}. Set

$$y_{\#,0} = ae^t + be^{-t}$$

Then

$$y_{\#,0} = ae^t - be^{-t}$$

We must see if we can choose a and b so that

$$a + b = 1$$
$$a - b = 0$$

This gives us two equations in the two unknowns a and b, and has the solution

$$a = 1/2, \quad b = 1/2$$

Therefore

$$y_{\#,0} = \frac{e^t + e^{-t}}{2}$$

Similarly we can see that

$$y_{\#,1} = \frac{e^t - e^{-t}}{2}$$

But now if we are given any solution of the differential equation we can write it as a linear combination of $y_{\#,0}$ and $y_{\#,1}$, according to reasoning I presented earlier, and then we can write it in turn as a linear combination of e^t and e^{-t}.

Much of what I have said here applies to more general linear differential equations with constant coefficients. The first step to make is to realize that solving such equations always involves exponential equations of some kind. To see why this should be so, and to understand why exponential functions arise in this context, is important and interesting in its own right.

Theorem. *Any homogeneous linear differential equation with constant coefficients has at least one solution of the form*

$$y = e^{\lambda t}$$

for a suitable choice of λ.

In solving the equation, we will therefore simply have to find what λ has to be.

I will prove this result in detail, because it illustrates an important way of thinking about linear differential equations with constant coefficients, and introduces for them a new principle.

The principle is **time invariance**. I have said that a differential equation is often the mathematical formualtion of Newton's Law. The coefficients which occur are physical constants associated to the system, and they will be constant precisely when the physical situation the differential equation encodes is one which does not vary with time. The *system* is time invariant, but this is not the same as saying that the *solutions of the system* are time-invariant. What it does mean is that if we shift the initial conditions in time, then the solutions will likewise be shifted in time. Mathematically, a shift by time interval h means replacing a function $f(t)$ by $f(t-h)$, and the principle of time invariance for constant coefficient equations is this:

($\bullet$) *If $y(t)$ is a solution of a linear homogeneous differential equation with constant coefficients, then for any constant h so is $y(t-h)$.*

A very closely related fact is this:

Theorem. *If $y(t)$ is a solution of a linear homogeneous differential equation with constant coefficients, then so is its derivative $y'(t)$.*

The way in which these two principles are related is that

$$y' = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

so that the derivative of a function is the limit of a linear combination of y and its shifts.

Both of these are straightforward to prove. The second, for example: Suppose that y is a solution of the linear equation

$$y'' + a(t)y' + b(t)y = 0$$

where for the moment I don't assume a and b to be constants. Take the derivative of both sides:

$$y''' + a'(t)y' + a(t)y'' + b'(t)y + b(t)y' = 0$$

But now if we assume $a(t)$ and $b(t)$ are constant, then their derivatives vanish, and this becomes

$$(y')'' + a(t)(y')' + b(t)(y') = 0$$

which means that y' is also a solution.

So we are now in the following circumstances: we have an operation $y \mapsto y'$ on the set of all solutions of the differential equation, and the set of solutions is a two dimensional vector space. The operator is linear. This means that if we express it in terms of coordinates, we get a matrix, in fact a 2×2 matrix. But now one thing we know about 2×2 matrices is that they always have at least one eigenvalue. In our circumstances, this means that there exists at least one solution y of the differential equation which is taken into a multiple of itself by the derivatation operator.

$$y' = \lambda y$$

for some constant λ, the eigenvalue. This is a first order differential equation, and we know that it has as solution the exponential function $e^{\lambda t}$.

In summary: *linearity together with time invariance implies that we can find exponential functions which are solutions.*

Explicit solutions

The basic idea in solving a linear homogeneous differential equation with constant coefficients

$$y'' + ay' + by = 0$$

we set $y = e^{\lambda t}$ and substitute it into the equation. We have

$$\begin{aligned} y &= e^{\lambda t} \\ y' &= \lambda e^{\lambda t} \\ y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

and

$$y'' + ay' + by = (\lambda^2 + a\lambda + b)e^{\lambda t}$$

and to obtain a solution of the equation we set λ equal to a root of the **characteristic equation**

$$\lambda^2 + a\lambda + b = 0 \ .$$

Example.

$$y'' - y = 0$$

Here the characteristic equation is

$$\lambda^2 = 1$$

and we get the same solutions

$$e^{-t}, \quad e^{-t}$$

we got before.

Example.

$$y'' - 3y' + y = 0$$

The characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

with roots $\lambda = 2, 1$ and we get solutions

$$e^{2t}, \quad e^{t}$$

Example.

$$y'' + y = 0$$

Characteristic equation

$$\lambda^2 + 1 = 0$$

Here the roots are $\lambda = \pm i$, and we get solutions

$$e^{it}, \quad e^{-it}$$

What good are these? We recall Euler's formula

$$e^{it} = \cos t + i \sin t$$

Then we write

$$\begin{aligned} e^{it} &= \cos t + i \sin t \\ e^{-it} &= \cos t - i \sin t \end{aligned}$$

We now recall that linear combinations of solutions are solutions. Adding the equations above and then subtracting them we get solutions

$$\begin{aligned} e^{it} + e^{-it} &= 2 \cos t \\ e^{it} - e^{-it} &= 2i \sin t \end{aligned}$$

Dividing by constants is also allowed, so we also get solutions

$$\begin{aligned} \frac{e^{it} + e^{-it}}{2} &= \cos t \\ \frac{e^{it} - e^{-it}}{2i} &= \sin t \end{aligned}$$

and these are the solutions which are of interest to us:

$$\cos t, \quad \sin t$$

In other words, *when we have complex roots of the characteristic equation we also have as solutions their real and imaginary parts, which we calculate according to Euler's formula.* More explicitly, *if $\lambda = a + ib$ is a root of the characteristic equation then*

$$e^{at} \cos bt, \quad e^{at} \sin bt$$

are solutions of the differential equation.

Example. For

$$y'' + y' + y = 0$$

the roots are

$$-\frac{1}{2} + \frac{\sqrt{-3}}{2}$$

and the solution are

$$e^{-t/2}\cos(\sqrt{3}t/2), \quad e^{-t/2}\cos(\sqrt{3}t/2)$$

Example.

$$y'' + 2y' + y = 0$$

Here the characteristic equation is

$$\lambda^2 + 2\lambda + 1 = 0$$

so we get a single root $\lambda = 1$, and a solution $y = e^{-t}$. How do we get another solution? The answer is suggested by the simplest example

$$y'' = 0$$

Here the root is $\lambda = 0$, and the exponential solution the constant 1. Of course the linear function t is also a solution. What happens in general is that *when the characteristic equation has just a single root λ then both $e^{\lambda t}$ and $te^{\lambda t}$ are solutions.*

When there is only a single root, the characteristic polynomial must be of the form

$$(\lambda - c)^2$$

and the differential equation looks like

$$y'' - 2c\,y' + c^2 = 0$$

In this case you can verify explicitly that te^{ct} does satisfy the equation.

Initial conditions

The procedure described above gives us two different solutions of the differential equation. All other solutions will be linear combinations of these two. Solving for initial conditions amounts to solving two equations for the two unknown coefficients.

Example. To find the solution of

$$y'' - 3y' + 2y = 0$$

such that $y(0) = 0$, $y'(0) = 1$ we set

$$y = Ae^t + Be^{2t}$$

Then $y' = Ae^t + 2Be^{2t}$ and we get

$$A + B = 0$$
$$A + 2B = 1$$

which leads to $A = -1$, $B = 1$. The solution we want is $e^{2t} - e^t$.