Mathematics 255—January 25, 1996

Second homework solutions

Exercise 1. *(a) Use a spread sheet and the improved Euler's method to calculate and graph the solution of the differential equation plus initial conditions in the range indicated:*

$$y' = xy + 1, \quad y(0) = 1, \quad 0 \le x \le 1$$

Use intervals $\Delta x = 0.5, 0.25, 0.125, 0.0625, .03125$.

(b) Estimate the error in each of the values you get for $y(1)$.

Not very exciting, but here are the values of (x, y) for $N = 2, \ldots, 32$.

$N = 2$

0.00000000	1.00000000
0.50000000	1.68750000
1.00000000	3.05078125

$N = 4$

0.00000000	1.00000000
0.25000000	1.28906250
0.50000000	1.68057251
0.75000000	2.23629367
1.00000000	3.05914605

$N = 8$

0.00000000	1.00000000
0.12500000	1.13378906
0.25000000	1.28759217
0.37500000	1.46676149
0.50000000	1.67802984
0.62500000	1.92999587
0.75000000	2.23378212
0.87500000	2.60393906
1.00000000	3.05970103

$N = 16$

0.00000000	1.00000000
0.06250000	1.06457520
0.12500000	1.13357333
0.18750000	1.20756148
0.25000000	1.28716995
0.31250000	1.37310274
0.37500000	1.46614969
0.43750000	1.56720045
0.50000000	1.67726066
0.56250000	1.79747094
0.62500000	1.92912889
0.68750000	2.07371508
0.75000000	2.23292349
0.81250000	2.40869747

0.87500000	2.60327220
0.93750000	2.81922511
1.00000000	3.05953567

$N = 32$

0.00000000	1.00000000
0.03125000	1.03175354
0.06250000	1.06454640
0.09375000	1.09844421
0.12500000	1.13351598
0.15625000	1.16983436
0.18750000	1.20747593
0.21875000	1.24652157
0.25000000	1.28705672
0.28125000	1.32917181
0.31250000	1.37296263
0.34375000	1.41853072
0.37500000	1.46598388
0.40625000	1.51543660
0.43750000	1.56701064
0.46875000	1.62083555
0.50000000	1.67704931
0.53125000	1.73579899
0.56250000	1.79724146
0.59375000	1.86154416
0.62500000	1.92888597
0.65625000	1.99945810
0.68750000	2.07346508
0.71875000	2.15112586
0.75000000	2.23267493
0.78125000	2.31836367
0.81250000	2.40846165
0.84375000	2.50325820
0.87500000	2.60306401
0.90625000	2.70821296
0.93750000	2.81906403
0.96875000	2.93600345
1.00000000	3.05944704

Here are the values of $y(1)$ we get, and indeed for a wider range of values for N:

2	3.05078125
4	3.05914605
8	3.05970103
16	3.05953567
32	3.05944704
64	3.05941831
128	3.05941026
256	3.05940813
512	3.05940759
1024	3.05940745

We now use the rule

$$\text{error} \cong c\Delta^2$$

where

$$c = \frac{y_2 - y_1}{\Delta_2^2 - \Delta_1^2}$$

to get successive estimates

N	Δ	$y_N(1)$	$y_N(1) - y_{N-1}(1)$	c	error in $y_N(1)$
2	0.50000000	3.05078125	−		
4	0.25000000	3.05914605	0.00836480	0.04461225	0.00278827
8	0.12500000	3.05970103	0.00055498	0.01183958	0.00018499
16	0.06250000	3.05953567	−0.00016536	−0.01411095	−0.00005512
32	0.03125000	3.05944704	−0.00008863	−0.03025096	−0.00002954
64	0.01562500	3.05941831	−0.00002873	−0.03922871	−0.00000958
128	0.00781250	3.05941026	−0.00000805	−0.04396045	−0.00000268
256	0.00390625	3.05940813	−0.00000212	−0.04638912	−0.00000071
512	0.00195312	3.05940759	−0.00000054	−0.04761942	−0.00000018
1024	0.00097656	3.05940745	−0.00000014	−0.04823859	−0.00000005

The point of this example is that sometimes we cannot really get a good estimate for the error unless the interval is already small enough.

Exercise 2. *The same for the equation*

$$y' = x^2 + y^2, \quad y(0) = 1$$

in as much of the range $[0, 1]$ *as you can.*

Here is what I get from a run for $N = 32$:

0.00000000	1.00000000
0.03125000	1.03225708
0.06250000	1.06672445
0.09375000	1.10369156
0.12500000	1.14348513
0.15625000	1.18647622
0.18750000	1.23308911
0.21875000	1.28381225
0.25000000	1.33921217
0.28125000	1.39995110
0.31250000	1.46680965
0.34375000	1.54071617
0.37500000	1.62278538
0.40625000	1.71436976
0.43750000	1.81712884
0.46875000	1.93312428
0.50000000	2.06495245
0.53125000	2.21593309
0.56250000	2.39038362
0.59375000	2.59402819
0.62500000	2.83462515
0.65625000	3.12296192
0.68750000	3.47449403

0.71875000	3.91217103
0.75000000	4.47158189
0.78125000	5.21097166
0.81250000	6.23244494
0.84375000	7.73198378
0.87500000	10.13593687
0.90625000	14.55924537
0.93750000	24.92634722
0.96875000	65.42400951
1.00000000	752.42333418

This and similar runs suggest strongly that the solution goes off to ∞ before $x = 1$. Here are the last few figures for a run of $N = 64$

0.93750000	28.61065583
0.95312500	48.41941392
0.96875000	123.28208651
0.98437500	1258.88722264
1.00000000	5303569.60932311

which is even more suggestive.

Exercise 3. *The solution of the equation in #2 goes off to ∞ at some finite time t. Estimate to 3 decimals what that time is, by any method you can think of. Explain your method.*

The only trick I can think of which you might have come up with is this. If we have a calculated solution of the differential equation which passes through (x_*, y_*) then from that point on its slope is greater than the solution to the equation

$$y' = x_*^2 + y^2$$

since $x \geq x_*$. Therefore the solution of our equation passes off to infinity *before* the solution to this comparison equation. The comparison equation can be solved, since it is separable.

$$\frac{dy}{x_*^2 + y^2} = dx$$

$$\frac{1}{x_*}\mathrm{atan}\left(\frac{y}{x_*}\right) = x + C$$

$$y = x_* \tan(x x_* + C)$$

where the (different) integration constant C is chosen to match the initial conditions

$$C = \mathrm{atan}\left(\frac{y_*}{x_*}\right) - x_*^2$$

This comparison equation goes off to infinity where

$$x_\infty + C = \frac{\pi}{2}, \quad x_\infty = \frac{\pi}{2} + x_*^2 - \mathrm{atan}\left(\frac{y_*}{x_*}\right)$$

Therefore what I suggest is a small modification of your calculation routine in which you calculate this estimated value of x_∞ every time you calculate a value of y, taking $x_* = x$, $y_* = y$. Here is what I get for the last part of a run with $N = 1024$, for example.

0.93554688	29.13149492	0.96986219
0.93750000	30.88780948	0.96986530
0.93945312	32.86883673	0.96986880
0.94140625	35.12056953	0.96987278
0.94335938	37.70244052	0.96987732
0.94531250	40.69260463	0.96988257
0.94726562	44.19591778	0.96988869
0.94921875	48.35636098	0.96989590
0.95117188	53.37708242	0.96990452
0.95312500	59.55409904	0.96991502
0.95507812	67.33583841	0.96992806
0.95703125	77.43482694	0.96994468
0.95898438	91.05331065	0.96996655
0.96093750	110.38403186	0.96999655
0.96289062	139.86822852	0.97004010
0.96484375	189.94362163	0.97010842
0.96679688	291.40314330	0.97022853
0.96875000	578.51332780	0.97047857
0.97070312	2388.03796607	0.97112188
0.97265625	186626.45893194	0.97266161
0.97460938	4543974853062.09	0.97460938

I would estimate $x_\infty = 0.970$, but not with tremendous confidence. We shall see later on that adjusting step size automatically is the best way to deal with a problem like this.

Exercise 4. *Use Newton's method and the implicit equation for the height $h(t)$ in the solution of the problem*

$$h' = 2 - \sqrt{0.196\,h}, \quad h(0) = 10$$

to calculate $h(t)$ exactly for $t = 1, 2, 10, 20$.

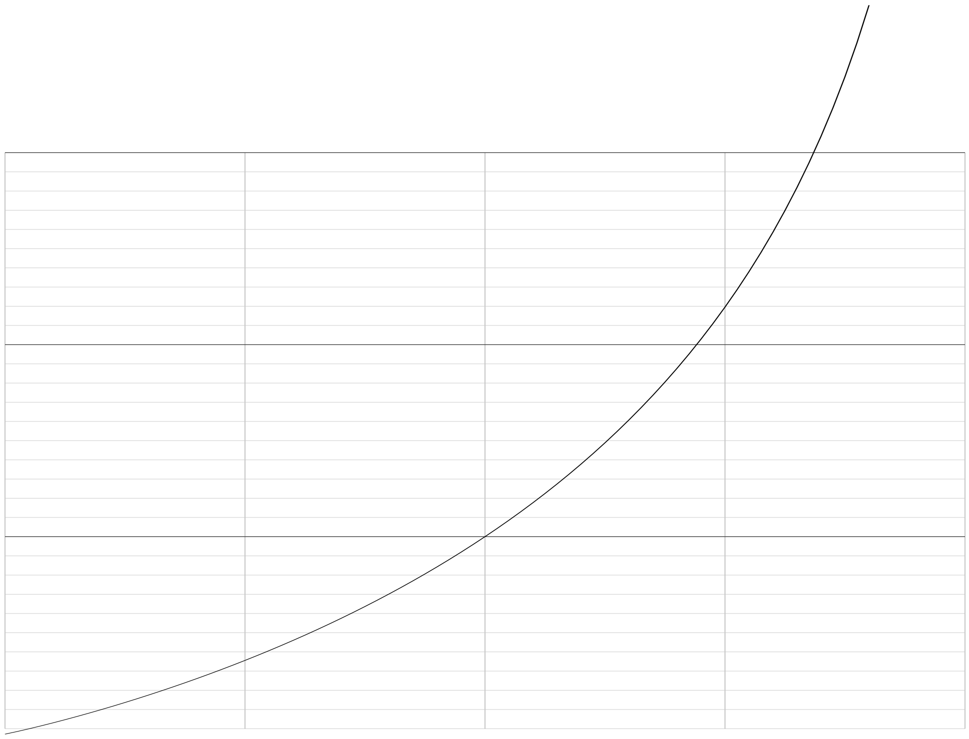
The equation for h is

$$F(h) = y(h) - y_0 - \frac{t}{c}$$

where $c = 2/0.196$ and

$$y(h) = \left(2 - \sqrt{0.196h}\right) - 2\log\left(2 - \sqrt{0.196h}\right), \quad y_0 = y(h_0)$$

The graph of $y(h) - y_0$ is somewhat unusual, and because of this some values of t will lead to trouble. Therefore it is important to choose the initial guess rather carefully.



This shows up in my runs of Newton's method. Here I chose x_0 according to the solution to the linear approximation near equlibrium.

$t = 1$
10.01905124 -0.09488483
10.59868822 0.00333149
10.57969528 0.00000385
10.57967331 0.00000000
strut
$t = 2$
10.03632403 -0.19005448
11.19499623 0.01387862
11.12153724 0.00006468
11.12119166 0.00000000
strut
$t = 10$
10.12748753 -0.95902096
15.91199016 0.53720441
14.62926838 0.06893608
14.41354148 0.00136952
14.40908060 0.00000056
14.40907879 0.00000000
strut
$t = 20$
10.17533502 -1.93106586
log: DOMAIN error

So we need to find a way to make an especially good guess. The best way to do this is to plot some values of $F(h)$. We have

x	$(y - y_0) * 10.20$
16.0	15.8618
16.5	18.1866
17.0	20.8492

so we choose $x_0 = 17.0$ which gives a run

17.00000000	0.08322017
16.85171242	0.00184386
16.84827678	0.00000093
16.84827503	0.00000000

Exercise 5. *A drop of water evaporates at a rate proportional to the surface area of the drop. If the drop is spherical, and in the time span $t = 0$ to $t = 30$ minutes goes in size from a 3mm radius to a 2mm radius, find a formula for the radius of the drop at all times $t > 0$. Graph it as a function of t.*

The equation we get is

$$\frac{dV}{dt} = 4\pi k r^2$$

where $V(r) = 4\pi r^3/3$ so that

$$\frac{dr}{dt} = k, \quad r = kt + C$$

The two constants C and k can be determined by two independent measurements.

$$C = 3, \quad k = -1/30 \ .$$

This is just a linear function of t, and in particular the drop is gone after 90 minutes.

Exercise 6. *Newton's law of cooling says that the rate of change of temperature of a cooling object is proportional to the difference between the temperature of the object and the surrounding temperature.*

(a) Write down the corresponding differential equation, if $\theta(t)$ is the temperature, T is the surrounding temperature, and k is the constant of proportion.

(b) If a cup of coffee has initial temperature 96° and at time $t = 1$ minute has cooled to 92° in a room at temperature 25°, find a formula for $\theta(t)$. Graph $\theta(t)$ as a function of t.

(a) We have

$$\theta'(t) = -k(\theta(t) - T) = k\theta - kT$$

This is a linear equation in θ and has general solution according to the formula

$$\theta = Ce^{-kt} + e^{-kt} \int^t e^{ks} kT \, ds = T + Ce^{-kt}$$

which makes sense, since it says that the temperature of the object approaches the surrounding temperature exponentially.

(b) To get a precise description of the solution we need to find what the two constants k and C are. From the data we have

$$96 = C + 25$$

$$92 = Ce^{-k} + 25$$

Therefore $C = 96 - 25 = 71$ and

$$e^{-k} = \frac{92-25}{71}, \quad k = -\log_e 0.943662 = .05799$$

The graph, as I have mentioned already, approaches $\theta_{\mathrm{eq}} = 25^\circ$ exponentially.

Exercise 7. *An object of mass m dropped from a height in free air falls under the acceleration of gravity, and also feels resistance $k|v|$ proportional to the magnitude of its velocity v.*

(a) Write down a first order differential equation for v.

(b) Write down a formula for its solution, assuming $v = 0$ at $t = 0$.

(c) Graph the solution.

(d) Find a formula for the length of time it takes to achieve 90% of its limiting velocity.

(a) The equation follows from Nwton's Law $F = ma$

$$mv' = -kv + mg$$

since $v' = a$ and the object feels two forces—air resistance $-kv$ and gravity mg.

(b) The equation is linear

$$v' = -\frac{k}{m}v + g$$

and has general solution

$$v = Ce^{-kt/m} + v_{\mathrm{eq}}$$

where v_{eq} is the terminal velocity, where air resistance and gravity cancel:

$$kv_{\mathrm{eq}} = mg, \quad v_{\mathrm{eq}} = \frac{mg}{k}$$

With the given initial conditions we have

$$C + v_{\mathrm{eq}}, \quad C = -v_{\mathrm{eq}}$$

so the solution is explicitly

$$v_{\mathrm{eq}}(1 - e^{-kt/m})$$

(c) As before, asymptotically tending to terminal velocity.

(d) We have

$$\begin{aligned} 0.90v_{\mathrm{eq}} &= v_{\mathrm{eq}}(1 - e^{-kt/m}) \\ 1 - e^{-kt/m} &= 0.90 \\ e^{-kt/m} &= 0.10 \\ kt/m &= -\log_e 0.10 \\ t &= -\frac{m\log_e 0.10}{k} \end{aligned}$$