Mathematics 255—February 15, 1996

Fifth homework solutions

Exercise 1. *Graph the functions*

(a) $\cos 2x - \sin 2x$

(b) $2\cos 3x - \sin 3x$

(c) $\cos(x/2) + 3\sin(x/2)$

(d) The function

$$\frac{\cos at - \cos bt}{a^2 - b^2}$$

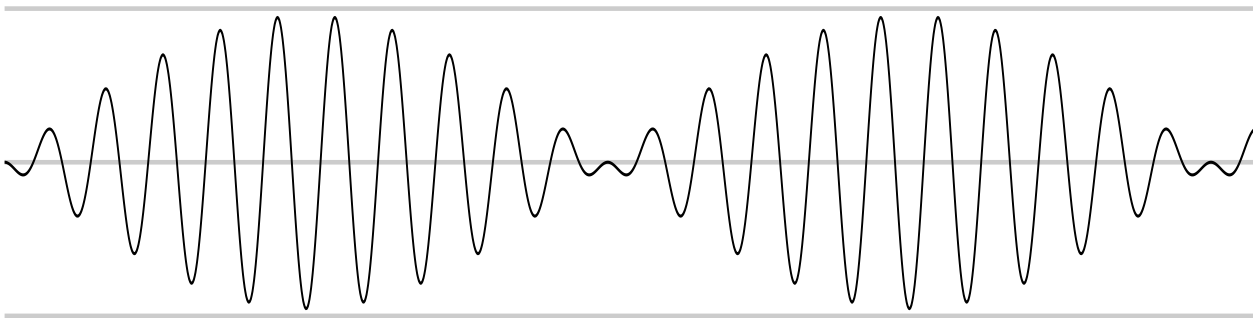
where $a = 1.1$, $b = 1.0$.

(a) $\sqrt{2}\cos(2x + \pi/4)$

(b) $\sqrt{5}\cos(3x + 0.4636)$

(c) $\sqrt{10}\cos(x/2 - 1.249)$

(d) This doesn't fall into the same category. You just have to plot it directly. Its graph looks like this:



(Question: what vertical scale do the horizontal lines represent?)

Exercise 2. *Solve*

$$y'' + 2y' + 2y = f(t), \quad y(0) = y'(0) = 0$$

where

(a) $f(t) = 1$

(b) $f(t) = e^t$

(c) $f(t) = \sin t$

(d) $f(t) = 1$ *in the range* $[0, 1]$ *and* 0 *elsewhere.*

The general solution to the homogeneous is

$$\begin{aligned} y(t) &= ae^{-t}\cos t + be^{-t}\sin t + \frac{1}{2} \\ y'(t) &= -ae^{-t}\cos t - ae^{-t}\sin t - be^{-t}\sin t + be^{-t}\cos t \\ y(0 &= a \\ y'(0) &= -a + b \end{aligned}$$

(a) The general solution is

$$y(t) = ae^{-t}\cos t + be^{-t}\sin t + \frac{1}{2}$$

We solve

$$\begin{aligned} a + \frac{1}{2} &= 0 \\ -a + b &= 0 \\ b &= -\frac{1}{2} \\ a &= -\frac{1}{2} \end{aligned}$$

(b) The general solution is

$$ae^{-t}\cos t + be^{-t}\sin t + \frac{e^t}{5}$$

We solve

$$\begin{aligned} a + \frac{1}{5} &= 0 \\ -a + b + \frac{1}{5} &= 0 \\ b &= -\frac{2}{5} \\ a &= -\frac{1}{5} \end{aligned}$$

(c) The general solution is

$$ae^{-t}\cos t + be^{-t}\sin t + \left(\frac{-2\cos t + \sin t}{5}\right)$$

We solve

$$\begin{aligned} a - \frac{2}{5} &= 0 \\ -a + b + \frac{1}{5} &= 0 \\ b &= \frac{1}{5} \\ a &= \frac{2}{5} \end{aligned}$$

(d) Here we use the basic formula. The fundamental solution is

$$e^{-t}\sin t$$

so the formula gives

$$\int_0^t e^{-(t-s)}\sin(t-s)\,f(s)\,ds = e^{-t}\sin t\int_0^t e^s\cos s\,f(s)\,ds - e^{-t}\cos t\int_0^t e^s\sin s\,f(s)\,ds$$

The answer has three forms, depending on cases $t<0$, $0\le t\le 1$, $1<t$. The solution is a flat 0 when $t<0$. For t in $[0,1]$ we get

$$e^{-t}\sin t\int_0^t e^s\cos s\,ds - e^{-t}\cos t\int_0^t e^s\sin s\,ds$$

And for $t > 1$ we get

$$e^{-t} \sin t \int_0^1 e^s \cos s \, ds - e^{-t} \cos t \int_0^1 e^s \sin s \, ds$$

Recall that

$$\int e^s \cos s \, ds = e^s \frac{\cos s + \sin s}{2}$$
$$\int e^s \sin s \, ds = e^s \frac{-\cos s + \sin s}{2}$$

(which I calculated by complex exponentials).

We get

$$e^{-t} \sin t \left[\frac{e^t \cos t + e^t \sin t - 1}{2} \right] - e^{-t} \cos t \left[\frac{-e^t \cos t + e^t \sin t + 1}{2} \right]$$

and

$$e^{-t} \sin t \left[\frac{e \cos 1^r + e \sin 1^r - 1}{2} \right] - e^{-t} \cos t \left[\frac{-e \cos 1^r + e \sin 1^r + 1}{2} \right]$$

Exercise 3. *Find the general solution of*

(a) $y'' + \omega_0^2 y = \cos \omega t$

(b) $y'' + \omega_0^2 y = \sin \omega t$

(c) $y'' + 3y' + 2y = e^{-t} + 1 + 2 \sin t$

See the notes for (a) and (b).

For (c) we get

$$ae^{-t} + be^{-2t} + te^{-t} + \frac{1}{2} + \frac{-3 \cos t + \sin t}{5}$$