Mathematics 255—January 11, 1996

Complex exponentials

Exponential integrals

We will often see exponential integrals like

$$\int x^n e^{cx}\,dx$$

in this course. If $n = 0$ then we have

$$\int e^{cx}\,dx = \frac{e^{cx}}{c}$$

while if $n > 0$ we can use integration by parts to get

$$\int x^n e^{cx}\,dx = \frac{x^n e^{cx}}{c} - \frac{n}{c}\int x^{n-1} e^{cx}\,dx$$

which allows us to go from n to $n-1$ to ... the integral with $n = 0$.

There is a simple trick which will make the calculations slightly easier. Namely, if we consider

$$\int \frac{x^n}{n!}\, e^{cx}\,dx$$

instead then we get

$$\int \frac{x^n}{n!}\, e^{cx}\,dx = \frac{x^n}{n!}\,\frac{e^{cx}}{c} - \frac{1}{c}\int \frac{x^{n-1}}{(n-1)!}\, e^{cx}\,dx$$

which means if we first **normalize** by dividing by $n!$ then we don't have to worry about keeping track of the terms involving n. Thus we can continue directly

$$\int \frac{x^n}{n!}\, e^{cx}\,dx = \frac{x^n}{n!}\,\frac{e^{cx}}{c} - \frac{x^{n-1}}{(n-1)!}\,\frac{e^{cx}}{c^2} + \ldots \pm \frac{e^{cx}}{c^{n+1}}\,.$$

Complex exponentials

Throughout this course, we will often see integrals like

$$\int e^x \cos x\,dx, \quad \int e^x \sin x\,dx$$

or even worse

$$\int x e^x \cos x\,dx, \quad \int x e^x \sin x\,dx.$$

The usual way way of finding these is to apply integration by parts twice and do some complicated algebra. For example, if we let

$$I_1 = \int e^x \cos x\,dx, \quad I_2 = \int e^x \sin x\,dx$$

Then integration by parts with $u = e^x$, $v = \sin x$ gives us

$$I_1 = \int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx = e^x \sin x - I_2$$

and with $u = e^x$, $v = -\cos x$

$$I_2 = \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + I_1$$

from which we can solve the pair of linear equations

$$I_1 + I_2 = e^x \sin x$$
$$I_1 - I_2 = e^x \cos x$$

for I_1 and I_2. The second pair of integrals are even more complicated.

There is a way to find all integrals of the form

$$\int x^n e^{ax} \cos bx \, dx, \quad \int x^n e^{ax} \sin bx \, dx$$

which is formally simpler if conceptually more complicated. But the complication involves a definition which we will have to learn later on anyway, so I will introduce it now.

The point is that $\cos x$ and $\sin x$ are really disguised forms of e^x, as the book explains in Chapter 3 (page 137 in the fifth edition). **This is an extremely important idea.** The basic equation involves $i = \sqrt{-1}$ and is

$$e^{ix} = \cos x + i \sin x$$

In other words, $\cos x$ is the real part of the complex exponential function e^{ix}, and $\sin x$ is its imaginary part. The complex exponential obeys all the usual rules that the real one does.

This formula can be proven by using Taylor's series. If we recall that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and we set $z = ix$ then we get since $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc.

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \cdots \\ &= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right] + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] \\ &= \cos x + i \sin x \end{aligned}$$

From this we see that if $c = a + ib$ is any complex number then

$$e^{cx} = e^{ax+ibx} = e^{ax}e^{ibx} = e^{ax} \cos bx + ie^{ax} \sin bx,$$

which says that $e^{ax} \cos bx$ is the real part of e^{ax+ibx} and $e^{ax} \sin bx$ is its imaginary part. Therefore the integrals

$$\int e^{ax} \cos bx \, dx, \quad \int e^{ax} \sin bx \, dx$$

are the real and imaginary parts of the complex integral

$$\begin{aligned}
\int e^{(a+ib)x}\,dx &= \int e^{cx}\,dx \quad (c = a + ib) \\
&= \frac{e^{cx}}{c} \\
&= \frac{e^{(a+ib)x}}{a+ib} \\
&= \frac{(a-ib)e^{(a+ib)x}}{a^2+b^2} \\
&= \frac{e^{ax}}{a^2+b^2}(a-ib)(\cos bx + i\sin bx) \\
&= \frac{e^{ax}}{a^2+b^2}\big((a\cos bx + b\sin bx) + i(a\sin bx - b\cos bx)\big)
\end{aligned}$$

Taking real and imaginary parts, we get

$$\int e^{ax}\cos bx\,dx = e^{ax}\,\frac{a\cos bx + b\sin bx}{a^2+b^2}$$

$$\int e^{ax}\sin bx\,dx = e^{ax}\,\frac{a\sin bx - b\cos bx}{a^2+b^2}$$