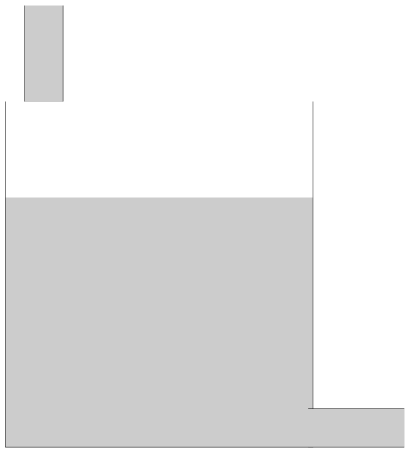
Mathematics 255—January 21, 1996

First order differential equations

Flow in and out of a tank

Consider a circular tank of water with water flowing into it at the top and out through a small hole at the bottom. More particularly, let

flow in:	1 litre every 5 seconds
effective area of the exit hole:	$1\,\mathrm{cm}^2$
cross-section of the tank:	$100\,\mathrm{cm}^2$
height of the water in the tank at time t:	$h(t)$
initial height of the water:	$h_0 = 10\,\mathrm{cm}$



We want to find a way to determine $h(t)$ for all $t > 0$. This is not a simple problem. Much of the theory and practice of solving differential equations, and much of the material in the rest of this course, is implicit in it.

First of all we shall see that the mathematical formulation of the situation is expressed naturally as a single differential equation.

There is one underlying physical principle we shall have to apply. *Torricelli's Theorem* asserts that if the height of the water is h then the velocity of the water flowing out is $\sqrt{2gh}$, where $g = 980\,\mathrm{cm/sec}^2$ is the acceleration of gravity. This value coincides with the velocity of an object which is dropped from a height h, and this is no coincidence. Water flowing out from the tank is effectively derived from the top of the tank, so Torricelli's formula follows from the principle of conservation of energy. At any rate, we can see that *as the height increases so does the velocity at which water flows out of its bottom.*

If the velocity of the flow out is v, then the volume of water flowing out in a short interval of time dt is $\alpha v\,dt$, where α is the area of the hole through which this flow takes place. If we take into account both the constant flow in and this variable flow out, we see that the rate at which the volume $V(t)$ of water in the tank changes is

$$V'(t) = C - \alpha\sqrt{2gh}$$

where

$$\begin{aligned} C &= \text{ flow rate in} \\ \alpha &= \text{ area of exit} \end{aligned}$$

In our case $C = 200\,\mathrm{cm}^3/\mathrm{sec}$. The tank is cylindrical, so $V = Ah$, where $A = 100\,\mathrm{cm}^2$. We have assumed that $\alpha = 1\,\mathrm{cm}^2$. Of course $g = 980\,\mathrm{cm}/\mathrm{sec}^2$. Therefore we obtain a single equation involving only h:

$$\begin{aligned} 100\,h' &= 200 - \sqrt{1960h} \\ h' &= 2 - \sqrt{0.196h} \end{aligned}$$

This is a **differential equation** involving the unknown function $h(t)$. It is of **first order** because it involves h and h'. We shall see later that most physical systems are of second order—involving h''—since they are often derived from Newton's second law of motion, which involves acceleration.

This differential equation is a natural way to express the mechanics of the situation at hand, but it is a little difficult to see quite how to use it to answer our original question: how does it help us to calculate $h(t)$ for various values of t?

Before I discuss this question, I want to point out that the differential equation is not itself sufficient to determine $h(t)$. We require in addition the **initial data** given in the problem, which tells us that

$$h(0) = 10 .$$

This sort of extra condition, which is called an **initial condition**, is always necessary to specify uniquely a solution of a first order differential equation. One of the basic facts about differential equations is that (with a few impractical exceptions) *given any first order differential equation*

$$h'(t) = F(t, h(t))$$

there is always a unique function h satisfying the equation and satisfying in addition any given initial condition $h(t_0) = h_0$ for any initial value t_0 of t. This at least assures us that the function we are now looking for is is determined by the data we have.

Solving the differential equation plus initial condition means describing, in more or less explicit terms, what this unique function looks like. You can do this in several different ways: (1) at best, you can find a completely explicit formula for the solution; (2) you can produce a method to calculate its values or draw its graph even without a formula; (3) you can describe roughly what the solution is like without going into details. Any of these are valid in various circumstances. In the rest of these notes I will show how these different techniques work out for the particular equation at hand, trying to put things in context as we go along.

Before I begin to discuss details, however, let me point out that intuitive reasoning can tell us something about what happens to the water in the tank. At $t = 0$ the height of the tank is $10\,\mathrm{cm}$. The rate of change of h at the starting moment is therefore

$$2 - \sqrt{0.196h} = 2 - \sqrt{1.96} = 2 - 1.4 = 0.6 .$$

It is therefore positive, which means that initially the height of the water is increasing. We can see, however, that the height cannot increase forever, since if $h = 100\,\mathrm{cm}$, for example, the flow out will exceed the flow in, which means that the height must decrease. There will be an **equilibrium height** at which the flow in will be balanced by the flow out. To find it we must solve

$$2 - \sqrt{0.196h_{\mathrm{eq}}} = 0, \quad h_{\mathrm{eq}} = \frac{4}{0.196} = 20.4082$$

For an initial height less than h_{eq}, the height will increase, and for one greater than h_{eq} it will decrease. In either case, as we shall see, the height will tend to but never reach the equilibrium height. One way we can judge the usefulness of a solution is by noting how well it explains this fundamental phenomenon.

Solution by formula—separable equations

Any differential equation

$$y' = F(x, y)$$

can be rewritten as

$$\frac{dy}{dx} = F(x, y) \ .$$

Sometimes we can rearrange it so all terms involving y are on one side, all involving x on the other. In these circumstances the equation is said to be **separable**. By integrating both sides, we can in some sense find an explicit solution of the differential equation.

The equation we are working with is separable. We shall look at it in detail in the next section, but it turns out to lead to some complications. A simpler example is one we shall see often in this course:

$$y'(x) = ay(x) + b$$

where a and b are constants and as the notation indicates the unknown function we are looking for is $y(x)$. The equation can be rewritten as

$$\frac{dy}{dx} = ay + b$$

and then

$$\frac{dy}{ay + b} = dx$$

which can be integrated to give

$$\frac{1}{a} \log(ay + b) = x + c$$

where c is a constant of integration. This leads to

$$ay + b = e^{ax+ac}$$

or

$$y = Ce^{at} - \frac{b}{a}$$

for some new constant $C = e^{ac}/a$. If we have the initial condition

$$y(0) = y_0$$

we can deduce that

$$C - \frac{b}{a} = y_0 , \quad C = y_0 + \frac{b}{a}$$

which leads to the final formula

$$y = y_0 + \frac{b}{a} \left(e^{ax} - 1 \right) \ .$$

What happens in our particular example

We can rewrite the differential equation as

$$\frac{dh}{dt} = 2 - \sqrt{0.196h}$$

and then get

$$\frac{dh}{2 - \sqrt{0.196h}} = dt$$

We can now take the indefinite integral of both sides:

$$\begin{aligned}\int \frac{dh}{2 - \sqrt{0.196h}} &= \int dt \\ &= t + c\end{aligned}$$

where c is a constant of integration. As for the left hand side, it can be integrated explicitly, although somewhat messily. We substitute

$$\begin{aligned}u &= 2 - \sqrt{0.196h} \\ h &= \frac{(2-u)^2}{0.196} \\ &= \frac{(u-2)^2}{0.196} \\ dh &= \frac{2(u-2)\,du}{0.196}\end{aligned}$$

$$\begin{aligned}\int \frac{dh}{2 - \sqrt{0.196h}} &= \frac{2}{0.196} \int \frac{(u-2)\,du}{u} \\ &= (10.20) \int \left(1 - \frac{2}{u}\right)\,du \\ &= (10.20)(u - 2\log u) \\ &= (10.20)\left(\left(2 - \sqrt{0.196h}\right) - 2\log\left(2 - \sqrt{0.196h}\right)\right)\end{aligned}$$

so that the differential equation is equivalent to an implicit equation for h

$$(10.20)\left(\left(2 - \sqrt{0.196h}\right) - 2\log\left(2 - \sqrt{0.196h}\right)\right) = t + c$$

The constant is not determined by the differential equation alone, but it is determined by the initial condition $h(0) = h_0$ (where in our case $h_0 = 10$):

$$c = (10.20)\left(\left(2 - \sqrt{0.196h_0}\right) - 2\log\left(2 - \sqrt{0.196h_0}\right)\right)$$

This leads to a final formula

$$\left(\sqrt{0.196h_0} - \sqrt{0.196h}\right) + 2\left(\log\left(2 - \sqrt{0.196h_0}\right) - \log\left(2 - \sqrt{0.196h}\right)\right) = \frac{t}{10.20}$$

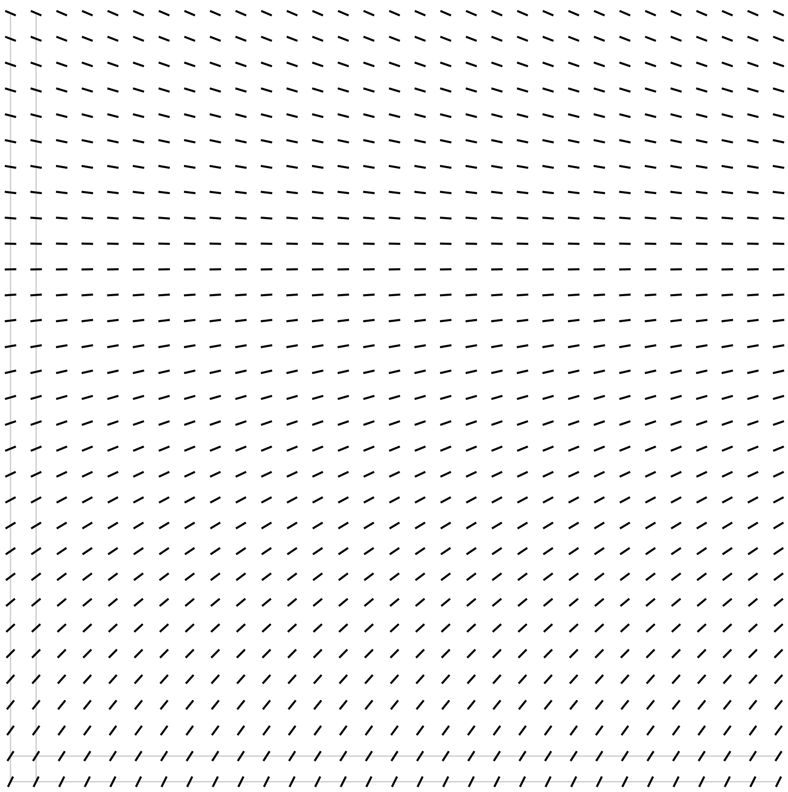
For the moment I point out that the implicit equation for h is not clearly very useful. In particular, it is not at all easy to read off from it the relationship between a solution and the equilibrium height. We shall come back later to the question of how to use this implicit formula to find explicit values for $h(t)$.

Graphical interpretation

A much less explicit, but in some ways more satisfactory, method of solving the differential equation

$$h' = F(h, t)$$

is to translate the differential equation into a picture. In looking for a solution to the equation we are looking for a function $h(t)$ whose derivative at any time t is determined by t and h itself. This can be put in graphic terms: the graph of $h(t)$ has at any point (t, h) in the plane a certain slope (which happens here to depend on h alone). We can get an idea of what the graph of h must look like by plotting these slopes. We can do this by drawing at points (t, h) in the plane a small segment with slope $F(h, t)$.



Slope field for $h' = 2 - \sqrt{0.196h}$.

This picture does not, at first sight, allow any quantitative estimate of $h(t)$, but it enables you to see easily what the general behaviour of h will be.

Numerical calculation

We can use common sense, at least, together with a great deal of computation, to obtain an estimate for $h(t)$. We know from calculus that for any function $f(x)$ and small quantity Δx that

$$f(x + \Delta x) \cong f(x) + f'(x)\Delta x$$

where $\cong$ means approximate equality. Furthermore, we know that this approximation is good up to order 2 in Δx so that it holds much better the smaller Δx is.

In our problem, we know that $h(0) = 10$ and from the differential equation we also know that

$$h'(0) = 2 - \sqrt{0.196(10)} = 2 - \sqrt{1.96} = 0.6 \; .$$

Therefore if Δt is small then by the approximation I mentioned a moment gao

$$h(\Delta t) \cong h(0) + h'(0)\Delta t = 0.6\Delta t \; .$$

In other words, at time $t = 0$ the height is increasing at the rate of 0.6 cm per sec, so that in an interval of time Δt it increases by approximately

$$\Delta h = 0.6\,\Delta t$$

Setting $\Delta t = 1$, for example, we can see that we have the approximate equation

$$h(1) \cong 10 + 0.6 = 10.6$$

But now we can repeat this analysis—unfortunately with more arithmetic. At height $h = 10.6$ the differential equation tells us that

$$h' = 2 - \sqrt{0.196(10.6)} = 0.5586$$

which makes

$$h(2) \cong 10.6 + 0.5586 = 11.1586$$

And so on. This is an ideal set-up for a spread sheet. We can lay out the calculations in a table:

t	h	$2 - \sqrt{0.196h}$
0	10.0000	0.6000
1	10.6000	0.5586
2	11.1586	0.5211
3	11.6797	0.4870
4	12.1667	0.4558
5	12.6225	0.4271
6	13.0496	0.4007
7	13.4503	0.3763
8	13.8266	0.3538
9	14.1804	0.3329
10	14.5133	0.3134

We shall see later on in the course a way to estimate how accurate this calculation is, and we shall also see much better ways to approximate the solution of a first order differential equation, but for now I shall lay out for comparison values which are accurate to 4 decimals, and for a larger range of t:

The method advocated here:

0	10.0000	0.6000
10	14.5133	0.3134
20	16.9595	0.1768
30	18.3602	0.1030
40	19.1823	0.0610
50	19.6712	0.0364
60	19.9639	0.0219
70	20.1400	0.0132
80	20.2461	0.0080
90	20.3102	0.0048
100	20.3489	0.0029
110	20.3723	0.0018
120	20.3865	0.0011
130	20.3950	0.0006
140	20.4002	0.0004
150	20.4034	0.0002
160	20.4053	0.0001
170	20.4064	0.0001
180	20.4071	0.0001
190	20.4075	0.0000
200	20.4078	0.0000
210	20.4079	0.0000
220	20.4080	0.0000
230	20.4081	0.0000
240	20.4081	0.0000

Exact solution:

0	10.0000
10	14.4091
20	16.8483
30	18.2662
40	19.1097
50	19.6177
60	19.9258
70	20.1133
80	20.2278
90	20.2978
100	20.3406
110	20.3668
120	20.3828
130	20.3926
140	20.3986
150	20.4023
160	20.4046
170	20.4060
180	20.4068
190	20.4073
200	20.4077
210	20.4079
220	20.4080
230	20.4080

240 20.4081

So you can see from this that our crude method is neither extremely accurate nor extremely inaccurate, and does give at least a rough idea of what the solution looks like. One obvious thing to do to improve its accuracy is to make the interval Δt shorter. We shall see later how much good this does.

Graphics and numerical approximation

This numerical calculation fits in nicely with the graphical method. In effect, we are sketching a solution by tracing along the line segments in the picture.

Behaviour of solutions near equilibrium

Near the equlibrium height the total flow is close to 0. If $f(h) = 2 - \sqrt{0.196h}$, then by definition $f(h_{\mathrm{eq}}) = 0$, and by Taylor's theorem we can get a better approximation

$$f(h_{\mathrm{eq}} + \eta) \cong f'(h_{\mathrm{eq}})\eta$$

Since

$$f'(h) = -\frac{\sqrt{0.196}}{2\sqrt{h}} = -\frac{\sqrt{0.196h}}{2h}$$

and

$$\sqrt{0.196h_{\mathrm{eq}}} = 2$$

we have

$$f'(h_{\mathrm{eq}}) = -\frac{1}{h_{\mathrm{eq}}} = -0.490\ .$$

Near equilibrium we can write the differential equation approximately as

$$h' = -\frac{1}{h_{\mathrm{eq}}}(h - h_{\mathrm{eq}}) = -\frac{h}{h_{\mathrm{eq}}} + 1$$

We know how to solve this equation, since we have seen something just like it in the section on separable equations. The solutions of this approximating equation are of the form

$$h = h_{\mathrm{eq}} + Ce^{-t/h_{\mathrm{eq}}}$$

where C is a constant determined by initial conditions. As $t \to \infty$ all of these solutions tend to the equilibrium height in an exponential fashion.

The same turns out to be true of solutions of the original equation.

Linear differential equations of first order

We have seen how to find the solutions of the differential equation

$$y'(x) = ay(x) + b$$

where a and b are constants. In fact we can write down a formula for solving any similar equation where a and b are allowed to be functions of x. Let's see how to understand the formula we're going to get.

First of all, consider the special case where $b = 0$ and a is a constant. Then

$$y' = ay$$

has the general solution

$$y = Ce^{ax}$$

where C is an arbitrary constant. If a is now $a(x)$ the formula for a solution is not so different from this. To get the general solution of

$$y' = a(x)y$$

we can separate x and y to get

$$\frac{dy}{y} = a(x)\,dx, quady = Ce^{A(x)}$$

where

$$A(x) = \int a(x)\,dx$$

that is to say the indefinite integral of $a(x)$. In other words, all solutions of the differential equation are of the form

$$y = Ce^{A(x)}\,.$$

For example the solutions of

$$y' = xy$$

are of the form

$$y = C\,e^{x^2/2}\,.$$

For the equation

$$y' = a(x)y + b(x)$$

we follow the following recipe.

Step (1) Find

$$A(x) = \int a(x)\,dx$$

as before.

Step (2) Then the general solution of the equation can be found by the formula

$$y = Ce^{A(x)} + e^{A(x)} \int^x e^{-A(t)} b(t)\,dt$$

where I use t as the variable of integration. It is important to keep t and x separate.

For example, to find the solutions of

$$y' = xy + x$$

we first set

$$A(x) = \frac{x^2}{2}$$

and then calculate

$$\begin{aligned} y &= Ce^{x^2/2} + e^{x^2/2} \int^x e^{-t^2/2} t \, dt \\ &= Ce^{x^2/2} - 1 \end{aligned}$$

(We can find the integral by substituting $u = t^2/2$.) *You can check this answer by calculating y' and substituting back into the differential equation!*

$$\begin{aligned} y' &= Cxe^{x^2/2} \\ yx &= Cxe^{x^2/x} - x \\ yx + x &= Cxe^{x^2/2} \end{aligned}$$

There is rarely any excuse for not knowing whether the solution you have is correct or not.

Equations of the form

$$y' = a(x)y + b(x)$$

are called **linear** since the expression on the right is linear. They make up essentially the only large class of differential equations whose solutions can be found by an explicit formula.

More about finding solutions numerically

The numerical method I have suggested above can be used very generally. Suppose we want to find values for the solution of the differential equation

$$y' = F(x, y)$$

with initial condition

$$y(x_0) = y_0 \ .$$

We first choose a sufficiently small interval Δx. Then we set in succession for $n = 0, 1, \ldots$

$$\begin{aligned} x_{n+1} &= x_n + \Delta x \\ y_{n+1} &= y_n + F(x_n, y_n) \, \Delta x \end{aligned}$$

to obtain a sequence of approximate values y_n for $y(x_n)$.